SYNCHRONIZATION OF TWO HINDMARSH-ROSE NEURONS WITH THRESHOLDS COUPLING

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Abstract
The Hindmarsh-Rose (HR) dynamical system is a well-known model of neuronal activity. This work addresses chaotic dynamic synchronization of two HR neurons via coupling based on the thresholds. The coupling is unidirectional and it is carried out by an underdamped signal, which activates every time that the master system crosses the threshold represented by means of a Poincaré plane. A novel master-slave system is presented, and the synchronization between the systems is detected via the auxiliary system approach. Numerical explorations verify such synchronization where the parameters of two HR neurons might differ. The result gives away a fundamental question on the valid interpretation of unidirectional links, their potential use, and if such chaotic synchronization is an active principle of biological neurons.

Key words
Chaos synchronization, Hindmarsh-Rose neuron, Poincaré plane, unidirectional coupling, nonlinear dynamics.

1 Introduction

In neuroscience, experimental evidence indicates that neuronal synchronization plays an important role in processing biological information in the brain [Purves *et. al.*, 2004], e.g. during the processing of olfactory information in the olfactory bulb [Desmaisons, Vincent and Lledo, 1999]. Furthermore, chaos has been found from neuronal network dynamics to macroscopical electroencephalography (EEG) both in theory and experimentally [Hrg, 2013; Nguyen and Hong, 2013; Makarenko and Llinás, 2013]. It is typically realized that synchronization of neuronal activities featured by chaotic synchronization is important for memory, learning, motion control and diseases such as epilepsy [Babloyantz and Destexhe, 1986]. Moreover, it plays an important role in the realization of associative memory, image segmentation and binding [Tsuda, 2001]. In order to ease the neural synchronization study, typical dynamical models of the electrical activity can be implemented, e.g. the Hindmarsh-Rose neuronal model.

The rich and interesting nonlinear dynamical behaviour of the Hindmarsh-Rose (HR) is one of the most popular and studied low-dimensional neuronal model in neurological sciences [Dtchetgnia Djeundam *et. al.*, 2013]. In this model, the action potential of a single neuron fires due to a sufficient stimulus (named threshold), induces several behaviour modes reflecting the daintiness activity of a genuine neuron: a succession of a rest state, firing action potential, and deactivation period. Also the system exhibits chaos for an appropriate choice of its intrinsic parameters [Storace, Daniele and Lange, 2008]. In nature, the action potential of a neuron is propagated as a wave of depolarization, followed closely by a corresponding wave of repolarization. When the membrane has just completed this cycle, it is in the refractory state for some milliseconds [Purves *et. al.*, 2004]. This delay prevents the action potential from spreading "backward" toward the body cell (i.e., antidromic impulse conduction), and ensures that under normal conditions the impulse conduction is unidirectional [Brodal, 2016]. Therefore, the HR model can be used in the simulation of the brain activity to investigate the chaotic synchronization. The idea of synchronization of two chaotic systems with common driving signals and different regular coupling schemes, was described by [Pecora *et. al.*, 1997]. The method is based on linking a trajectory of the master system to the same values of the slave one.

This work proposes a mechanism on how to couple two HR neurons and synchronize its electrical activity; to carry out this, the detection of a threshold of the electrical activity of a master neuron in regimen chaotic via *Poicaré* plane was implemented as previously defined by [Ontañón-García *et. al.*, 2013]. The idea is to generate a driving signal which is activated in discrete time events caused by the crossing of a specific threshold by some master neuron with a previously defined *Poincaré* plane.

The rest of this article is organized as follows: In the Section 2, the Hindmarsh-Rose neuronal model is briefly introduced. Section 3, the dynamical behaviour of a single Hindmarsh-Rose neuron and its bifurcation diagram are presented. In Section 4, it is proposed the tresholds coupling based on *Poincaré* plane. Section 5, contains chaotic synchonization of two HR neurons threshold coupled and numerical results about master-slave interconnection. Finally, conclusions are made in Section 6.

