Planning and Control of Spatial Motion of Flying Vehicles ⋆

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Abstract: The problem under consideration is planning of spatial trajectories for a flying vehicle. The methods are based on the six-dimensional model with the longitudinal overload, transversal overload and the roll angle as controls. The class of trajectories with the monotone variation of the mechanical energy of a flying vehicle is considered.
The method of constructing the multi-link trajectories by interfacing more than one interval with monotone energy variation is described. Attention is focused on the multi-link trajectories with monotone energy variations. Examples of the two-level algorithm to construct complex spatial trajectories are presented.

Keywords: Spatial trajectiry, flying vehicle, control, stabilization

1. INTRODUCTION

Complexity of the flight vehicle situation and the high cost of control decisions bring forth the task of planning the flight trajectory. The information about the feasible flight trajectories is of special importance both for making decisions about complex spatial maneuvers and in non-standard situations. In this and other cases, the acceptable variants of the flight trajectories must be analyzed in real time, which presents special requirements on the methods for seeking feasible flight trajectories.

Determination even of one trajectory is a mathematical challenge because the trajectory must connect the initial and final states, pass through some intermediate states, and be realizable by a particular flight vehicle. Hence the problem lies in developing special methods to solve a rather general problem of the trajectory motion control.

A bulky scientific literature deals with the problem of control of various flight vehicles [Taranenko V.T., Batenko A.P., Thomson D.G. and Bradley R.]. It deserves noting that the mathematical models of flight vehicle motion are well known. To solve a particular motion control problem, a simplified mathematical model is taken, and the control algorithm is constructed on its basis. It goes without saying that the decisions made in this way must be tested by mathematical modeling of more precise motion models. This approach was justified by solving control problems such as motion on the vertical plane, rectilinear motion on the horizontal plane, and vertical takeoff and landing.

For simple geometry of the flight trajectory, such problems can be solved using the linear motion models and linear methods of the control theory. However, at modeling of and seeking for the controls realizing complex spatial maneuvers of the flying vehicle, the linear models and linear control methods turn out to be insufficient. More sophisticated mathematical models allowing for the nonlinear nature of motion must be used.

This paper presents a solution to the problem of real-time construction of the trajectory with the given initial and final states. The solution is chosen in the class of trajectories with monotone variation of the mechanical energy of the flight vehicle. The trajectory determined is checked to verify that the overloads, the roll angle, and the state variables do not exceed the predefined critical values.

2. MOTION EQUATIONS

Consider the problem of flying vehicle motion control under the following assumptions: 1) mass is constant; 2) no wind; 3) terrestrial curvature is disregarded.

To describe the motion of the center of mass of a flying vehicle we take the trajectory reference frame.

By allowing for the representation of the forces acting on the flying vehicle through the overloads and adding three differential equations relating the velocity vector with the spatial coordinates, we obtain the following system of six differential equations describing the flying vehicle motion under the aforementioned assumptions:

$$\begin{cases} \dot V = (n_x - \sin\theta) g, & \dot H = V \sin\theta, \\ \dot\theta = \dfrac{(n_y \cos\gamma - \cos\theta) g}{V}, & \dot L = V\cos\theta\,\cos\psi, \\ \dot\psi = -\dfrac{n_y g \sin\gamma}{V\cos\theta}, & \dot Z = -V\cos\theta\,\sin\psi, \end{cases} \tag{1}$$

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where V is the velocity, *m/sec*; θ is the flight path angle, *rad*; ψ is the heading angle, *rad*; H is the altitude m; L is the along-track deviation, m; Z is the cross-track position, m; n_x is the longitudinal overload; n_y is the transversal overload; γ is the roll angle, *rad*; g is the sea-level acceleration of gravity, m/sec^2.

The along-track position L, altitude H, and cross-track position Z are the coordinates x_g, y_g, z_g of the position of the flying vehicle center of mass in the normal earth-fixed reference frame.

The overloads n_x, n_y and the roll angle γ are considered as the controls.

