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STABILIZING SADDLE STEADY STATES OF DYNAMICAL SYSTEMS WITH PARTIALLY UNCERTAIN MODEL BY MEANS OF PROPORTIONAL FEEDBACK

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Abstract

A simple zero-order proportional feedback technique for stabilizing unknown saddle type unstable fixed points is described. The technique employs either natural or artificially created stable fixed points to find unknown coordinates of the unstable fixed point. Two physical examples have been investigated, namely mechanical pendulum and autonomous Duffing-Holmes oscillator have been considered both analytically and numerically.

Key words

Control, steady states, saddles, proportional feedback.

1 Introduction

Stability of any either natural or artificial system is a valuable and desired property. Stabilization in particular of unstable fixed points (UFPs) of dynamical systems is an important problem in basic science and engineering applications if periodic or chaotic oscillations are unacceptable behaviours. Usual control methods, based on proportional feedback [Kuo, 1995; Nijmeijer and Schaft, 1996; Ogata, 1997] require knowledge of a mathematical model of a system or at least the exact location of the UFP in the phase space for the reference point. However, in many real complex systems, especially in biology, physiology, economics, sociology, chemistry, nuclear engineering neither the reliable models nor the exact coordinates of the UFPs are *a priori* known. Moreover, the position of the UFP may slowly vary with time because of external unknown and unpredictable, e.g. chaotic forces. Therefore model-independent and reference-free methods, automatically tracing unknown UFP are needed.

A number of adaptive, reference-free methods using either low-pass, high-pass or notch stable filters have been described in literature (many of the references can be found in a recent paper [Tamaševičius *et al.*, 2010]). However, they can stabilize unstable nodes and unstable spirals only, but fail to stabilize the saddle-type UFPs, more specifically the UFPs with an odd number of real positive eigenvalues. To solve the problem of the odd number limitation Pyragas *et al.* [Pyragas *et al.*, 2002] proposed to use an unstable filter. It was a bold idea to fight instability with another instability. The technique has been demonstrated to stabilize saddles in several mathematical models [Pyragas *et al.*, 2002; Pyragas *et al.*, 2004; Tamaševičius *et al.*, 2008; Tamaševičius *et al.*, 2010] also in the experiments with an electrochemical oscillator [Pyragas *et al.*, 2002; Pyragas *et al.*, 2004], the Duffing-Holmes-type electrical circuit [Tamaševičius *et al.*, 2008]. This advanced method is limited to dissipative dynamical systems only. It is not applicable to conservative systems. The limitation of the unstable filter method can be proved analytically using the Hurwitz stability criteria: the necessary condition for stabilizing a saddle UFP is that the cut-off frequency of the unstable filter is lower than the damping coefficient of the system [Pyragas *et al.*, 2002]. In conservative systems damping is zero under definition. Formally, the cut-off frequency could be set negative. However, this would mean that the unstable filter should become a stable one and, therefore, inappropriate to stabilize a saddle-type UFP. To get around the problem a conjoint filter, that involves unstable and stable subfilters, has been very recently suggested and demonstrated for the Lagrange point L2 of the Sun–Earth astrodynamical system [Tamaševičius *et al.*, 2010].

The control methods described in [Pyragas *et al.*, 2002; Pyragas *et al.*, 2004; Tamaševičius *et al.*, 2008; Tamaševičius *et al.*, 2010] are focused on designing complex unstable higher order controllers with several adjustable control parameters. Even linear analysis of the stability properties employs high order Hurwitz matrixes for determining the threshold values of the feedback coefficients, while finding optimal control parameters requires numerical solution of high order characteristic equations. Therefore the developed techniques are somewhat complicated for practical applications.

In this paper, we suggest simple zero-order stable proportional feedback technique, which employs either natural or artificially created stable fixed points (SFPs) to find unknown coordinates of the UFP.