2 The Hindsmarsh-Rose neuronal model

In nonlinear dynamics and neuroscience, the Hindsmarsh-Rose neuronal model is a simplified version of the physiologically realistic model proposed by Hodgkin and Huxley [Hodgkin and Huxley, 1952] and a modification of the FitzHugh-Nagumo [FitzHugh, 1961] equations. It was originally proposed to model the firing synchronization of two snail neurons [Coombes and Bressloff, 2005]. The HR neuronal model is given by

$$\dot{\boldsymbol{X}} = \begin{bmatrix} y - x^3 + bx^2 + I - z \\ 1 - dx^2 - y \\ r(s(x - x_r) - z) \end{bmatrix}, \tag{1}$$

where $\boldsymbol{X} = [x, y, z]^T \in \mathbb{R}^3$ is the vector of the state variables. The relevant state variable $x(t)$ is known as the membrane potential, $y(t)$ (spiking variable) is the recovery variable associated with the fast current of Na^+ or K^+ ions, $z(t)$ (bursting variable) is the adaptation current associated with the slow current of, for instance, Ca^{2+} ions. $I \in \mathbb{R}_0^+$ or $I(t)$ is the external current injected into the neuron, while $b \in \mathbb{R}^+$ represents the qualitative behaviour of the model. $r \in \mathbb{R}^+$ is a small parameter ($0 < r \ll 1$) that governs the bursting behaviour, $x_r \in \mathbb{R}$ is the x-coordinate of the point of stable equilibrium in the event that an external current is not applied and $d \in \mathbb{R}^+$. $s \in \mathbb{R}^+$ governs adaptation: smaller values of $s \approx 1$ result in fast spiking behaviour [Hindmarsh and Rose, 1984].

3 The dynamics of a single HR neuron

For certain values of the parameters r and I, various dynamic behaviours of the membrane potential $x(t)$ can be observed as shown in Figure 1, where the variety of dynamical behaviours, one may find some types:

1. Resting state: the stimulus to the neuron is below a certain threshold and the response reaches a stationary regime as shown in Figure 1(a).
2. Tonic spiking: the response is made up a regular series of equally spaced spikes as shown in Figure 1(b).
3. Regular bursting: the response is made up of groups of two or more spikes (called burst) separated by periods of inactivity as shown in Figure 1(c).
4. Chaotic bursting: the response is made up of an aperiodic series of burst as shown in Figure 1(d).

3.1 The bifurcation diagram

In order to identify the chaotic regions of the HR neuron, was plotted the bifurcation diagram as a function of r with different bursting periods. So it is define a *Poincaré map* as follows:

Definition 3.1. [Perko, 2013; Starke *et. al.*, 2010] *Consider the flow $\phi(t)$ as the solution of the system given by equation (1). A local cross section, the Poincaré section, $\Sigma \in \mathbb{R}^2$ is taken such that the flow $\phi(t)$ is everywhere transverse to it. Like a section through the unique point $\boldsymbol{X}^* = (x^*, y^*, z^*)^T = ((max(x(t)) + min(x(t)))/2, (max(y(t)) + min(y(t)))/2, (max(z(t)) + min(z(t)))/2)^T$ in the middle of attractor $\mathscr{A}_X$ of the system (1) to guarantee $\mathscr{A}_X \cap \Sigma \neq \emptyset$, is chosen:*

$$\Sigma = \{\boldsymbol{X} \in \mathbb{R}^3 | \boldsymbol{N} \cdot (\boldsymbol{X} - \boldsymbol{X}^*) = 0\}, \tag{2}$$

where $\boldsymbol{N} = [n_1, n_2, n_3]^T \in \mathbb{R}^3$ is the nonzero normal vector. Let p_1 be the point on Σ where $\phi(t)$ intersects with Σ, then the Poincaré map $P : \Sigma \to \Sigma$. Thus starting at point p_1 on Σ, the Poincaré map will define the next intersection p_2 of the flow $\phi(t)$ with Σ. This is called the first return map. Starting from point p_2, the second intersection of the flow with Σ gives the point p_3 and so on. The complete map is thus defined as

$$P : p_i \to p_{i+1}, i = 1, \ldots, \infty. \tag{3}$$

Hence, the Σ crosses the attractor $\mathscr{A}_x$, generating the points $\{p_1, p_2, p_3, \ldots\} \in \Sigma$ at each crossing event and therefore specify the following time series $\Delta_{X_0} = \{t_1, t_2, t_3, \cdots\} \in \mathbb{R}_0^+$ corresponding to each crossing time event.

