ELEMENTS OF ASYMPTOTIC CONTROL THEORY FOR A CLOSED STRING

Aleksey Fedorov
RQC
Russia
akf@rqc.ru

Alexander Ovseevich
IPMech, RAS
Russia
ovseev@ipmnet.ru

Abstract
We study the asymptotical control theory for one of the simplest distributed oscillating system — the closed string under a bounded load applied to a single distinguished point. We find exact classes of the string states that allows complete damping, and asymptotically exact value of the required time. We specify the structure of the asymptotically optimal feedback control, which is dry-friction like. We explicitly describe the singular arcs of the control.

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1 Introduction and the problem statement.

In this paper we apply a technique, introduced in [Ovseevich, Fedorov 2016; Ovseevich, Fedorov 2013], to control of a simple distributed system, the closed string under an impulsive control applied to a fixed point in the string. The phase space S of the system consists of pairs $\mathfrak{f} = (f_0, f_1)$ of distributions on a one-dimensional torus $\mathcal{T}$, and the motion is governed by the string equation

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + u\delta, \quad |u| \leq 1. \tag{1}$$

Here, $x \in [0, 2\pi]$ is the angle coordinate on the torus, t is time, $f_0 = f$, $f_1 = \frac{\partial f}{\partial t}$, δ is the Dirac δ-function. In other words, S is the space of initial data for (1). It is clear, that any solution of (1) with zero initial data is even. This is why we assume that S consists of pairs $\mathfrak{f} = (f_0, f_1)$ of even distributions. The equation describes oscillations of the closed homogeneous string under a bounded load applied to a fixed point (zero).

Our goal is to design an easily implementable feedback control for damping the oscillations. This means that we do not necessarily want to stop the motion of the string as a whole, so that that our target manifold $\mathcal{C}$ consists of pairs of constants $\mathcal{C} = \{(c_0, c_1)^* \in \mathbf{R}^2 \subset S\}$. Another useful point of view is to take the factor-space $\overline{S} = S/\mathcal{C}$ as the phase space of our system, and try to reach zero in this space. This is reasonable, because the target space $\mathcal{C}$ is invariant under the natural flow, associated with the string equation.

In what follows, we deal with a class of problems of minimum-time steering from the initial state to a terminal manifold $\mathcal{C}$ consisting of a pair of constants $\{(c_0, c_1)^* \in \mathbf{R}^2 \subset S\}$. More specifically, we study three problems:

1. Complete stop at a given point: $\mathcal{C} = 0$
2. Stop moving: $\mathcal{C} = \mathbf{R} \times 0$
3. Oscillation damping: $\mathcal{C} = \mathbf{R}^2$

2 String as a mechanical system.

The equations of motion of the free string (1) are that of the following Lagrangian system, where q is an even function such that $\frac{\partial q}{\partial x} \in L_2(\mathcal{T})$, and the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2} \int_{\mathcal{T}} |\dot{q}|^2(x)\, dx - \frac{1}{2} \int_{\mathcal{T}} \left| \frac{\partial q}{\partial x} \right|^2 (x)\, dx - uq(0), \tag{2}$$

so that $\frac{1}{2} \int_{\mathcal{T}} |\dot{q}|^2\, dx$ is the kinetic energy, and $\frac{1}{2} \int_{\mathcal{T}} \left| \frac{\partial f}{\partial x} \right|^2 dx$ is the potential energy of the system. The Lagrangian corresponds to the Hooke's law: the strain (deformation) is proportional to the applied stress. In terms of the Lagrangian the stress at point x is $\frac{\delta L}{\delta q}(x) = \frac{\partial q}{\partial x}(x)$, and the strain at x is $\frac{\partial q}{\partial x}(x)$, so that the coefficient of proportionality is 1.

The string also allows for a Hamiltonian description. The phase space is then the set of pairs (p, q), where p is an (even) function from $L_2(\mathcal{T})$, q is an (even) function from the space N of functions such that $\frac{\partial q}{\partial x} \in L_2(\mathcal{T})$, and the Hamiltonian

$$H(p, q) = \frac{1}{2} \int_{\mathcal{T}} |p|^2(x)\, dx + \frac{1}{2} \int_{\mathcal{T}} \left| \frac{\partial q}{\partial x} \right|^2 (x)\, dx + uq(0). \tag{3}$$

The canonical symplectic structure $\omega = dp \wedge dq$ is given by $\omega((X, X'), (Y, Y')) = \langle X, Y' \rangle - \langle X', Y \rangle$. Here $X, Y \in L_2(\mathcal{T})$, $X', Y' \in N$, and the angle brackets stand for the scalar product in $L_2(\mathcal{T})$.

Finally, the Pontryagin Hamiltonian H^{pont} in "coordinates" $\mathfrak{f} = (f_0, f_1)$ and adjoint variables $\mathfrak{x} = (\xi_0, \xi_1)$ takes the form

$$H^{\mathrm{pont}}(\mathfrak{f}, \mathfrak{x}) = \langle f_1, \xi_0 \rangle + \langle \Delta f_0, \xi_1 \rangle + |\xi_1(0)|. \quad (4)$$

3 The support function of reachable sets.

The first issue we deal with is that of controllability. We approach it by computation of the support function. The approach has much in common with that of [Lions 1988].