It is required to select a trajectory and corresponding controls such that by moving along it the flying vehicle passes from the initial state

$$x_0 = (V_0,\, \theta_0,\, \psi_0,\, H_0,\, L_0,\, Z_0)^{\mathrm{T}}, \tag{2}$$

to the given final state

$$x_* = (V_*,\, \theta_*,\, \psi_*,\, H_*,\, L_*,\, Z_*)^{\mathrm{T}}, \tag{3}$$

which must be realized with the given precision:

$$|\Delta x_i| = |x_i - x_{i*}| < \Delta_i, \quad i = \overline{1,\,6}.$$

At that, the state variables must lie within the given ranges of variation:

$$\begin{gathered} V \in [V_{\min},\, V_{\max}],\ |\theta| < \frac{\pi}{2},\ \theta \in [\theta_{\min},\, \theta_{\max}], \\ \psi \in [\psi_{\min},\, \psi_{\max}],\ H \in [H_{\min},\, H_{\max}], \\ L \in [L_{\min},\, L_{\max}],\ Z \in [Z_{\min},\, Z_{\max}]. \end{gathered} \tag{4}$$

Similar constraints are also imposed on the controls:

$$|\gamma| < \gamma_{\max},\ n_{x,\min} \le n_x \le n_{x,\max},\ n_{y,\min} \le n_y \le n_{y,\max}. \tag{5}$$

We also assume that in the initial and final states the values of controls

$$\gamma_0, \quad n_{x0}, \quad n_{y0}, \quad \gamma_*, \quad n_{x*}, \quad n_{y*} \tag{6}$$

and their tolerable deviations Δ_γ, Δ_x, and Δ_y in the final state are known.

3. INTRODUCTION OF THE VIRTUAL CONTROLS

We introduce the following variables as the virtual controls for system (1):

$$v_1 = n_x, \quad v_2 = n_y \cos\gamma, \quad v_3 = n_y \sin\gamma. \tag{7}$$

With these controls, (1) becomes an affine system of $n = 6$ equations with $m = 3$ controls:

$$\begin{cases} \dot V = -g\sin\theta + g v_1, & \dot H = V\sin\theta, \\ \dot\theta = -\dfrac{\cos\theta}{V} g + \dfrac{g}{V} v_2, & \dot L = V\cos\theta\,\cos\psi, \\ \dot\psi = -\dfrac{g}{V\cos\theta} v_3, & \dot Z = -V\cos\theta\,\sin\psi. \end{cases} \tag{8}$$

System (8) has the canonical form [Krishchenko A.P., 1985]

$$\ddot{\boldsymbol{y}} = A + B\boldsymbol{v} \tag{9}$$

where

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad A = \begin{pmatrix} -g \\ 0 \\ 0 \end{pmatrix},$$

$$B = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ \cos\theta\,\cos\psi & -\sin\theta\,\cos\psi & \sin\psi \\ -\cos\theta\,\sin\psi & \sin\theta\,\sin\psi & \cos\psi \end{pmatrix}.$$

The canonical state variables are

$$y_1 = H, \quad y_2 = L, \quad y_3 = Z, \tag{10}$$

$$\dot y_1 = V\sin\theta,\ \dot y_2 = V\cos\theta\cos\psi,\ \dot y_3 = -V\cos\theta\sin\psi. \tag{11}$$

In the domain which in source state variable is described by (4) system (9) is solvable with respect to the controls

$$\boldsymbol{v} = B^{-1}(\ddot{\boldsymbol{y}} - A). \tag{12}$$

4. PROGRAM TRAJECTORY AS TIME FUNCTION

Since the time interval is not defined, we take it equal to $[t_0, t_*]$ and determine the spatial trajectory $H = y_1(t)$, $L = y_2(t)$, $Z = y_3(t)$, $t \in [t_0, t_*]$, satisfying all boundary conditions, that is, the given boundary conditions for state and control. To this end, we use relations (7) to calculate the boundary values of the virtual controls $\boldsymbol{v}(t_0) = \boldsymbol{v}_0$, $\boldsymbol{v}(t_*) = \boldsymbol{v}_*$.

According to (9), (10), and (11), the boundary conditions for state and the virtual controls at the ends of the time interval $[t_0, t_*]$ define the boundary conditions for the vector function $\boldsymbol{y}(t)$ and their first and second derivatives. Thus for $t = t_0$ we establish that

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0, \quad \dot{\boldsymbol{y}}(t_0) = \dot{\boldsymbol{y}}_0, \quad \ddot{\boldsymbol{y}}(t_0) = \ddot{\boldsymbol{y}}_0, \tag{13}$$

and for $t = t_*$, similarly

$$\boldsymbol{y}(t_*) = \boldsymbol{y}_*, \quad \dot{\boldsymbol{y}}(t_*) = \dot{\boldsymbol{y}}_*, \quad \ddot{\boldsymbol{y}}(t_*) = \ddot{\boldsymbol{y}}_*, \tag{14}$$