In this study, the values $b = 3$, $d = 5$, $I = 3$, $s = 4$ and $x_r = -8/5$ are used. The bifurcation diagram shown in Figure 2, will be obtained by calculating the

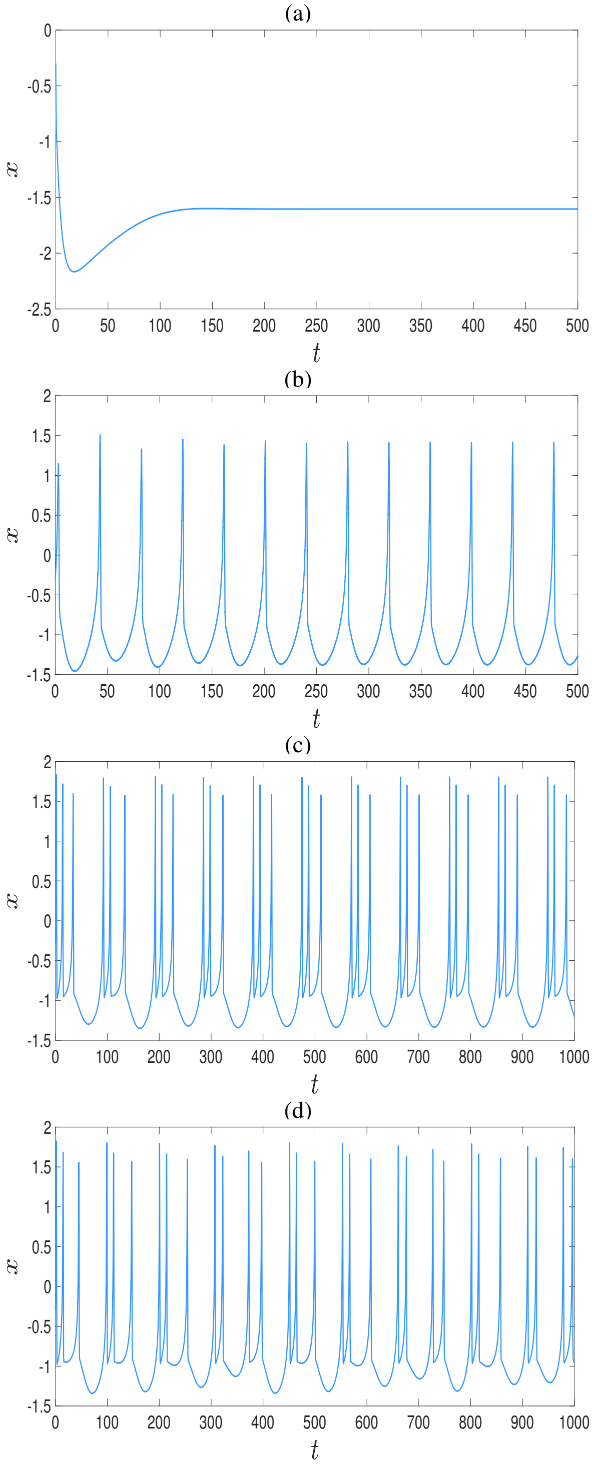

Figure 1: Dynamic behaviours of a single Hindsmarsh-Rose neuron: (a) Resting state with $r = 0.01325$ and $I = 0$. (b) Tonic spiking with $r = 0.045$ and $I = 3$. (c) Regular bursting with $r = 0.011$ and $I = 3$. (d) Chaotic bursting with $r = 0.01325$ and $I = 3$.

intersections of the trajectories of the system (1) with a plane $\Sigma = \{(x, y, z) \in \mathbb{R}^3 | x + y - 3 = 0\}$ satisfying Definition 3.1. The Figure 2 shows the transition from simple bursting to complex bursting oscillation via intermittent chaos as r decreased, according by [Fan and Holden, 1993] for $r \geq 0.016$ and $0 < r \leq 0.0045$,

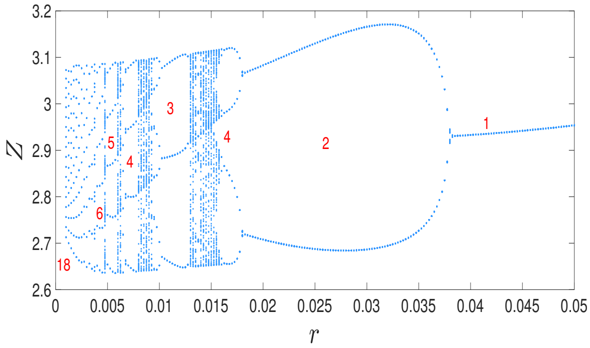

Figure 2: The bifurcation diagram of a single HR neuron model with $I = 3$.

the HR neuronal model exhibits continuous periodic spiking, marked with red numbers some of the periodic states of the system, e.g. when $r = 0.001$ the bursting have period 18. The chaotic behaviour is found in the regions $r = 0.00475$, $0.00575 < r < 0.0065$, $0.00775 < r < 0.0095$ and $0.01275 < r < 0.016$.