To make a comparison with the finite-dimensional case clear, we rewrite the governing equation in the form of first-order system

$$\frac{\partial \mathfrak{f}}{\partial t} = A\mathfrak{f} + Bu, \quad |u| \le 1. \quad (5)$$

where $A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$, $\Delta = \frac{\partial^2}{\partial x^2}$, and $B = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$. We commence with computing the support function $H = H_{D(T)}(\xi)$ of the reachable set $D(T)$, $T \ge 0$ of system (5) with zero initial condition. In other words, we have to find $H = \sup_u \langle \mathfrak{f}(T), \xi \rangle$, where $\xi \in S^*$ is a dual vector, and the sup is taken over admissible controls. Towards this end, we extend $\xi = (\xi_0, \xi_1)$ to the solution of the Cauchy problem

$$\frac{\partial \xi}{\partial t} = -A^* \xi, \quad \xi(T) = \xi, \quad (6)$$

where $A^* = \begin{pmatrix} 0 & \Delta \\ 1 & 0 \end{pmatrix}$ is the adjoint operator to A. These equations are exactly the equations for the adjoint variables of the Pontryagin maximum principle. We have

$$\tfrac{d}{dt} \langle \mathfrak{f}(t), \xi(t) \rangle = \langle A\mathfrak{f} + Bu, \xi \rangle - \langle \mathfrak{f}, A^* \xi \rangle = \langle Bu, \xi \rangle = u(t)\xi_1(0, t).$$

Now, a standard formal computation shows that

$$H = H_{D(T)}(\xi) = \sup_{|u| \le 1} \int_0^T u(t)\xi_1(0, t)dt = \int_0^T |\xi_1(0, t)|dt. \quad (7)$$

The reachable sets $D(T)$, $T \ge 0$ are closed in the standard topology of distributions, and of course they are convex. Therefore they are uniquely defined by their support functions.

Now we can characterize the vectors $\mathfrak{f}$ reachable from zero in time T as follows:

$$\mathfrak{f} \in D(T) \Leftrightarrow \langle \mathfrak{f}, \xi \rangle \le \int_0^T |\xi_1(0, t)|dt \text{ for any } \xi \in S^*. \quad (8)$$

Here, the function $x, t \mapsto \xi_1(x, t)$ is defined via solution of Cauchy problem (6).

In particular, the space $\mathcal{D}$ generated by vectors $\mathfrak{f} \in \bigcup_{T \ge 0} D(T)$ reachable from zero in an arbitrary time $T \ge 0$ is the dual space to the Frechet space of vectors ξ with finite norms

$$\|\xi\| = \|\xi\|_T = \int_0^T |\xi_1(0, t)|dt \quad (9)$$

for any $T > 0$. This space $\mathcal{D}$ coincides with the set of vectors reachable from zero in an arbitrary time $T \ge 0$ by means of a bounded (not necessarily by 1) control.

It is not difficult to compute $\xi_1(0, t)$ in terms of the Fourier coefficients of the functions $\psi = \xi_1$ and $\phi = \xi_0$. Suppose that

$$\psi(x, t) = \sum_{-\infty}^{\infty} \psi_n(t) e^{inx} \quad (10)$$

is the Fourier expansion of ξ_1. Since ψ is an even and real distribution, the coefficients ψ_n are real, and $\psi_n = \psi_{-n}$, so that (10) is, in fact, the cosine-expansion:

$$\psi(x, t) = \sum_{n=0}^{\infty} \psi_n(t) \cos nx. \quad (11)$$

The quantity we want to compute is

$$\|\xi\| = \int_0^T |\xi_1(0, t)|dt = \int_0^T \left| \sum \psi_n(t) \right| dt.$$

From (6) we immediately conclude that for $n \neq 0$

$$\psi_n(t) = e^{int} a_n + e^{-int} b_n,$$

where a_n, b_n are constants. For $n = 0$ we have $\psi_0(t) = a_0 + b_0 t$. It is clear that for $n \neq 0$

$$a_n = \frac{1}{2}\left(\psi_n + \frac{\phi_n}{in}\right), \quad b_n = \frac{1}{2}\left(\psi_n - \frac{\phi_n}{in}\right),$$

where ϕ_n is the nth Fourier coefficient of ϕ, and

$$a_0 = \psi_0, \; b_0 = \phi_0.$$

The Fourier coefficients of ϕ, like that of ψ, are real, and even with respect to n.

3.1 Natural norm in the dual space

By an easy computation we conclude that

$$\|\xi\| = \|\xi\|_T =$$

$$\int_0^T \left|\sum_{n\neq 0}\left(\psi_n \cos nt + \frac{\phi_n}{n}\sin nt\right) + \psi_0 + \phi_0 t\right| dt. \tag{12}$$

Suppose that T is $\geq 2\pi$. Then, the Banach norm $\|\xi\|$ is equivalent to the following more familiar Sobolev-type norm

$$\|\xi\|' = \|\xi_1\|_1 + \|\eta\|_1. \tag{13}$$

Here, $\|g\|_1 = \int_{-T/2}^{T/2} |g|\, dt$ is the usual L_1-norm, and $\eta(t) = \int_0^t \xi_0(x)dx$. Indeed, denote by f the integrand

$$f(t) = \sum_{n\neq 0}\left(\psi_n \cos nt + \frac{\phi_n}{n}\sin nt\right) + \psi_0 + \phi_0 t.$$