Each of the components $y_i(t)$, $i = 1, 2, 3$, of the smooth vector function $\boldsymbol{y}(t)$, satisfying the boundary conditions (13), (14) may be taken independently. For example, all of them may be found among the polynomials of the variable t of degree five. Indeed, let for the smooth function $f(t)$ the boundary conditions

$$f(t)|_{t=t_0} = f_0, \quad \dot f(t)|_{t=t_0} = \dot f_0, \quad \ddot f(t)|_{t=t_0} = \ddot f_0, \tag{15}$$

$$f(t)|_{t=t_*} = f_*, \quad \dot f(t)|_{t=t_*} = \dot f_*, \quad \ddot f(t)|_{t=t_*} = \ddot f_*. \tag{16}$$

be defined over the interval $[t_0, t_*]$. We consider the polynomial of the fifth degree

$$p(t) = \sum_{j=0}^{2} \frac{f_0^{(j)}}{j!}(t - t_0)^j + \sum_{j=1}^{3} c_j (t - t_0)^{2+j}. \tag{17}$$

For any values of the constants c_j, the polynomial $p(t)$ satisfies the boundary conditions (15) for $t = t_0$. For $t = t_*$, conditions (16) can always be satisfied by an appropriate choice of the constants c_j. It is sufficient to substitute the polynomial in (16) and solve the resulting system of linear algebraic equations with respect to the unknowns c_j:

$$\begin{cases} \Delta^3 c_1 + \Delta^4 c_2 + \Delta^5 c_3 = f_* - f_0 - \dot f_0 \Delta - \dfrac{\ddot f_0}{2}\Delta^2, \\ 3\Delta^2 c_1 + 4\Delta^3 c_2 + 5\Delta^4 c_3 = \dot f_* - \dot f_0 - \ddot f_0 \Delta, \\ 6\Delta c_1 + 12\Delta^2 c_2 + 20\Delta^3 c_3 = \ddot f_* - \ddot f_0, \end{cases} \tag{18}$$

where $\Delta = t_* - t_0 \ne 0$. Solution of this square system always exists and is unique because the determinant of the system matrix is equal to $2\Delta^9 \ne 0$.

5. CONSTRUCTION OF THE PROGRAM TRAJECTORY AS THE FUNCTION OF ENERGY

To realize the above procedure for construction of the program control it is necessary to know the length $t_* - t_0$ of the time interval. However, this instant is not given in advance. The problem can be circumvent in part by passing to a new variable

$$E = H + \frac{V^2}{2g} \tag{19}$$

which is the total energy of the system reduced to the dimensionless form (E also will be referred to as energy). It is assumed that E is a monotone function of time. Since

$$\dot{E} = V n_x = V v_1, \tag{20}$$

the energy varies monotonically over the trajectory if the velocity V does not go to zero and the overload n_x retains its sign. In what follows, we denote by stroke the derivatives of various variables with respect to the variable E (for example, $V' = \frac{dV}{dE}$). By passing in system (8) to the new independent variable E, we obtain the following system of differential equations describing the motion of the flight vehicle along a trajectory section with monotone variation of energy:

$$\begin{cases} V' = \dfrac{(v_1 - \sin\theta)g}{V v_1}, & H' = \dfrac{\sin\theta}{v_1}, \\ \theta' = \dfrac{(v_2 - \cos\theta)g}{V^2 v_1}, & L' = \dfrac{\cos\theta\,\cos\psi}{v_1}, \\ \psi' = -\dfrac{v_3 g}{V^2 v_1 \cos\theta}, & Z' = -\dfrac{\cos\theta\,\sin\psi}{v_1}. \end{cases} \tag{21}$$

The initial (2) and final (3) states of system (1) define the initial E_0 and final E_* values of the variation of E. Therefore, states (2)–(3) may be regarded as the initial (for $E = E_0$) and final (for $E = E_*$) states of system (21).