4 Thresholds coupling of two HR neurons

The threshold associated to the master system, given by the HR neuron (4), will be defined as a *Poincaré section* in the phase space of the master and slave HR neurons with unidirectional coupling, where neurons interact only when an orbit of the master HR neuron crosses this section. So, consider the following master-slave systems representation of the HR neuron (1):
Master neuron:

$$\dot{\boldsymbol{X}}_m = F(\boldsymbol{X}_m) = \begin{bmatrix} y_m - x_m^3 + 3x_m^2 + I - z_m \\ 1 - 5x_m^2 - y_m \\ r_m\left(4(x_m + \frac{8}{5}) - z_m\right) \end{bmatrix}. \tag{4}$$

Slave neuron:

$$\dot{\boldsymbol{X}}_s = G(\boldsymbol{X}_m, \boldsymbol{X}_s) = F(\boldsymbol{X}_s) + kH(\boldsymbol{X}_m, \boldsymbol{X}_s), \tag{5}$$

where the state vectors are represented by $\boldsymbol{X}_m = [x_m, y_m, z_m]^T \in \mathbb{R}^3$ and $\boldsymbol{X}_s = [x_s, y_s, z_s]^T \in \mathbb{R}^3$ with corresponding functions $F(\boldsymbol{X}_m) : \mathbb{R}^3 \to \mathbb{R}^3$ for the master system and $G(\boldsymbol{X}_m, \boldsymbol{X}_s) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ for the slave system, k is the coupling strength and the coupling based on thresholds correspond to $H(\boldsymbol{X}_m, \boldsymbol{X}_s) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. Which is given by:

$$H(\boldsymbol{X}_m, \boldsymbol{X}_s) = S(\boldsymbol{X}_m - \boldsymbol{X}_s). \tag{6}$$

Where S is a scalar function which is defined in a given interval and is determined as follows:

$$S(t, t_i) = \begin{cases} e^{-\tau(t-t_i)}\cos(t - t_i) & \text{, if } t \in [t_i,\ t_{i+1}) \\ 0 & \text{, otherwise} \end{cases}, \tag{7}$$

where $t_i \in \Delta_{X_0}$ from Definition 3.1, $\tau \in \mathbb{R}^+$ represents an underdamping factor which modulate the magnitude of the signal and its frequency. Note that for a value of $t = t_i$, the coupling starts.

Thus, having this in consideration it is define finally the thresholds coupling:

Definition 4.1. *Let $\boldsymbol{X}_0$ be a point in the phase space of the master system (4) and $\Delta_{X_0} = \{t_1, t_2, \ldots\}$ be a time series comprised of the events generated each time that the trajectory of the master system with initial condition $\boldsymbol{X}_0$ crosses the plane Σ according the Definition 3.1. If the coupling $H(\boldsymbol{X}_m, \boldsymbol{X}_s)$ from equation (6) depends on the time series Δ_{X_0} then the coupling is called a threshold coupling.*

5 Chaotic synchronization of two HR neurons threshold coupled

In this section, synchronization of two threshold coupled chaotic HR neurons with different parameters that governs the bursting bahaviour is presented, the synchronization is defines as follows:

Definition 5.1. [Zhang, Liu and Ma, 2007] *It is said that the systems (4) and (5) are generalized synchronous (GS) with respect to the vector map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$, if there exist a function $H(\boldsymbol{X}_m, \boldsymbol{X}_s)$ from equation (6) such that the solutions $\boldsymbol{X}_m(t)$ and $\boldsymbol{X}_s(t)$ of systems (4) and (5), respectively, satisfy the following property:*

$$\lim_{t\to\infty} \|\boldsymbol{X}_s(t) - \Phi(\boldsymbol{X}_m(t))\| = 0, \qquad (8)$$

where the map Φ is an arbitrary continuously differentiable function. Note that if Φ is the identity function, then the systems (4) and (5) are completely synchronized.