The norm $\|\xi\|$ is equivalent to $\|f\|_1 = \int_{-T/2}^{T/2} |f|\, dt = \int_{-T/2}^{T/2} |f^+ + f^-|\, dt$, where

$$f^+(t) = \sum_{n\neq 0} \psi_n \cos nt + \psi_0 = \xi_1(t),$$

resp.

$$f^-(t) = \sum_{n\neq 0} \frac{\phi_n}{n}\sin nt + \phi_0 t = \eta(t)$$

is even, resp. odd part of the function f. Indeed, put $g(t) = f(t) - \phi_0 t$. Then, $\|\xi\|$ is equivalent to $\int_0^T |g|dt + |\phi_0|$, while $\|f\|_1$ is equivalent to $\int_{-T/2}^{T/2} |g|dt + |\phi_0|$. Since the function g is 2π-periodic, and intervals of integration have length $T \geq 2\pi$, both integrals $\int_0^T |g|dt$, and $\int_{-T/2}^{T/2} |g|dt$ are equivalent to $\int_0^{2\pi} |g|dt$.

The L_1-norms of functions $f^\pm$ can be estimated via the L_1-norm of f:

$$\|f^\pm\|_1 \leq \|f\|_1.$$

Therefore, $\|\xi\|' = \|f^+\|_1 + \|f^-\|_1 \leq 2\|f\|_1$. On the other hand, it is obvious that

$$\|f\|_1 \leq \|f^+\|_1 + \|f^-\|_1 = \|\xi_1\|_1 + \|\eta\|_1 = \|\xi\|'.$$

We conclude that the norms $\|\xi\|'$ and $\|\xi\|$ are equivalent indeed. Therefore, if $T \geq 2\pi$ the dual space to the Banach space with norm $\|\xi\|$ coincides with the space of pairs $\mathfrak{f} = (f_0, f_1)$, where $\frac{\partial f_0}{\partial x} \in L_\infty$, and $f_1 \in L_\infty$. Thus, it is possible to damp the string, where the initial state $\mathfrak{f} = (f_0, f_1)$ possesses these properties, by a bounded load applied to a fixed point. Here, by damping we mean the complete stop, when not just oscillations, but also the displacement of the string as a whole is forbidden.

Remark. Our arguments show that the equivalence class of the norm $\|\xi\|_T$ does not depend on T provided that $T \geq 2\pi$. Theorems 1, 2 give a more quantitative statement of this independence of T.

3.2 Damping the oscillations

In order to deal with damping oscillations only, it suffices to make an analog of previous computations in the factor-space $S/\mathcal{C}$. The corresponding support functions are almost the same as those previously found. We just have to assume that the zero-mode coefficients ϕ_0, ψ_0 of the dual vectors ξ are zero. Then, the formula for support functions of the corresponding reachable sets $\overline{D}(T)$ takes the form:

$$H_{\overline{D}(T)}(\xi) = \int_0^T \left|\sum \left(\psi_n \cos nt + \frac{\phi_n}{n}\sin nt\right)\right| dt$$
$$= \int_0^T \left|\xi_1(y) + \int_0^y \xi_0(x)dx\right| dy. \tag{14}$$

Basically the same, but simpler arguments than that of previous subsection 3.1, prove that states $\mathfrak{f} = (f_0, f_1)$, where $\frac{\partial f_0}{\partial x} \in L_\infty$, and $f_1 \in L_\infty$ are exactly those that can be damped.

4 The shape of the reachable set $D(T)$

By following [5] one can easily find an asymptotic formula for the above support function (12). For $T > 0$ define a linear isomorphism $C(T)$ from S to S by $C(T)\mathfrak{f} = \frac{1}{T}(f_0, f_1^T)^*$, where $g^T(t) = \sum_{n\neq 0} g_n \cos nt + \frac{1}{T} g_0$, if $g(t) = \sum_{n\neq 0} g_n \cos nt + g_0$. It is clear that

$$H_{C(T)D(T)}(\xi) =$$

$$\frac{1}{T}\int_0^T \left|\sum_{n\neq 0}\left(\psi_n \cos nt + \frac{\phi_n}{n}\sin nt\right) + \psi_0 + \frac{\phi_0}{T}t\right| dt,$$

where ξ is the pair (ξ_0, ξ_1), and $\xi_0(x) = \sum \phi_n \cos nx$, $\xi_1(x) = \sum \psi_n \cos nx$.

Theorem 1. *Consider problem (1) from the Introduction, corresponding to the terminal manifold $\mathcal{C} = 0$. Then, as $T \to \infty$ we have the following limit formula:*

$$\lim_{T\to\infty} H_{C(T)D(T)}(\xi) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 |f(t,\tau)| dt d\tau \tag{15}$$

where $f = \sum_{n\neq 0}\left(\psi_n \cos nt + \frac{\phi_n}{n}\sin nt\right) + \psi_0 + \phi_0 \tau$.

