A CRITERION FOR NONEXISTENCE OF LIMIT CYCLES FOR DIFFERENTIAL INCLUSIONS

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Abstract: A condition which ensures the absence of periodic orbits for nonsmooth dynamical systems is presented. A connection to methods for estimating the Hausdorff dimension is emphasized. For a class of hybrid systems described by a linear system with relay feedback the conditions are presented in the form of linear matrix inequalities.

Keywords: linearization, periodic solutions, direct Lyapunov method

Discontinuous dynamical systems and, particularly, relay systems have attracted considerable attention over the last decades. While the mathematics of smooth dynamical systems still produces new and interesting discoveries, in applied disciplines it has been realized that for many applications discontinuities should be taken into account. For example, discontinuities can be used to simplify modeling of friction in mechanical systems, to design disturbance tolerant sliding mode controllers, to deal with a switching control strategy in manufacturing systems, and so on. A hot topic in research in the control community is formed by the class of so called hybrid dynamical systems, which combine continuous and discrete dynamics (Filippov, 1988; Utkin, 1992; Tsypkin, 1984; Clarke *et. al.*, 1998; Matveev & Savkin, 2000).

The main purpose of this paper is to present a possible generalization of the Bendixson result to arbitrary dimension taking into account the possible discontinuity of the right hand side. There are several higher dimensional generalizations of this criterion, see, e.g. (Smith, 1981; Muldowney, 1990; Li & Muldowny, 1993; Li & Muldowny, 1996; Starkov, 2005). Muldowney and Li (Muldowney, 1990; Li & Muldowny, 1993; Li & Muldowny, 1996) used an approach based on compound matrices to prove a negative Bendixson-like criterion. In this paper we investigate this question by a method which allows to estimate the Hausdorff dimension of invariant compact sets (Douady and Osterlé, 1980; Smith, 1986; Témam, 1988; Leonov *et. al.*, 1996; Pogromsky and Nijmeijer, 2000; Reitmann and Schnabel, 2000; Booichenko *et. al.*, 2005).

The paper is organized as follows. In Section II we present some necessary background material. Section III contains some result on the estimation of the Hausdorff dimension of invariant sets. Based on these results in Section IV we present a new version of a generalized Bendixson's criterion. Particular attention is then drawn to LMI-based results for linear systems with relay feedback.

1. HAUSDORFF DIMENSION

Consider a compact subset K of $\mathbb{R}^n$. Given $d \geq 0$, $\varepsilon > 0$, consider a covering of K by open spheres B_i with radii $r_i \leq \varepsilon$. Denote by

$$\mu(K, d, \varepsilon) = \inf \sum_i r_i^d \tag{1}$$

the d-measured volume of covering of the set K. Here the infimum is calculated over all finite ε-coverings of K. There exists a limit, which may be infinite,

$$\mu_d(K) := \sup_{\varepsilon > 0} \mu(K, d, \varepsilon) = \lim_{\varepsilon \to 0} \mu(K, d, \varepsilon).$$

Definition 1. The measure μ_d is called the *Hausdorff d-measure.*

Some properties of the measure μ_d can be summarized as follows. There exists a single value $d = d_*$, such that for all $d < d_*$, $\mu_d(K) = +\infty$ and for all $d > d_*$, $\mu_d(K) = 0$, with

$$d_* = \inf\{d \, : \, \mu_d(K) = 0\} = \sup\{d \, : \, \mu_d(K) = +\infty\}.$$

(see Proposition 5.3.2 in (Leonov *et. al.*, 1996)).

Definition 2. The value d_* is called the *Hausdorff dimension* of the set K.

In the sequel, we will use the notation $\dim_H K$ for the Hausdorff dimension of the set K.

For the control community the notions of Hausdorff measure and Hausdorff dimension are not common and we like to clarify the above definitions.

Suppose we have a two-dimensional bounded surface S with area $m(S)$. We cover this surface by open spheres as required in the definition of the Hausdorff measure. Then, for $d = 1$ and $d = 3$ we have

$$\mu_1(S) = \lim_{\varepsilon \to 0} \mu(S, 1, \varepsilon) = +\infty,$$
$$\mu_3(S) = \lim_{\varepsilon \to 0} \mu(S, 3, \varepsilon) = 0,$$

while for $d = 2$ we have

$$\mu_2(S) = \frac{m(S)}{\pi}$$

This example illustrates the behavior of $\mu_d(K)$ for a given K as a function of d. Namely, for values of d less than $\dim_H K$, $\mu_d(K)$ is infinite and for all values of d greater than $\dim_H K$ $\mu_d(K)$ is zero (see Proposition 5.3.2 in (Leonov *et. al.*, 1996)).