With the purpose of analysis and detection of GS between the master and the slave HR neurons, the auxiliary system approach was defined by [Abarbanel, Rulkov and Sushchik, 1996]. Here it is considered an auxiliary system identical to the slave system (5), and coupled in the same way to the system (4), but with different set of initial conditions $\boldsymbol{X}_s(0) \neq \boldsymbol{X}_{aux}(0)$. For practical purposes is defined as the synchronization error between system (5) and the auxiliary system, as follows:

$$\boldsymbol{\xi}(t) = \boldsymbol{X}_s(t) - \boldsymbol{X}_{aux}(t), \qquad (9)$$

where $\boldsymbol{\xi}(t) = [\xi_x, \xi_y, \xi_z]^T : \mathbb{R}^3 \to \mathbb{R}^3$, so $\xi_x = x_s - x_{aux}$, $\xi_y = y_s - y_{aux}$ and $\xi_z = z_s - z_{aux}$.

The synchronization can be easily detected by this method, if the coupled systems present only one basin of attraction (see [Ontañón-García *et. al.*, 2013]) and satisfy the asymptotic condition (8) from Definition 5.1.

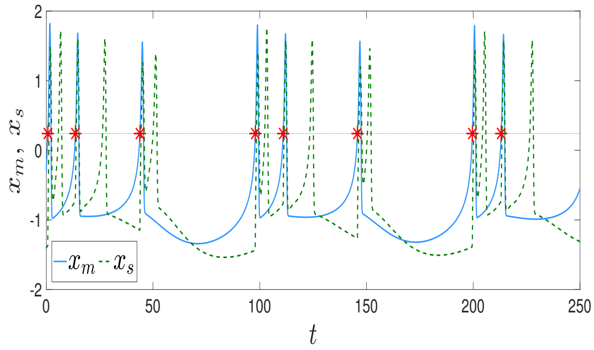

Figure 3: Time evolution of the master neuron (4) and slave neuron (5), with solid line and green dashed line, respectively.

If there exists only one basin of attraction and GS is ensured, $\lim_{t\to\infty} \|\boldsymbol{\xi}(t)\| = 0$ is satisfied for any initial state $\boldsymbol{X}_s(0)$ and $\boldsymbol{X}_{aux}(0)$.

5.1 Numerical simulations

In this section, numerical simulations are performed to synthesize the proposed strategy. The same set of model parameters as provided in Section 3.1, but different values of the parameter r, i.e., for the master HR neuron $r_m = 0.01325$ and the slave HR neuron $r_s = 0.008$ (both in chaotic regimen). The selected initial conditions of the master, slave and auxiliary neurons are set as $\boldsymbol{X}_m(0) = [-0.2984, 0.0001, 2.5915]^T$, $\boldsymbol{X}_s(0) = [-1.4084, -8.992, 2.4947]^T$ and $\boldsymbol{X}_{aux}(0) = [-1.4913, -10.108, 2.6267]^T$, respectively, are used throughout the section.

In order to meet the requirements of plane Σ from the Definition 3.1, the parameters take following values $\boldsymbol{X}^* = [0.2406, -3.6525, 2.9169]^T$ with $\boldsymbol{N} = [1, 0, 0]^T$, and so the condition $\mathscr{A}_X \cap \Sigma \neq \emptyset$ is fullfilled. This location has been chosen in order to detect the membrane potential (threshold) of the master HR neuron (4). Figure 3 depicts the x_m and x_s states of the coupled master and slave HR neurons, represented as continuous blue line and dashed green line, respectively, the black dashed-dot line marks the plane Σ, and each crossing event is marked with red asterisks. When the coupling signal from equation (7) is zero because no exist crosses with a *Poicaré* plane (i.e., $\mathscr{A}_X \cap \Sigma = \emptyset$), the slave HR neuron oscillates chaotically independently. Each crossing event is marked with a red asterisk. So the time series Δ_{X_0} contains each crossing event that satisfies $\dot{x}_m > 0$. Figure 4 depicts the signal of equation (5) along with the crossing events. The coupling strength and external applied current are $k = 5$ and $I = 3$, respectively.

The results of the computer simulations at $\tau = 0.9$ and $k = 5$ are presented in Figure 5. Figure 5(a) and Figure 5(b) show the projections of the attractor from its phase space onto the planes (x_s, y_s) and (x_s, x_{aux}), respectively. One can see from plot shown in Figure 5(b) that the manifold $\boldsymbol{X}_s = \boldsymbol{X}_{aux}$, the slave HR neu-