Remind, that the shape $\operatorname{Sh}\Omega$ of a set $\Omega \subset S$ is the orbit of the group of linear (topological) isomorphisms of the space S acting on Ω. In terms of shapes one can say that the limit shape $\mathrm{Sh}_\infty = \lim_{T\to\infty} \mathrm{Sh}\, D(T)$ is related to the convex body Ω corresponding to the support function

$$H_\Omega(\xi) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 |f(t,\tau)| dt d\tau, \qquad (16)$$

where $f = \sum_{n\neq 0}\left(\psi_n \cos nt + \frac{\phi_n}{n}\sin nt\right) + \psi_0 + \phi_0\tau$. This means that the shape of the convex set with support function (16) coincides with Sh_∞. Similarly, in the reduced space $\overline{S}$ we have

Theorem 2. *Consider problems (2)-(3) from the Introduction, corresponding to the terminal manifolds $\mathcal{C} = \mathbf{R}\times 0$, or $\mathcal{C} = \mathbf{R}^2$. Then, the following limit formula holds:*

$$\begin{aligned} &\lim_{T\to\infty} \tfrac{1}{T} H_{\overline{D}(T)}(\xi) = \\ &\tfrac{1}{2\pi}\int_0^{2\pi}\left|\sum\left(\psi_n\cos nt + \tfrac{\phi_n}{n}\sin nt\right)\right| dt = \\ &\tfrac{1}{2\pi}\int_0^{2\pi}|\zeta(t)|\, dt. \end{aligned} \qquad (17)$$

where $\zeta(t) = \xi_1(t) + \int_0^t \xi_0(x)dx$.

Note that the operator of multiplication by $\frac{1}{T}$ in the factor-space $\overline{S}$ is induced by the operator $C(T)$, and (17) describes the limit shape $\lim_{T\to\infty} \mathrm{Sh}\,\overline{D}(T)$. Denote by Ω the convex body such that its support function is given by the right-hand side of (17):

$$\begin{aligned} H_\Omega(\xi) &= \tfrac{1}{2\pi}\int_0^{2\pi}\left|\sum\left(\psi_n\cos nt + \tfrac{\phi_n}{n}\sin nt\right)\right| dt \\ &= \tfrac{1}{2\pi}\int_0^{2\pi}|\zeta(t)|\, dt. \end{aligned} \qquad (18)$$

According to theorem 2 the set $T\Omega$ is an approximation of $D(T)$ if T is large.

5 Dry-friction control

Our control design is based on the following idea: The optimal control at state $\mathfrak{f}$ implements the steepest descent in the direction normal to boundaries of the reachable sets $D(T)$. Our control implements the steepest descent in the direction normal to boundaries of the approximate reachable sets $T\Omega$, where Ω is defined via (18). This means that in notations (17)

$$u(\mathfrak{f}) = -\operatorname{sign}\langle B, \xi\rangle = -\operatorname{sign}\xi_1(0) = -\operatorname{sign}\zeta(0), \qquad (19)$$

where the momentum ξ is to be found via the equation

$$T^{-1}\mathfrak{f} = \frac{\partial H_\Omega}{\partial \xi}(\xi), \qquad (20)$$

or, equivalently, $\mathfrak{f} = (f_0, f_1)$, where

$$\begin{aligned} T^{-1}f_0(x) &= -\textstyle\int_0^x (\operatorname{sign}\zeta(y))^- dy, \\ T^{-1}f_1(x) &= (\operatorname{sign}\zeta(x))^+, \end{aligned} \qquad (21)$$

where the notation $f^\pm$ stands for even/odd part of the function f:

$$f^\pm(x) = \frac{1}{2}(f(x) \pm f(-x)). \qquad (22)$$

These identities are to be understood as inclusions, because the sign-map is multivalued. Namely, their precise meaning is

$$\begin{aligned} T^{-1}f_0(x) &= -\textstyle\int_0^x \phi(y)^- dy, \\ T^{-1}f_1(x) &= \psi(x)^+, \end{aligned} \qquad (23)$$

where $\phi(y) \in \operatorname{sign}\zeta(y)$, and $\psi(x) \in \operatorname{sign}\zeta(x)$.

6 Duality transform

We discuss a general duality transformation related to equation (20). Toward this end we denote the function H_Ω just by $H = H(\xi)$, and the factor T by $\rho(\mathfrak{f})$. Then, the relation between H and ρ is similar to the Legendre transformation:

$$\begin{aligned} \langle \mathfrak{f}, \xi\rangle = \rho(\mathfrak{f})H(\xi), \quad \rho(\mathfrak{f}) = \max\nolimits_{H(\xi)\le 1}\langle \mathfrak{f}, \xi\rangle, \\ H(\xi) = \max\nolimits_{\rho(\mathfrak{f})\le 1}\langle \mathfrak{f}, \xi\rangle, \end{aligned} \qquad (24)$$

where the correspondence $\mathfrak{f} \rightleftarrows \xi$ has the form

$$\mathfrak{f} = \rho(\mathfrak{f})\frac{\partial H}{\partial \xi}(\xi), \quad \xi = H(\xi)\frac{\partial \rho}{\partial \mathfrak{f}}(\mathfrak{f}). \qquad (25)$$

Here, ξ and $\mathfrak{f}$ are the points where the maximums in (24) are attained. These relations make sense provided that H and ρ are norms, i.e., homogeneous of degree 1 convex functions such that the sublevel sets $\{H(\xi)\le 1\}$ and $\{\rho(\mathfrak{f})\le 1\}$ are convex bodies. These sublevels are mutually polar to each other. In other words, if $\Omega = \{\rho(\mathfrak{f})\le 1\}$, and $\Omega^\circ = \{H(\xi)\le 1\}$, then $\Omega = \{\mathfrak{f} : \langle \mathfrak{f}, \xi\rangle \le 1,\ \xi\in\Omega^\circ\}$ and vice versa. In the language of Banach spaces, the normed spaces $(\mathbb{V}, \rho)$ and $(\mathbb{V}^*, H)$ are dual to each other. The derivatives in (25) should be understood as subgradients. If the functions H and ρ are differentiable the equation (25) has the classical meaning. If one of the functions H and ρ is differentiable and strictly convex, then, the other one is also so.