According to (20), for a trajectory with monotone variation of energy connecting the initial (2) and final (3) system states to exist, it is necessary that for $E_0 > E_*$ the control $v_1 = n_x$ be negative over the interval $[E_*, E_0]$ and positive for $E_0 < E_*$, that is, the values of control $v_1(E)$ must be coordinated with the relation between E_0 and E_*. To take this situation into consideration, we introduce the constant δ assuming that

$$\delta = \mathrm{sign}(E_* - E_0) = \begin{cases} -1, & E_0 > E_*, \\ 1, & E_0 < E_*, \end{cases} \tag{22}$$

and T_E is the interval of variations of the variable E bounded by the points E_0 and E_*. Then, the desired condition to conform to the control $v_1(E)$ with E_0 and E_* lies in satisfying the inequality

$$v_1(E) = \delta |v_1(E)|, \quad E \in T_E. \tag{23}$$

To stress this fact, we call $v_1(E)$ the *conforming control.*

When defining the boundary condition for the control $n_x = v_1$, one must bear in mind that the trajectory with monotone variation of energy connecting the initial (2) and final (3) states of the system exists only if the corresponding conforming condition is satisfied. For $E = E_0$, it has the form $n_{x0} = \delta|n_{x0}|$, and $n_{x*} = \delta|n_{x*}|$ for $E = E_*$. If the boundary conditions for control $n_x = v_1$ are not conforming with variations of energy, then in the class of continuous controls there exists no solution of the terminal problem to which a trajectory with monotone variation of energy corresponds.

We assume that the boundary conditions n_{x0} and n_{x*} defined in the states (2) and (3) are conforming with variation of energy. We refer to such boundary conditions for control n_x as the *conforming conditions.*

Using (21) we may determine the boundary values H'_0, L'_0, Z'_0 and H'_*, L'_*, Z'_* of the functions H', L', Z' for $E = E_0$ and $E = E_*$.

We denote by $S^2(T_E)$ the set of twice continuously differentiable on T_E functions

$$r : T_E \to \mathbb{R}^3, \; r(E) = (h(E), l(E), z(E))^{\mathrm{T}}$$

which together with their first derivatives at the ends of the interval T_E satisfy the following boundary conditions:

$$r(E_0) = (H_0,\ L_0,\ Z_0)^{\mathrm{T}}, \quad r(E_*) = (H_*,\ L_*,\ Z_*)^{\mathrm{T}},$$
$$r'(E_0) = (H'_0,\ L'_0,\ Z'_0)^{\mathrm{T}}, \quad r'(E_*) = (H'_*,\ L'_*,\ Z'_*)^{\mathrm{T}}.$$

Indices such as $l''_* = l''(E_*)$ will be used to denote the values of the functions $h(E)$, $l(E)$, and $z(E)$ and their derivatives with respect to E at the ends E_0 and E_* of the interval of energy variations. The same index at the state or control variable will denote their given value at the corresponding boundary point. For the rest of the variables, these indices will be used to denote their desired or calculated values at the boundary points.

Theorem 1. Let the boundary conditions for control be defined only for n_x and be conforming, and the function $r(E) = (h(E), l(E), z(E))^{\mathrm{T}} \in S^2(T_E)$ satisfy the conditions

$$l'^2(E) + z'^2(E) \neq 0, \quad E \in T_E; \tag{24}$$
$$h(E) < E, \quad E \in T_E. \tag{25}$$

Then, there exist virtual controls (7) continuous on T_E such that system (8) passes from the initial state (2) to the final state (3) along the spatial trajectory $r(T_E)$, monotone variation of energy corresponding to this transition.

We substitute the functions $H = h(E)$, $L = l(E)$, and $Z = z(E)$ in system (21) and obtain the equations

$$\begin{cases} V' = \dfrac{(v_1 - \sin\theta)g}{V v_1}, & h' = \dfrac{\sin\theta}{v_1}, \\ \theta' = \dfrac{(v_2 - \cos\theta)g}{V^2 v_1}, & l' = \dfrac{\cos\theta\,\cos\psi}{v_1}, \\ \psi' = -\dfrac{v_3 g}{V^2 v_1 \cos\theta}, & z' = -\dfrac{\cos\theta\sin\psi}{v_1} \end{cases} \tag{26}$$

in the functions $V = V(E)$, $\theta = \theta(E)$, $\psi = \psi(E)$, $v_1 = v_1(E)$, $v_2 = v_2(E)$ and $v_3 = v_3(E)$ where the argument E of all functions is omitted for simplicity.

Due to (26) and conforming condition we have

$$v_1 = v_1(E) = \frac{\delta}{\sqrt{h'^2 + l'^2 + z'^2}}, \tag{27}$$

and this function according to (24) is continuous on T_E.