2 Simple mathematical models

To illustrate the idea we start with an extremely simple mathematical example

$$\dot{x} = x - \xi. \tag{1}$$

Here ξ is *a priori* unknown parameter. The corresponding UFP, $x_0 = \xi$ is also unknown and therefore the proportional feedback cannot be applied directly. However, we demonstrate that this unknown UFP can be still stabilized by two-step proportional feedback. In the first step we apply proportional feedback with an arbitrarily chosen reference point r_1:

$$\dot{x} = x - \xi + k(r_1 - x), \tag{2}$$

where r_1 is any real, either positive or negative (zero value is also applicable) constant. For $k > 1$ the feedback creates an artificial SFP:

$$x_1 = (kr_1 - \xi)/(k - 1).$$

Note, that the control term $k(r_1 - x)$ in Eq. (2), in general, does not vanish, because r_1 is not the natural UFP of the original Eq. (1). An exception is a "resonant" value $r_1 = x_0$. It means that we are lucky to guess the right reference point x_0 and the procedure is accomplished in one step. Otherwise the unknown parameter ξ should be found from the steady-state case of Eq. (2):

$$\xi = x_1 + k(r_1 - x_1).$$

In the second step we simply replace r_1 in Eq. (2) with $x_0 = \xi$ found in the first step:

$$\dot{x} = x - \xi + k(x_0 - x) \tag{3}$$

and readily stabilize the initially unknown UPF $x_0 = \xi$.

If a dynamical system has two fixed points, specifically one UFP and one SFP, the latter can be employed to find the position of the first one. In this case stabilization can be achieved in one step only, without creating an artificial SFP. The following nonlinear equation is an example:

$$\dot{x} = x - x^2 - \xi. \tag{4}$$

For $\xi < 0.25$ it has two real fixed points:

$$x_{01} = 0.5 + (0.25 - \xi)^{1/2},$$
$$x_{02} = 0.5 - (0.25 - \xi)^{1/2}.$$

The x_{01} is a SFP, while x_{02} is a UFP. Note an important feature:

$$x_{01} + x_{02} = 1. \tag{5}$$

Thus, the natural SFP, x_{01} can be immediately used to find the UFP, $x_{02} = 1 - x_{01}$ to be inserted in the feedback term:

$$\dot{x} = x - x^2 - \xi + k(1 - x_{01} - x). \tag{6}$$

Now we can generalize the above specific examples in the following form:

$$\dot{x} = F(x) - \xi, \tag{7}$$

where $F(x)$ is either linear or nonlinear function. Depending on $F(x)$ Eq. (7) can have several fixed points, which satisfy the steady-state equation $F(x_0) = \xi$. The fixed points are either UFP or SFP depending on the derivative of $F(x)$ with respect to x, the $F'(x)$ at $x = x_0$. If $F'(x_0) > 0$ the x_0 is UFP, and if $F'(x) < 0$ the x_0 is SFP. We recall here that all the fixed points are unknown because of the unknown parameter ξ. Let us consider a UFP and apply two-step procedure. The first step similarly to Eq. (2) is given by

$$\dot{x} = F(x) - \xi + k(r_1 - x). \tag{8}$$

The unknown parameter ξ is found from the steady-state equation:

$$\xi = F(x_1) + k(r_1 - x_1)$$

and then is inserted into the steady-state equation of the uncontrolled system:

$$F(x_0) - F(x_1) - k(r_1 - x_1) = 0.$$

If the $F(x)$ is well defined the latter equation can be solved with respect to x_0 and, finally, the second step is applied:

$$\dot{x} = F(x) - \xi + k(x_0 - x). \tag{9}$$

3 Mechanical pendulum

The first example is a mechanical pendulum given by

$$\ddot{\varphi} + \beta\dot{\varphi} + \sin\varphi = \xi. \tag{10}$$

In Eq. (10) φ is the angle between the downward vertical and the rod, β is the damping coefficient, and ξ is a constant, but generally unknown torque. For small torque $\xi < 1$, the system has two fixed points $(\varphi_{01,02}, \dot{\varphi}_{01,02}) = (\varphi_{01,02}, 0)$, where