The above discussion of duality is absolutely correct in the finite dimensional setting, but is not literally correct in infinite dimension, because there is no good duality theory for general Banach spaces: double dual to a Banach space is not necessarily isomorphic to the initial one. Still, it is a good starting point for intuition.

In the cases at hand we need to calculate the dual function ρ for the function $H = H_\Omega$ from (18).

Theorem 3. *Consider problems (2) – (3) from the Introduction, corresponding to the terminal manifolds $\mathcal{C} = \mathbf{R} \times 0$, or $\mathcal{C} = \mathbf{R}^2$.*

1. *If $\mathcal{C} = \mathbf{R}^2$, then $\rho(\mathfrak{f}) = 2\pi \left|\frac{\partial f_0}{\partial x} + f_1\right|_\infty$, where the norm $|\phi|_\infty$ of a function ϕ on the torus $\mathcal{T} = \mathbf{R}/2\pi\mathbf{Z}$ is $\inf_c \sup_{x\in\mathcal{T}} |\phi(x) + c|$, where $c \in \mathbf{R}$ is an arbitrary constant.*
2. *If $\mathcal{C} = \mathbf{R} \times 0$, then $\rho(\mathfrak{f}) = 2\pi \left|\frac{\partial f_0}{\partial x} + f_1\right|_\infty$, where the norm $|\phi|_\infty$ is the sup-norm of the function ϕ on the torus $\mathcal{T} = \mathbf{R}/2\pi\mathbf{Z}$.*

We consider only the case (2) of Theorem 3, because the case (3) is quite similar. Note that

$$|\phi|_\infty = \frac{1}{2}\left(\sup\phi - \inf\phi\right). \qquad (26)$$

We note that $\frac{\partial f_0}{\partial x}$ is an odd function, while f_1 is even. This implies that the norm $\rho(\mathfrak{f})$ is equivalent to (although does not coincide with) $\max\left(\left|\frac{\partial f_0}{\partial x}\right|_\infty, |f_1|_\infty\right)$.

To prove the theorem we define $\rho_0(\mathfrak{f})$ by the formula $\rho_0(\mathfrak{f}) = \left|\frac{\partial f_0}{\partial x} + f_1\right|_\infty$, and check that $H(\xi) = \frac{1}{2\pi}\max_{\rho_0(\mathfrak{f})\le 1}\langle \mathfrak{f}, \xi\rangle$. Toward this end, put

$$\psi_0(t) = \int_0^t \xi_0(x)dx,\ \psi_1(t) = \xi_1(t),\ \psi = \psi_0 + \psi_1,$$

and

$$\phi_0 = \frac{\partial f_0}{\partial x},\ \phi_1 = f_1,\ \phi = \phi_0 + \phi_1.$$

Denote by $\int_\mathcal{T} f$, where $\mathcal{T} = \mathbf{R}/2\pi\mathbf{Z}$, the normalized integral $\frac{1}{2\pi}\int_0^{2\pi} f(t)dt$. Remind that $\langle \mathfrak{f}, \xi\rangle$ stands for $\int_0^{2\pi}\langle \mathfrak{f}(t), \xi(t)\rangle dt$.

Then,

$$\int_\mathcal{T}\phi\psi = \int_\mathcal{T}\phi_0\psi_0 + \int_\mathcal{T}\phi_1\psi_1 = \frac{1}{2\pi}\langle \mathfrak{f}, \xi\rangle, \qquad (27)$$

because the integrals $\int_\mathcal{T}\phi_0\psi_1 dt$, $\int_\mathcal{T}\phi_1\psi_0 dt$ vanish, being integrals of odd functions over $\mathcal{T} = \mathbf{R}/2\pi\mathbf{Z}$. It is clear from (27), that the maximum of $\int_\mathcal{T}\phi\psi dt$, taken over ϕ such that $|\phi|_\infty \le 1$, coincides with the maximum of $\langle \mathfrak{f}, \xi\rangle$, taken over $\mathfrak{f}$ such that $\rho_0(\mathfrak{f}) \le 1$. However, it is trivial that the maximum of $\int_\mathcal{T}\phi\psi = \int_\mathcal{T}|\psi|$. The latter value, according to (18), equals $H(\xi)$.

One can regard our computation of the norm ρ as an a priori estimate for solutions of the wave equation:

Theorem 4. *Suppose $\mathfrak{f} = (f_0, f_1)$, is a solution of the Cauchy problem*

$$\frac{\partial \mathfrak{f}}{\partial t} = A\mathfrak{f} + Bu, \quad |u| \le 1, \quad \mathfrak{f}(0) = 0, \qquad (28)$$

where $B = (0, \delta)$, $u = u(t)$. Then, if $T \ge 2\pi$ we have $\rho(\mathfrak{f}(T)) = 2\pi\left|\frac{\partial f_0}{\partial x} + f_1\right|_\infty \le T$.