It follows from the expression for h' in system (26) that

$$\sin\theta = h' v_1. \tag{28}$$

Therefore, with regard for the constraint $|\theta| < \pi/2$,

$$\cos\theta = \frac{\sqrt{l'^2 + z'^2}}{\sqrt{h'^2 + l'^2 + z'^2}} = \delta v_1 \sqrt{l'^2 + z'^2}. \qquad (29)$$

It follows from the expressions for l' and z' in system (26) and equality (29) that

$$\cos\psi = \frac{\delta l'}{\sqrt{l'^2 + z'^2}}, \quad \sin\psi = -\frac{\delta z'}{\sqrt{l'^2 + z'^2}}. \qquad (30)$$

Let us determine the velocity $V = V(E) > 0$ by the formula

$$V = V(E) = \sqrt{2g(E - h)}, \qquad (31)$$

which is correct with respect to (25). With this choice the first equation of system (26) will fulfilled.

By differentiating equalities (28) and (30) with respect to E we get

$$\theta' = \theta'(E) = -\delta v_1^2 \frac{h'(l'l'' + z'z'') - h''(l'^2 + z'^2)}{\sqrt{l'^2 + z'^2}}. \qquad (32)$$

and

$$\psi' = \psi'(E) = -\frac{z''l' - l''z'}{l'^2 + z'^2}. \qquad (33)$$

It now becomes possible to determine the controls v_2 and v_3 from the expressions for θ' and ψ' in system (26):

$$v_2 = v_2(E) = -2(E-h)\delta v_1^3 \frac{h'(l'l''+z'z'') - h''(l'^2+z'^2)}{\sqrt{l'^2+z'^2}} + \\ + \delta v_1 \sqrt{l'^2 + z'^2}, \qquad (34)$$

$$v_3 = v_3(E) = 2(E-h)\delta v_1^2 \frac{z''l' - l''z'}{\sqrt{l'^2 + z'^2}}. \qquad (35)$$

According to the above formulas and condition (24), the functions $v_2 = v_2(E)$ and $v_3 = v_3(E)$ are continuous on T_E.

All controls were determined as the functions of energy: $v_i = v_i(E)$, $i = 1, 2, 3$. The same refers to the state variables. Although for the angles θ and ψ only their sines and cosines were determined, these values—with regard for the initial state and choice of the function $r(E)$—enable unique calculation of the functions $\theta(E)$ and $\psi(E)$ at variations of E from E_0 to E_*. Since this solution of system (26) was determined by squaring, it is required to verify that it is namely the solution of system (26) that was established and that it satisfies the boundary conditions for state and control v_1.

It is possible to verify directly that the functions $H = h(E)$, $L = l(E)$, and $Z = z(E)$ together with $V = V(E)$ (31), $\theta = \theta(E)$ (28) – (29), $\psi = \psi(E)$ (30) and the controls $v_1(E)$ (27), $v_2(E)$ (34) and $v_3(E)$ (35) convert the system equations (21) into identities.

Let us check the boundary conditions. Since $r(E) = (h(E), l(E), z(E))^{\mathrm{T}} \in S^2(T_E)$, the values of the functions $H = h(E)$, $L = l(E)$, and $Z = z(E)$ at the ends of the interval of energy variations coincide with the values of the spatial coordinates at the boundary points.

For the function $V(E)$, we establish that

$$V(E_0) = \sqrt{2g(E_0 - h_0)} = \sqrt{2g(E_0 - H_0)} = V_0$$

and similarly $V(E_*) = V_*$.

With regard for (21), we obtain for the control $v_1(E)$ that

$$v_1(E_0) = \frac{\delta}{\sqrt{H_0'^2 + L_0'^2 + Z_0'^2}} = \delta|v_{10}| = v_{10}$$

owing to the fact that the boundary values are conforming for it; similarly, $v_1(E_*) = v_{1*}$.

We determine in the same way that

$$\sin\theta(E_0) = H_0' v_{10} = \sin\theta_0,$$

$$\cos\theta(E_0) = \delta v_{10}\sqrt{{L_0'}^2 + {Z_0'}^2} = \frac{\delta v_{10}\cos\theta_0}{|v_{10}|} = \cos\theta_0.$$

Similarly, $\sin\theta(E_*) = \sin\theta_*$, and $\cos\theta(E_*) = \cos\theta_*$.

The corresponding equalities for the angle ψ are verified in the same manner.

The dependences of the state variables on the energy define in the state space the curve passing through the initial state (2) for $E = E_0$ and the final state (3) for $E = E_*$. This curve is the trajectory of system (21) because the determined dependences of the state variables and control on energy convert all equations of system (21) into identities. Consequently, the controls $v_i = v_i(E)$, $i = 1, 2, 3$, are the program solutions of the terminal problem (2)–(3) for system (21).