2. UPPER ESTIMATES FOR THE HAUSDORFF DIMENSION OF INVARIANT COMPACT SETS

Consider a system of differential equations

$$\dot{x} = f(x), \quad x \in \Omega \subset \mathbb{R}^n, \quad x_0 \in \Omega \tag{2}$$

where $f \; : \; \Omega \to \mathbb{R}^n$ is a (possibly) discontinuous vector field defined on some open positively invariant set Ω, and which satisfies conditions guaranteeing the existence of solutions $x(t, x_0)$ in Ω in some reasonable sense, that is, if the function f is discontinuous and satisfies some mild regularity assumptions, one can construct a set-valued function $\mathbf{f}$ according to numerous possible definitions (e.g., Filippov convex definition, Utkin's equivalent control, etc.) such that an absolutely continuous solution of the differential inclusion

$$\dot{x} \in \mathbf{f}(x)$$

is called a solution for system (2). We assume that the set-valued function $\mathbf{f}$ is bounded, upper semicontinuous with closed convex values.

Later on (Lemma 1) we impose conditions that guarantee uniqueness of the solutions (2) in positive time. The parameterized mapping $x_0 \mapsto x(t, x_0)$, $t \geq 0$, or the semi-flow will be denoted as $\varphi^t \; : \Omega \to \Omega$.

Consider a scalar differentiable function $V \; : \; \Omega \times \Omega \to \mathbb{R}$, $V(x, x) = 0$.

Define the time derivative of the function V along *two* solutions $x(t, x_1)$, $x(t, x_2)$ of (2) as follows

$$\dot{V}(x_1,x_2) := \frac{\partial V(x_1,x_2)}{\partial x_1}\dot{x}_1 + \frac{\partial V(x_1,x_2)}{\partial x_2}\dot{x}_2.$$

Since V is Lipschitz continuous and the solutions $x(t,x_i)$ are absolutely continuous functions of time, the derivative

$$\dot{V}(x(t,x_1),x(t,x_2))$$

exists almost everywhere in $[0,\min_i \bar{T}_i)$, where $\bar{T}_i$ is the maximal interval of existence of the solution $x_i(t,x_{i0})$ in Ω.

For the function V we can also define its upper derivative as follows

$$\dot{V}^*(x_1,x_2) = \sup_{\xi_i\in\mathbf{f}(x_i)}\left(\frac{\partial V(x_1,x_2)}{\partial x_1}\xi_1 + \frac{\partial V(x_1,x_2)}{\partial x_2}\xi_2\right).$$

Then for almost all $t \geq 0$ it follows that

$$\dot{V}(x(t,x_1),x(t,x_2)) \leq \dot{V}^*(x(t,x_1),x(t,x_2)).$$

By the same token, the lower derivative of the function V is defined as

$$\dot{V}_*(x_1,x_2) = \inf_{\xi_i\in\mathbf{f}(x_i)}\left(\frac{\partial V(x_1,x_2)}{\partial x_1}\xi_1 + \frac{\partial V(x_1,x_2)}{\partial x_2}\xi_2\right).$$

and satisfies

$$\dot{V}(x(t,x_1),x(t,x_2)) \geq \dot{V}_*(x_1(t,x_1),x_2(t,x_2)).$$

almost everywhere.

We formulate the following hypothesis:

H1. There exists a $n\times n$ symmetric positive definite matrix P, such that the function

$$V(x_1,x_2) = (x_1-x_2)^\top P(x_1-x_2) \qquad (3)$$

satisfies the following inequality

$$\dot{V}^*(x_1,x_2) \leq (x_1-x_2)^\top Q(x_1)(x_1-x_2)+o(||x_1-x_2||^2)$$

for all $x_1,x_2\in\Omega$ with a symmetric differentiable matrix valued function Q, bounded on Ω and with the function of higher order terms o obeying

$$\frac{o(||x_1-x_2||^2)}{||x_1-x_2||^2} \to 0, \quad \text{as} \quad ||x_1-x_2|| \to 0$$

uniformly over x_1,x_2 from any compact subset of Ω.

H1a. The lower derivative of the function $W(x_1,x_2) = (x_1-x_2)^\top Q(x_1)(x_1-x_2)$ satisfies the following condition: for any compact subset of Ω there is a number M that

$$\dot{W}_*(x_1,x_2) \geq MV(x_1,x_2).$$

Note that assumption H1a is always satisfied if f is locally Lipschitz continuous.

H2. All solutions starting in Ω are defined for all $t\geq 0$.

We begin with the following preliminary result:

Lemma 1. Suppose the assumptions H1 and H2 are satisfied. Then any solution $x(t,x_0)$ to (2), with $x_0\in\Omega$ is right-unique (for the definition of right-uniqueness see (Filippov, 1988), Chapter 2, page 106)) and depends continuously on the initial conditions.

The previous lemma shows that the Cauchy problem (2) is well-posed and continuous dependence on initial conditions follows.