The following trivial corollary, where previous notations are retained, is quite important for us:

Corollary 1. *Suppose $\mathfrak{f} = (f_0, f_1)$, is a solution of $\frac{\partial \mathfrak{f}}{\partial t} = A\mathfrak{f} + Bu$, $|u| \le 1$, while $\widetilde{\mathfrak{f}}$ is control-free: $\frac{\partial \mathfrak{f}}{\partial t} = A\mathfrak{f}$, and $\widetilde{\mathfrak{f}}(0) = \mathfrak{f}(0)$. Then, provided that $T \ge 2\pi$ we have $\left|(f_1 - \widetilde{f}_1)(T)\right|_\infty \le \frac{1}{2\pi}T$.*

7 Computation of the basic control

Consider problem (2) from the Introduction, corresponding to the terminal manifolds $\mathcal{C} = \mathbf{R}\times 0$. In order to find the control we need to solve equations (21) as explicitly as possible. In other words, we have to find function ζ such that

$$T^{-1}\frac{\partial f_0}{\partial x}(x) \in -\operatorname{sign}\zeta(x), \quad T^{-1}f_1(x) \in \operatorname{sign}\zeta(x). \qquad (29)$$

Our discussion of duality, in particular the second equation (25), shows that the solution is given by

$$T = \rho(\mathfrak{f}) = 2\pi\left|\frac{\partial f_0}{\partial x} + f_1\right|_\infty, \quad \zeta = \frac{\partial\rho}{\partial \mathfrak{f}_1}(\mathfrak{f}).$$

The final expression for the control is

$$u\left(\mathfrak{f}\right) = -\operatorname{sign}\zeta(0) = -\operatorname{sign} f_1(0), \qquad (30)$$

where we take into account the second equation (21) and the vanishing at 0 of the odd part of the function $x \mapsto \operatorname{sign}\zeta(x)$. Thus, we obtain indeed a generalization of the dry friction, for it acts with maximal possible amplitude against the velocity, because $f_1(0)$ is exactly the velocity of the point, where the load is applied. The control (30) leads to the nonlinear wave equation

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} - \operatorname{sign}\left(\frac{\partial f}{\partial t}(0)\right)\delta \qquad (31)$$

governing the damping process.

8 Restatement of the model

Previous considerations stress the importance of the function

$$g = \frac{\partial f_0}{\partial x} + f_1. \qquad (32)$$

Knowledge of this function is almost equivalent to the knowledge of both functions f_0 and f_1. Indeed, the function $\frac{\partial f_0}{\partial x}$ is odd and f_1 is even. Therefore, knowledge of these functions is equivalent to the knowledge of the function g. On the other hand the knowledge of $\frac{\partial f_0}{\partial x}$ gives a complete information on f_0 up to an additive constant. This constant is irrelevant if the goal of our damping process is to stop oscillation, or to stop motion of the string at an unspecified point. The law of the controlled motion (1), (30), (31) can be restated as follows:

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) g(x,t) = \delta(x)u(t), \quad |u| \leq 1. \tag{33}$$

This form of the governing law has its advantages. In particular, it can be made rather explicit: One can rewrite (33) as

$$\frac{d}{dt}\, g(x-t,t) = \delta(x-t)u(t), \tag{34}$$

which means that

$$\begin{aligned} g(x-t,t) &= g(x,0) + \textstyle\int_0^t \delta(x-s)u(s)ds \\ &= g(x,0) + \textstyle\sum_I u(x+2k\pi), \end{aligned} \tag{35}$$

where the summation is over the set $I = I_t$ of $k \in \mathbf{Z}$ such that $x + 2k\pi \in [0,t]$. By the change of variables $z = x - t$ we come to

$$g(z,t) = g(z+t,0) + \sum_J u(z+t+2k\pi), \tag{36}$$

where the summation is over the set $J = J_t$ of $k \in \mathbf{Z}$ such that $z + 2k\pi \in [-t,0]$. Equation (36) should be understood as follows: Here, g is a bounded measurable function of x, t and u is a bounded measurable function of t, the curve $t \mapsto g(\cdot,t)$ is continuous as a map from real line to distributions depending on the space variable x. Equation (36) does not hold pointwise, but expresses an equality in the space of curves of distributions of x.

9 Existence of the motion under dry-friction control

We have to obtain an existence theorem for initial value problem for the nonlinear wave equation (31). By using transformation (32) the task reduces to solution of the functional equation

$$g(z,t) = g(z+t,0) - \sum_J \operatorname{sign} g(0,z+t+2k\pi), \tag{37}$$

which in turn can be reduced to the search for the function $g(0,t)$, $t \geq 0$, because this defines the control low $u(t) = -\operatorname{sign} g(0,t)$. This is quite nontrivial, because the the function $\phi(t) = g(0,t)$, we are looking for, should satisfy a functional equation. The first step in establishing the desired functional equation is to make equation (37) hold pointwise. It is explained in the previous section that this equation expresses an equality in the space of curves of distributions of x. In order to make equation (37) hold pointwise we consider the one-sided averaging operators

$$\begin{aligned} \mathrm{Av}^{\pm\epsilon} &: f(z,t) \mapsto \tfrac{1}{\epsilon}\textstyle\int_0^{\pm\epsilon} f(z+x,t)dx, \\ \mathrm{Av}^{\pm} &: f \mapsto \lim_{\epsilon\to 0} \mathrm{Av}^{\pm\epsilon}(f), \end{aligned} \tag{38}$$

and more standard two-sided operator

$$\begin{aligned} \mathrm{Av}^{\epsilon} &: f(z,t) \mapsto \tfrac{1}{2\epsilon}\textstyle\int_{-\epsilon}^{\epsilon} f(z+x,t)dx, \\ \mathrm{Av} &: f \mapsto \lim_{\epsilon\to 0} \mathrm{Av}^{\epsilon}(f). \end{aligned} \tag{39}$$