We fix the initial time instant t_0 and determine the dependence of energy vs. time. According to (20)

$$\frac{dt}{dE} = \frac{1}{V(E)v_1(E)} = \delta \frac{\sqrt{(h'(E))^2 + (l'(E))^2 + (z'(E))^2}dE}{\sqrt{2g(E - h(E))}}$$

and therefore

$$t = t_0 + \delta \int_{E_0}^{E} \frac{\sqrt{(h'(E))^2 + (l'(E))^2 + (z'(E))^2}dE}{\sqrt{2g(E - h(E))}}. \qquad (36)$$

The value of $E = E(t)$ is determined uniquely from (36) as equation in E.

The terminal time instant t_* corresponding to the function $r(E)$ is equal to

$$t_* = t_0 + \delta \int_{E_0}^{E_*} \frac{\sqrt{(h'(E))^2 + (l'(E))^2 + (z'(E))^2}dE}{\sqrt{2g(E - h(E))}}. \qquad (37)$$

The dependence of energy vs. time enables one to consider the controls $v_i = v_i(E)$, $i = 1, 2, 3$, as the time functions: $v_i = v_i(E(t))$, $t \in [t_0, t_*]$, $i = 1, 2, 3$. For system (8), these functions will be the program solution of the terminal problem (2)–(3) where t_* was determined from (37).

In what follows, we assume that the boundary conditions for control $n_x = v_1$ are defined and conforming with the boundary states.

Relations (32) and (33) can be represented as

$$\begin{aligned} &h'(z'z'' + l'l'') - h''(l'^2 + z'^2) = b_1(E), \\ &l'z'' - z'l'' = b_2(E), \end{aligned} \qquad (38)$$

where

$$b_1(E) = -\delta \frac{\theta'\sqrt{l'^2 + z'^2}}{v_1^2}, \quad b_2(E) = -(l'^2 + z'^2)\psi'.$$

Let us determine the boundary values of controls (34):

$$v_2(E_0) = \frac{V_0^2}{g}\theta'(E_0)v_{10} + \cos\theta_0,$$

where

$$\theta'(E_0) = -\delta v_{10}^2 \frac{h_0'(l_0' l_0'' + z_0' z_0'') - h_0''(l_0'^2 + z_0'^2)}{\sqrt{l_0'^2 + z_0'^2}} = $$
$$= -\delta v_{10}^2 \frac{H_0'(L_0' l_0'' + Z_0' z_0'') - h_0''(L_0'^2 + Z_0'^2)}{\sqrt{L_0'^2 + Z_0'^2}}.$$

Similarly,

$$v_2(E_*) = \frac{V_*^2}{g}\theta'(E_*)v_{1*} + \cos\theta_*,$$

where

$$\theta'(E_*) = -\delta v_{1*}^2 \frac{h_*'(l_*' l_*'' + z_*' z_*'') - h_*''(l_*'^2 + z_*'^2)}{\sqrt{l_*'^2 + z_*'^2}} = $$
$$= -\delta v_{1*}^2 \frac{H_*'(L_*' l_*'' + Z_*' z_*'') - h_*''(L_*'^2 + Z_*'^2)}{\sqrt{L_*'^2 + Z_*'^2}}.$$

Additionally,

$$v_3(E_0) = -\frac{V_0^2}{g}\psi'(E_0)v_{10}\cos\theta_0,$$

where

$$\psi'(E_0) = -\frac{z_0'' L_0' - l_0'' Z_0'}{L_0'^2 + Z_0'^2}$$

and

$$v_3(E_*) = -\frac{V_*^2}{g}\psi'(E_*)v_{10}\cos\theta_*,$$

where

$$\psi'(E_*) = -\frac{z_*'' L_*' - l_*'' Z_*'}{L_*'^2 + Z_*'^2}.$$

These boundary values of the controls $v_2(E)$ and $v_3(E)$ depend on the values at the boundary points of the second derivative of the function $r(E)$ satisfying the condition of Theorem 1. One can readily see that the function $r(E)$ can be always selected with the value $r''(E)$ at the boundary point such that at this point the boundary condition for $v_2(E)$ and/or $v_3(E)$ will be satisfied.

6. POLYNOMIAL DEFINITION OF THE TRAJECTORY

As follows from Theorem 1, any smooth curve $r(E)$ with the values of function, its derivative and, possibly, second derivative defined at the ends of the interval can be used as the trajectory of a flying vehicle. It is only natural to use polynomials for this purpose.