Let $\lambda_1(x)\geq\lambda_2(x)\geq\ldots\geq\lambda_n(x)$, $x\in\Omega$ be the ordered solutions of the following generalized eigenvalue problem

$$\det(Q(x)-\lambda P) = 0$$

which are real since both Q and P are symmetric.

Consider a compact set S of finite Hausdorff d-measure for some $d = d_0+s$, $d\leq n$, where $d_0\in\mathbb{N}$ and $s\in[0,1)$. Suppose that $S\subset\Omega$, then $\varphi^t(S)\subset\Omega$ for all positive t. Now we formulate the following result.

Theorem 2. Suppose hypotheses H1, H1a and H2 are satisfied. If for some $d = d_0+s$, $0<d_0\leq n$, $0\leq s<1$ it follows that

$$\sup_{x\in\Omega}(\lambda_1(x)+\ldots+\lambda_{d_0}(x)+s\lambda_{d_0+1}(x)) < 0. \quad (4)$$

Then

$$\lim_{t\to\infty}\mu_d(\varphi^t(S)) = 0.$$

The proof is based on the construction of a finite set of affine maps $\mathbb{R}^n\to\mathbb{R}^n$ which locally approximate φ^t. Then, using the linear part of those maps we approximate how the d-measured volume is changed under those maps to compute the change of $\mu_d(\varphi^t(S))$. The main result of this section is the following theorem.

Theorem 3. Suppose hypotheses H1, H1a and H2 are satisfied, and there exist positive integer d_0

and real $s \in [0, 1)$ such that

$$\sup_{x\in\Omega} (\lambda_1(x) + \ldots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x)) < 0. \quad (5)$$

Suppose that there is an invariant compact set $K \in \Omega$.

Then $\dim_H K \leq d_0 + s$.

3. A HIGHER-DIMENSIONAL GENERALIZATION OF BENDIXSON'S CRITERION

We begin with some definitions.

Definition 3. (Federer, 1969) A set $S \subset \mathbb{R}^n$ is called a d-dimensional rectifiable set, $d \in \mathbb{N}$ if $\mu_d(S) < \infty$ and μ_d-almost all of S is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^d$ to $\mathbb{R}^n$.

The rectifiable sets are generalized surfaces of geometric measure theory. Any 1-dimensional closed rectifiable contour γ bounds some two-dimensional rectifiable set, for example the cone over γ.

A set is said to be simply connected if any simple closed curve can be contracted to a point continuously in the set.

Theorem 4. Suppose that assumptions H1, H1a and H2 are satisfied, let Ω be a simply connected set. Suppose that

$$\sup_{x\in\Omega}(\lambda_1(x) + \lambda_2(x)) < 0. \quad (6)$$

Then Ω does not contain whole periodic orbits.

The proof of Theorem 4 follows an idea used in the proof of the Leonov theorem ((Leonov, 1991), see also Theorem 8.3.1 in (Leonov *et. al.*, 1996)).

It is worth noting that this theorem being applied to smooth systems together with its time reversed version (for smooth systems we have local right and left uniqueness) gives the classical Bendixson divergency condition.

The main idea of the proof (see (Leonov, 1991)) is based on the existence of a surface with minimal area given its boundary. Although the mathematical problem of proving existence of a surface that has minimal area and which is bounded by a prescribed curve, has a long defied mathematical analysis, an experimental solution is easily obtained by a simple physical device. Plateau, a Belgian physicist, studied the problem by dipping an arbitrarily shaped wire frame into a soap solution. The resulting soap film corresponds to a relative minimum of area and thus produces a minimal surface spanned by that wire contour. A classical solution to Plateau's problem can be found, for example, in (Courant, 1950) with some regularity assumptions on the contour γ that can be violated if γ is a closed orbit corresponding to a periodic solution of a system of differential equations with discontinuous right hand sides. Fortunately, the argument based on geometric measure theory allows to overcome this difficulty.

3.1 *Example*

Consider the following system:

$$\dot{x} = Ax + Bu, \quad u = -\text{sign}(y), \quad y = Cx \quad (7)$$

where $x \in \mathbb{R}^3$, $u, y \in \mathbb{R}^1$ and the matrices A, B, C are given as follows

$$A = \begin{pmatrix} \alpha & 1 & 1 \\ -1 & \beta & -1 \\ -1 & 1 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}, \ C = (0\ 0\ 1)$$

with positive b. Consider the smooth function (3) in the form

$$V = (x_1 - x_2)^\top (x_1 - x_2)$$

For this system the corresponding solution according to the Filippov convex definition coincides with the Utkin solution (Filippov, 1988). At the discontinuity points of the right-hand side, the corresponding set valued function in the differential inclusion is obtained by the closure of the graph of the right hand side and by passing over to the convex hull. As shown in (Filippov, 1988), p.155, these procedures do not increase the upper value of $\dot{V}^*$ and hence it is sufficient to compute the derivative of V only in the area of continuity of the right hand side. The derivative of V in this area satisfies

$$\dot{V} \leq 2(x_1 - x_2)^\top \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -1 \end{pmatrix} (x_1 - x_2)$$