$$\varphi_{01} = \arcsin\xi,$$
$$\varphi_{02} = \pi - \arcsin\xi.$$

The φ_{01} is SFP (lower position of the pendulum), the φ_{02} is a saddle UFP (upper position of the pendulum).One can see that the sum of the two angles is a constant value:

$$\varphi_{01} + \varphi_{02} = \pi. \tag{11}$$

Thus we can apply a simplified one-step procedure, similarly to the first-order mathematical example given by Eq. (4). Here we exploit the existing natural SFP of the pendulum to determine the position of the UFP, without creating any artificial point. The coordinate of the unknown UFP is readily obtained from the coordinate of the known (observed) stable point, $\varphi_{02} = \pi - \varphi_{01}$, independently of the unknown parameter ξ. Then we apply the proportional feedback:

$$\ddot{\varphi} + \beta\dot{\varphi} + \sin\varphi = \xi + k(\pi - \varphi_{01} - \varphi). \tag{12}$$

Linearization of Eq. (12) around φ_{02} gives the characteristic equation:

$$\lambda^2 + \beta\lambda + k + \cos(\pi - \varphi_{01}) = 0. \tag{13}$$

For small ξ the angle $\varphi_{01} \ll \pi$, thus $\lambda_{1,2} = -\beta/2 \pm [\beta^2/4 - (k-1)]^{1/2}$. The threshold value of the feedback coefficient is $k_{th} = 1$ for which the largest eigenvalue λ_1 crosses zero from positive to negative values. The optimal value of the feedback coefficient $k_{opt} = 1 + \beta^2/4$; the eigenvalues are both negative and equal to each other, $\lambda_1 = \lambda_2 = -\beta/2$. Further increase of k makes the eigenvalues complex, but does not change their real parts. So, for higher feedback coefficients the convergence rate saturates with k and is fully determined by the damping coefficient β. Results of numerical integration of Eq. (12), shown in Fig. 1, demonstrate dynamics of stabilization (including transient process) of saddle-type UFP.

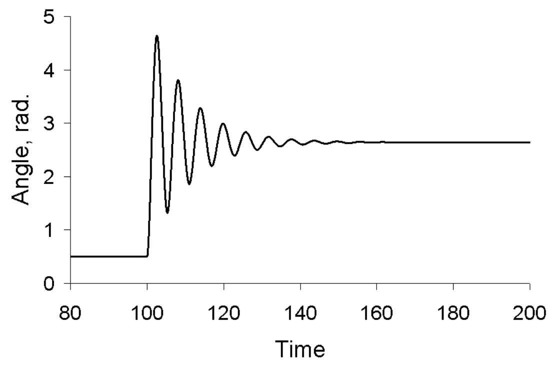

Figure 1. One-step stabilization of the upper position of mechanical pendulum by proportional feedback given in Eq. (12); the dynamics of the angle φ. The control is switched on at $t = 100$. The parameters are $\beta = 0.2$, $k = 2$. The stable angle observed before switching the control $\varphi_{01} = 0.5$, extracted unknown parameter $\xi = \sin\varphi_{01} = 0.47943$, stabilized UFP and angle calculated from the relationship $\varphi_{02} = \pi - \varphi_{01} = 2.64$.

4 Duffing–Holmes system

The second physical example is the Duffing-Holmes nonlinear damped oscillator, which in contrast to classical Duffing system [Ott, 1993] lacks external periodic driving force, but includes *apriori* unknown constant bias ξ:

$$\ddot{x} + b\dot{x} - x + x^3 = \xi. \tag{14}$$

Here b is the damping coefficient. For $|\xi| < 2/\sqrt{27}$ Eq. (14) has three fixed points. Two side points are SFPs, while the middle one is a saddle-type UFP. Their coordinates for non-zero ξ are rather cumbersome:

$$\begin{aligned} x_{01} &= -h\cos\frac{\pi-\theta}{3}, \\ x_{02} &= -h\cos\frac{\pi+\theta}{3}, \\ x_{03} &= h\cos\frac{\theta}{3}, \\ h &= \frac{2}{\sqrt{3}}, \\ \theta &= \arccos\frac{\xi\sqrt{27}}{2} \end{aligned} \tag{15}$$