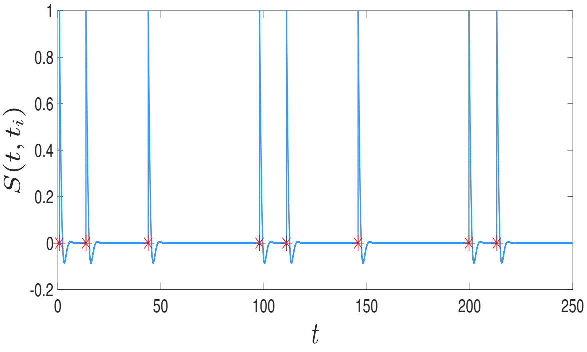

Figure 4: Coupling thresholds function (7) in solid line for $\tau = 0.9$ and $k = 5$. Marked with red asterisk the events t_i of each intersection of the master system with the plane Σ according the Definition 4.1.

ron and the auxiliary system are completely synchronized, and therefore the manifold of synchronized motions specified by $\boldsymbol{X}_s = \Phi(\boldsymbol{X}_m)$, using the auxiliary system approach, the master and the slave HR neurons with chaotic regimen are generalized synchronous satisfy Definition 5.1. Also, corroborated with the synchronization error given by equation (9), between the slave neuron (5) and system auxiliary, which converge asymptotically to zero shown in Figure 5(c).

The (y_s, y_{aux}) and (z_s, z_{aux}) projections of the attractor look identical to the one in Figure 5(b) and were not included in the article in order to avoid redundancy.

6 Conclusion

This article investigates the synchronization states of two HR neurons coupled by thresholds. The synchronization behaviour of the Hindmarsh-Rose neurons is investigated and is an important topic to consider because due to this phenomenon, the neural processing is carried out in real biological systems. When HR neurons are synchronized in the generalized sense. The motion in the combined phase space of the master HR neuron and the slave HR neuron collapses in a stable way onto a manifold dictated by the synchronization relationship $\boldsymbol{X}_s(t) = \Phi(\boldsymbol{X}_m(t))$, and when the orbits reach this synchronization manifold they remain there. Also, the synchronization between slave and auxiliary system HR neurons is complete. The validity of this approach is verified numerically.

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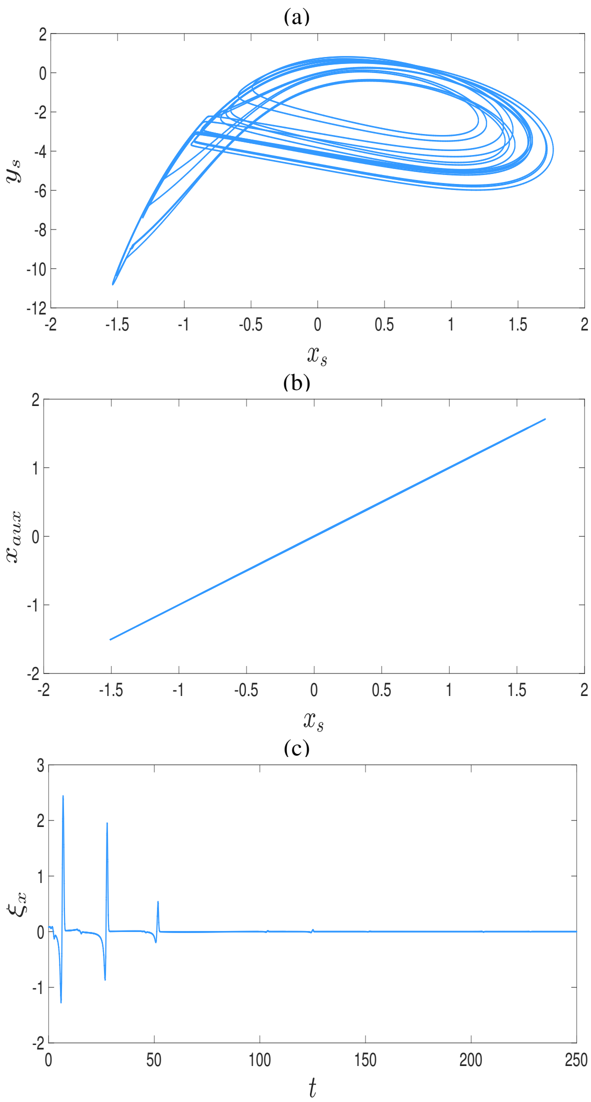

Figure 5: Projections of the synchronized attractor by the coupled thresholds between (4) and (5) HR neurons. For $\tau = 0.9$ and $k = 5$ values, for some time. (a) The projection of the slave neuron onto the (x_s, y_s) plane. (b) At these values of k and τ the systems are synchronized, view this in the projection onto (x_s, x_{aux}) plane. (c) Time evolution of synchronization error ξ_x (9) satisfies the asymptotic condition in equation (8).

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