Let us consider the construction of the polynomial $y(t)$ over the interval $[t_1, t_2]$ for which defined are at the initial point t_1 of the interval the values of all their derivatives from the zero to the $(r{-}1)$st order and at the terminal point t_2, from the zero to the $(s{-}1)$st order. Within the framework of the paper, the numbers s and r may assume values 2 or 3.

We have $r + s$ conditions for coefficients of a polynomial $y(t)$. Therefore we may select the polynomial order $r{+}s{-}1$. If we denote

$$y(t) = d_0 + d_1(t-t_1) + d_2(t-t_1)^2 + \dots + d_{r+s-1}(t-t_1)^{r+s-1},$$

we have from boundary conditions

$$\begin{aligned} & d_0 = y_0, \\ & \vdots \\ & A_{r-1}^{r-1} d_{r-1} = y_0^{(r-1)}, \\ & d_0 + d_1\Delta + \dots + d_{r+s-1}\Delta^{r+s-1} = y_*, \\ & \vdots \\ & A_{s-1}^{s-1} d_{s-1} + \dots + A_{r+s-1}^{s-1} d_{r+s-1}\Delta^r = y_*^{s-1}, \end{aligned} \qquad (39)$$

where $A_n^m = n(n-1)\dots(n-m+1)$.

System (39) is linear and has a unique solution. With this solution we determine a unique polynomial $y(t)$ satisfying given boundary conditions.

Construction of the spatial trajectory. Let us consider determination of the spatial trajectory

$$H = h(E), \quad L = l(E), \quad Z = z(E)$$

from the boundary conditions for state under different requirements on the values of control at the ends of the interval T_E of energy variation.

Theorem 1 and analysis of its proof provide the following results in case when the boundary conditions for states and the conforming boundary conditions for control $n_x = v_1$ are defined.

1. The functions $h(E)$, $l(E)$, and $z(E)$ can be defined as polynomials.

2. If the boundary conditions for the controls v_2 and v_3 are not defined, then the functions $h(E)$, $l(E)$, and $z(E)$ can be taken as the third-degree polynomials of E with the given values of the polynomial and its derivative at the ends of the interval $[E_0, E_*]$. If the condition (24) is violated, then $\epsilon(E-E_0)^2(E-E_*)^2$ with some $\epsilon \neq 0$ may be added to $l(E)$ or $z(E)$.

3. If the values $v_2(t_0) = v_{20}$ and $v_3(t_0) = v_{30}$ of the controls v_2 and v_3 are defined at the initial point t_0, then for the functions $h(E)$, $l(E)$ and $z(E)$ their values and their derivatives are known at both ends of the interval. Moreover the three values of the second derivatives at the initial point will be related by two linear equations (38). Any three values of h_0'', l_0'', and z_0'' satisfying this system of equations can be taken.

4. If the values of the controls v_2 and v_3 are defined both at the initial and final points, then by two conditions will be imposed on the second derivatives of the functions $h(E)$, $l(E)$, and $z(E)$ at the initial and final points. Thus it allows one to take functions from the third-fifth-degree polynomials.

7. STABILIZATION OF THE PROGRAM CONTROL

A certain program motion of the canonical system (9) will be denoted by $(\tilde{\boldsymbol{y}}(t), \tilde{\boldsymbol{v}}(t))$, $t \geq t_0$. This may be, for example, any of the two above program motions of this system (in time and energy). We determine a continuously differentiable control $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{y}, t)$ in the form of (nonstationary) state feedback of system (9) [Krishchenko A.P., 1994] such that its values at the points of the program trajectory coincide with the values of the corresponding program control

$$\boldsymbol{v}(\tilde{\boldsymbol{y}}(t), t) = \tilde{\boldsymbol{v}}(t)$$

and system (9) closed by the control $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{y}, t)$ in the variables of the perturbed motion

$$z_i = y_i - \tilde{y}_i(t), \;\; \dot{z}_i = \dot{y}_i - \dot{\tilde{y}}_i(t), \quad i = 1, 2, 3, \qquad (40)$$

is of the form

$$\ddot{z}_i + k_{i1}\dot{z}_i + k_{i0}z_i = 0, \quad i = 1, 2, 3, \qquad (41)$$

where the constants k_{ij} are positive.

We notice that the matrix $G(\boldsymbol{y}) = \dfrac{1}{g}B(\boldsymbol{y})$ is orthogonal and, therefore, $G^{-1} = G^{\mathrm{T}}$.