While for $\xi = 0$ they become: $x_{01} = -1$, $x_{02} = 0$, $x_{03} = 1$. There is a simple relationship between the three coordinates:

$$x_{01} + x_{02} + x_{03} = 0, \tag{16}$$

which is valid also for non-zero ξ. Therefore one can think about the one-step algorithm ($x_{02} = -x_{01} - x_{03}$), similarly to the case of the pendulum. However from a

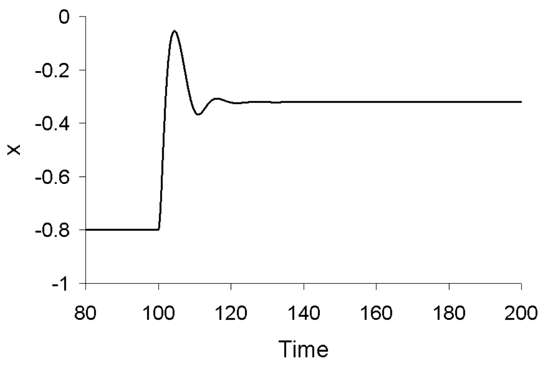

Figure 2. One-step stabilization of the UFP of the Duffing-Holmes oscillator by proportional feedback given in Eq. (18); the dynamics of the variable $x(t)$. The control is switched on at $t = 100$. The parameters are $b = 0.5$, $k = 1.1$. SFP observed before switching the control $x_{01} = -0.8$, extracted unknown parameter $\xi = x_{01}^3 - x_{01} = 0.288$, stabilized UFP and coordinate calculated from formula (15) $x_{02} = -0.321$.

practical point of view the procedure is not convenient, since one needs to find (to observe) two remote SFPs, separated by UFP. So, if a system is located at one of the SFP, say x_{01}, we have to switch it to another SFP (x_{03}) by applying some rather strong external force. Alternatively, we can use only one SFP, either x_{01} or x_{03}. From the corresponding formula (15) we can extract ξ and to use it for finding x_{02}, again from the formulas (15). However this formal way requires rather long and complicated calculations. There is a shorter way. Indeed, the SFP x_{01} satisfies the steady-state equation:

$$x_{01}^3 - x_{01} - \xi = 0. \qquad (17)$$

From here the unknown parameter ξ is readily derived as $\xi = x_{01}^3 - x_{01}$ and is used to calculate x_{02} from the appropriate formula (15). Then this coordinate is employed in the proportional feedback:

$$\ddot{x} + b\dot{x} - x + x^3 = \xi + k(x_{02} - x). \qquad (18)$$

Linearization of Eq. (18) around x_{02} provides the characteristic equation:

$$\lambda^2 + b\lambda + k - 1 + 3x_{02}^2 = 0. \qquad (19)$$

Its two eigenvalues are given by $\lambda_{1,2} = -b/2 \pm [b^2/4 - (k - 1 + 3x_{02}^2)]^{1/2}$. For small ξ the coordinate of the UFP $|x_{02}| \ll 1$. Then stabilization parameters are the same as that for the pendulum: the threshold coefficient $k_{th} = 1$, the optimal value $k_{opt} = 1 + b^2/4$, and the best pair of the real negative eigenvalues $\lambda_{1,2} = -b/2$. Numerical results for the Duffing-Holmes oscillator obtained by integrating Eq. (18) are presented in Fig. 2.

5 Conclusion

We have suggested simple proportional feedback technique for stabilizing uncertain saddle type fixed points of dynamical systems. The method involves either one or two step algorithm of stabilization. It makes use of either natural or of artificially created stable fixed points in order to find the hidden coordinates of the unstable fixed point. Two simple mathematical examples have been presented and two physical examples have been investigated. Specifically, mechanical pendulum and the autonomous Duffing-Holmes oscillator with unknown external forces are considered analytically and numerically. Though the examples represent the second order damped systems, we believe that similar technique can be applied to higher order active systems, including autonomous chaotic oscillators as well.

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