Note that, according to the Lebesgue differentiation theorem, the limit averaging operators $\mathrm{Av}^{\pm}$ and Av are identities when applied to any L_1-function. The reason for application of these operators is that, if operators $\mathrm{Av}^{\pm\epsilon}$ are applied to the right-hand and the left-hand sides of equation (37), the obtained equation holds *pointwise*. In particular,

$$\begin{aligned} &\mathrm{Av}^{\epsilon}(g)(0,t) = \mathrm{Av}^{\epsilon}(g)(t,0) - \\ &\textstyle\sum \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} u(z+t+2k\pi)1_{[-t,0]}(z+t+2k\pi)dz = \\ &\mathrm{Av}^{\epsilon}(g)(t,0) - \tfrac{1}{2}\textstyle\sum \mathrm{Av}^{-\epsilon}(u)(t+2k\pi), \end{aligned} \tag{40}$$

where $u(t) = -\operatorname{sign} g(0,t)$, summation is over the set of $k \in \mathbf{Z}$ such that $2k\pi \in [-t,0]$, and $\epsilon < t$.

To state the desired functional equation consider the function $\phi(t) = \mathrm{Av}(g)(0,t)$.

It follows from (40) by passing to the limit $\epsilon \to 0$ that

$$\phi(t) = G(t) - \frac{1}{2}\sum_{2k\pi\in[-t,0]} \operatorname{sign}\phi(t+2k\pi), \tag{41}$$

where $G(t) = g(t,0)$ is the given initial 2π-periodic function.

Solution of this equations gives at the same time a rigorously defined solution to the nonlinear wave equation (31).

Thus, we have to solve the equation

$$\phi(t) + \frac{1}{2}\sum_{2k\pi\in[-t,0]} \operatorname{sign}\phi(t+2k\pi) = G(t), \tag{42}$$

where G is a given function, and ϕ is unknown. Note that the function ϕ need not be periodic. It should be defined for nonnegative t. Note also that if $t < 2\pi$, the latter equation reduces to a very simple one:

$$\phi(t) + \frac{1}{2}\operatorname{sign}\phi(t) = G(t), \qquad (43)$$

which, obviously, has a unique solution, since the map $x \mapsto x + \operatorname{sign} x$ is a strictly monotone increasing (multivalued) function. More explicitly, the solution $\phi(t) = G(t) - \frac{1}{2}$ if $G(t) > \frac{1}{2}$, and $\phi(t) = G(t) + \frac{1}{2}$ if $G(t) < -\frac{1}{2}$. Otherwise, $\phi(t) = 0$. Note that $|\phi(t)| \le |G(t)|$ in the considered interval $[0, 2\pi)$ of values of the argument t.

It is better rewrite the above equation (43) in the form

$$\phi(t) + \frac{1}{2}v(t) = G(t), \quad v(t) = \operatorname{sign}\phi(t), \qquad (44)$$

where sign-function is regarded as multivalued: $\operatorname{sign}(0) = [-1, 1]$. Then, the a priori multivalued $\operatorname{sign}\phi(t)$ is defined by (44) uniquely. If $t < 2\pi$ we obtain from (41) and periodicity $G(t + 2\pi) = G(t)$ that

$$\phi(t + 2\pi) + \frac{1}{2}\operatorname{sign}\phi(t + 2\pi) = \phi(t) \qquad (45)$$

which allows to extend by the preceding arguments the function $\phi(t)$ from $t \in [0, 2\pi)$ to any positive value of t. By the already used arguments we obtain that $|\phi(t)| \le |G(t)|$ for all $t \ge 0$.

Theorem 5. *The Cauchy problem for the nonlinear wave equation*

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) g(x, t) = -\delta(x)\operatorname{sign} g(0, t), \qquad (46)$$

where $g(x, 0)$ is a given bounded (Borel-measurable) function possesses a unique bounded solution for $t \ge 0$. The functions $\phi(t) = g(0, t)$ and $u(t) = -\operatorname{sign} g(0, t)$ form a unique solution of functional equation (41).

We call the flow $g = g(\cdot, 0) \mapsto \Phi_t(g) = g(\cdot, t)$, where $t \ge 0$, in the space of measurable bounded functions the dry-friction flow.

10 Asymptotic optimality of control: polar-like coordinate system

In this section we present at an intuitive level reasons for asymptotic optimality of the control law (30). The rigorous treatment of asymptotic optimality is performed in the next Section 11. We define a polar-like coordinate system, well suited for representation of the motion under the control u. Every state $0 \ne \mathfrak{f}$ of the string can be represented uniquely as

$$\mathfrak{f} = \rho\phi, \text{ where } \rho = \rho(x) \text{ is positive, and } \phi \in \partial\Omega. \qquad (47)$$