The condition $H1a$ is satisfied since $b > 0$. The previous theorem suggests that if $\min\{\alpha, \beta\} \geq -1$, a sufficient condition for the absence of periodic solutions is

$$\alpha + \beta < 0 \tag{8}$$

To demonstrate that violation of the condition (8) can result in oscillatory behavior we perform a computer simulation for the following parameter values: $\alpha = 1, \beta = -1/2, b = 1$. The results of the simulation are presented in Figure 1. It is seen that the system possesses an orbitally stable limit cycle.

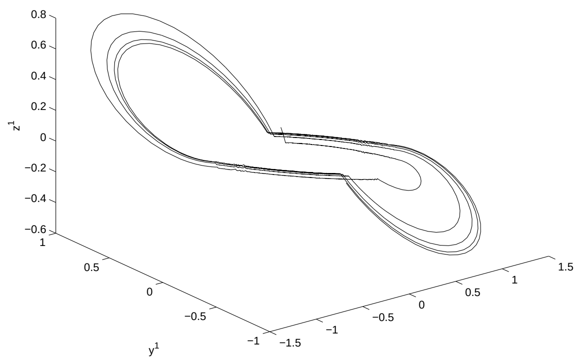

Fig. 1. Oscillatory behavior of (7) for $\alpha + \beta > 0$.

3.2 *An LMI based criterion for Lur'e systems with discontinuous right hand side*

In the previous example the matrix A was chosen as a sum of a diagonal and a skew-symmetric matrix that made all necessary calculations trivial. Next we present an LMI based criterion which ensures the absence of periodic solutions for the following system:

$$\dot{x} = Ax + Bu, \quad u = -b\mathrm{sign}(y), \quad y = Cx \tag{9}$$

where $x \in \mathbb{R}^n$, $n \geq 2$, $u, y \in \mathbb{R}^1$, $b > 0$ and the matrices A, B, C are of corresponding dimensions.

Theorem 5. Suppose there exists μ and positive definite matrix P such that the following inequality

$$\begin{pmatrix} P(A - \mu I_n) + (A - \mu I_n)^\top P & * & * \\ B^\top P - C & 0 & 0 \\ B^\top (PA + A^\top P) + \gamma C & 0 & 0 \end{pmatrix} \geq 0 \tag{10}$$

is satisfied for some $\gamma \geq 0$. Then if

$$\mathrm{tr}A - (n-2)\mu < 0$$

the system (9) does not have periodic solutions.

Proof: According to (10) the matrix P satisfies the following equation $PB = C^\top$. Thus taking the derivative of the following function

$$V = (x_1 - x_2)^\top P (x_1 - x_2)$$

yields (as in the previous example it is sufficient to compute the derivative in the area of continuity of the right hand side)

$$\begin{aligned} \dot{V} = & (x_1 - x_2)^\top (PA + A^\top P)(x_1 - x_2) \\ & -2b(Cx_1 - Cx_2)(\mathrm{sign}Cx_1 - \mathrm{sign}Cx_2) \\ \leq & (x_1 - x_2)^\top (PA + A^\top P)(x_1 - x_2) \end{aligned} \tag{11}$$

Let us verify the condition H1a with the function

$$W = (x_1 - x_2)^\top (PA + A^\top P)(x_1 - x_2).$$

With similar calculations as above the function W satisfies the condition H1a since $(PA + A^\top P)B = -\gamma C^\top$.

Now consider the smallest solution λ_n of the following equation

$$\det(PA + A^\top P - \lambda P) = 0 \tag{12}$$

From the hypothesis it follows that $\lambda_n \geq 2\mu$. On the other hand if λ_i, $i = 1, \ldots, n$ are the solutions of (12) then

$$\lambda_1 + \ldots + \lambda_n = 2\mathrm{tr}A$$

Since $\lambda_i \geq \lambda_n$ it follows that

$$\lambda_1 + \lambda_2 \leq 2(\mathrm{tr}A - (n-2)\mu) < 0$$

and according to Theorem 4 the system (9) has no periodic solutions. $\square$

4. CONCLUSIONS

In this paper we presented a new *discontinuous version of a Bendixson like criterion.* The criterion is based on a new result on the estimation of the Hausdorff dimension of invariant sets for (possibly) discontinuous systems. The new criterion can be applied for the design and control of discontinuous systems when the requirement of global stability is too restrictive. Our study is based on dichotomy-like properties of solutions of dynamical systems with respect to each other rather than with respect to some invariant sets.

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