The identity

$$\ddot{\tilde{\boldsymbol{y}}}(t) = A + g\,G(\tilde{\boldsymbol{y}}(t))\,\tilde{\boldsymbol{v}}(t) \qquad (42)$$

is valid for the program motion. By subtracting (42) from (9), we establish that

$$\ddot{\boldsymbol{y}} - \ddot{\tilde{\boldsymbol{y}}}(t) = g\,G(\boldsymbol{y})\,\boldsymbol{v} - g\,G(\tilde{\boldsymbol{y}}(t))\tilde{\boldsymbol{v}}(t),$$

Consequently,

$$\boldsymbol{v} = G^{\mathrm{T}}(\boldsymbol{y})\,G(\tilde{\boldsymbol{y}}(t))\tilde{\boldsymbol{v}}(t) + \frac{1}{g}G^{\mathrm{T}}(\boldsymbol{y})\big(\ddot{\boldsymbol{y}} - \ddot{\tilde{\boldsymbol{y}}}(t)\big).$$

With allowance for (40) and (41), we finally obtain

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{y}, t) = G^{\mathrm{T}}(\boldsymbol{y})\,G(\tilde{\boldsymbol{y}}(t))\tilde{\boldsymbol{v}}(t) - \\ - \frac{1}{g}G^{\mathrm{T}}(\boldsymbol{y})\big(K_1(\dot{\boldsymbol{y}} - \dot{\tilde{\boldsymbol{y}}}(t)) + K_0(\boldsymbol{y} - \tilde{\boldsymbol{y}}(t))\big), \qquad (43)$$

where $K_1 = \mathrm{diag}(k_{11}, k_{21}, k_{31})$, $K_0 = \mathrm{diag}(k_{10}, k_{20}, k_{30})$ are diagonal matrices. For this control, the program trajectory $\tilde{\overline{y}}(t)$ of the canonical system is asymptotically stable.

8. RECALCULATION OF CONTROLS

The vector function $v = v(\overline{y}, t)$ defined by (43) is a set of auxiliary relations providing solution of the terminal control problem. The initial controls (longitudinal and transversal overloads and the roll angle) can be established from the virtual control using relations 7):

$$n_x = v_1, \quad n_y = \sqrt{v_1^2 + v_2^2}, \quad \gamma = \arctan\frac{v_3}{v_2}. \qquad (44)$$

If the control constraints (5) reflecting the constructive characteristics of the helicopter are disregarded, then, according to Theorem 1, under any boundary conditions for the state variables and controls that meet the conforming conditions, the problem of terminal control is solvable in the class of continuous controls. This is true for any variant of the boundary conditions, that is, independently of whether the boundary conditions for the transversal overload and the roll angle are given or not. At that, the flight trajectory as a function of E will be twice continuously differentiable. However, condition (25) may lead to an unsuccessful attempt to establish the solution in the class of trajectories obeying the small-degree polynomials of E.

We note that under the determined controls $v = v(\overline{y}, t)$ the program trajectory of system (1) corresponding to $\tilde{\overline{y}}(t)$ will be asymptotically stable in the canonical variables.

The established controls need not satisfy constraints (5). It is planned that in real fact the controls will be specified as follows:

$$\begin{aligned} \tilde{n}_x &= \mathrm{sat}(n_x; n_{x,\min}, n_{x,\max}), \\ \tilde{n}_y &= \mathrm{sat}(n_y; n_{y,\min}, n_{y,\max}), \\ \tilde{\gamma} &= \mathrm{sat}(\gamma; -\gamma_{\min}, \gamma_{\max}), \end{aligned} \qquad (45)$$

where $\mathrm{sat}(x; a, b) = \min\{\max\{x, a\}, b\}$ is the saturation function.

Since all calculations rely on the virtual controls v_1, v_2, and v_3 (see Sec. 3), to take into consideration the constraints on the original controls, the current values of the virtual controls are recalculated into the main controls which are then corrected and recalculated back into the virtual controls.

Adjustment of the controls by the saturation function brings about an additional error in the result of motion modeling. This error can be so high that the motion trajectory does not reach the final point. Yet in some cases this distortion of the program controls can be eliminated using the stabilization mechanism so as the resulting trajectory is acceptable. Potential distortions in controls give rise to the need for additional testing of the determined trajectory. This testing is done by means of direct modeling of motion and analysis of its results.

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