The pair ρ, ϕ is the coordinate representation for x, and $\rho(\phi) = 1$ is the equation of the "sphere" $\omega = \partial\Omega$. It is important that the set ω is invariant under free (uncontrolled) motion of our system (5). This follows from the similar invariance of the support function $H_\Omega(p)$ under evolution governed by $\dot{p} = -A^*p$ (system (6)). The latter invariance is clear, because the support function is an ergodic mean of the function $|\xi_1(0, t)|$ under the free motion. This implies invariance of the dual function $\rho = \rho(\mathfrak{f})$, so that $\langle \partial\rho/\partial\mathfrak{f}, A\mathfrak{f}\rangle = 0$. Therefore, under the control u from (30) the total (Lie) derivative of ρ takes the form

$$\dot{\rho} = \left\langle \frac{\partial\rho}{\partial\mathfrak{f}}, A\mathfrak{f} + Bu \right\rangle = \left\langle \frac{\partial\rho}{\partial\mathfrak{f}}, Bu \right\rangle = -\left|\left\langle \frac{\partial\rho}{\partial\mathfrak{f}}, B \right\rangle\right|, \qquad (48)$$

where the last identity holds because $\partial\rho/\partial\mathfrak{f}$ is the outer normal to the set $\rho\Omega$. In particular, the "radius" ρ is a monotone nonincreasing function of time. Moreover, the RHS of (48) necessarily equals -1 if $f_1(0) \ne 0$. For any other admissible control, we have

$$\dot{\rho} \ge -\left|\left\langle \frac{\partial\rho}{\partial\mathfrak{f}}, B \right\rangle\right|. \qquad (49)$$

The evolution of ϕ by virtue of system (5) is described by

$$\begin{aligned} \dot{\phi} &= A\phi + \tfrac{1}{\rho}(Bu - \phi\dot{\rho}) = \\ & A\phi + \tfrac{1}{\rho}\left(Bu + \phi\left|\left\langle \tfrac{\partial\rho}{\partial\mathfrak{f}}, B\right\rangle\right|\right). \end{aligned} \qquad (50)$$

We note that the right-hand side $-\left|\left\langle \frac{\partial\rho}{\partial\mathfrak{f}}(\mathfrak{f}), B\right\rangle\right|$ of (48) equals $-\left|\left\langle \frac{\partial\rho}{\partial\mathfrak{f}}(\phi), B\right\rangle\right|$. Thus, the evolution of the RHS of (48) is determined by the evolution of ϕ by virtue of (50). It is clear that if ρ is large, then the second term in the RHS of (50) is $O(1/\rho)$ and affects the motion of ϕ over the "sphere" ω only slightly. Our next task is to compute approximately the "ergodic mean"

$$E_T = \frac{1}{T}\int_0^T \left|\left\langle \frac{\partial\rho}{\partial\mathfrak{f}}, B\right\rangle\right| dt$$

of the RHS of (48) provided that ρ is large. Here, B is a constant vector, while, according to the preceding arguments, the vector function $\frac{\partial\rho}{\partial\mathfrak{f}}(t) := \frac{\partial\rho}{\partial\mathfrak{f}}(\mathfrak{f}(t))$ behaves approximately as $e^{A^*t}\frac{\partial\rho}{\partial\mathfrak{f}}(0)$. Therefore, the ergodic mean E_T is well approximated by

$$E_T = \frac{1}{T}\int_0^T \left|\left\langle e^{A^*t}\xi, B\right\rangle\right| dt,$$

where $\xi = \frac{\partial\rho}{\partial\mathfrak{f}}(0)$. We know from Theorem 2 that as $T \to \infty$ the ergodic mean E_T tends to $H(\xi) = H(\frac{\partial\rho}{\partial\mathfrak{f}})$. But, according to one of the basic "duality relation" (25), we know that $H(\frac{\partial\rho}{\partial\mathfrak{f}}) = 1$.

Therefore, we conclude, by using abbreviation $\rho(t) = \rho(\frac{\partial\rho}{\partial\mathfrak{f}}(\mathfrak{f}(t)))$, that

$$(\rho(0) - \rho(T))/T = 1 + o(1), \text{ as } T \to \infty,$$

provided that we use the dry-friction control (30).

Under any other admissible control, according to Theorem 4,

$$(\rho(0) - \rho(T))/T \leq 1 + o(1).$$

This expresses the asymptotic optimality we sought for.

11 Asymptotic optimality: proof

Here we prove the asymptotic optimality of the control (30) via the use of the function $g(x,t)$ from (32). The law of motion (36) is

$$g(z,t) = g(z+t,0) - \sum_J \operatorname{sign} g(0, z+t+2k\pi), \quad (51)$$

where the set $J = J_t$ consists of $k \in \mathbf{Z}$ such that $z + 2k\pi \in [-t, 0]$. The functional ρ has the form $\rho(g) = 2\pi \sup_{x\in\mathbf{R}/2\pi\mathbf{Z}} |g(x,t)|$. The control $\operatorname{sign} g(0, z+t)$ is not affected by the scaling transformation

$$g \mapsto \Phi = g/\rho.$$

However, if ρ is large, then our previous considerations reveal that the function $\Phi = g/\rho$ moves in al almost uncontrollable mode. The latter means that approximately

$$\Phi(x,t) \approx \Phi(x+t, 0),$$

so that we come to the approximate equality

$$\operatorname{sign} g(0,z) \approx \operatorname{sign} g(z,0).$$

More precisely, suppose that in the time-interval $[0,T]$ we have $\rho(g_t) \geq 2\pi M$, where M is a (large) constant. In view of (41) we have

$$g(0,t) = g(t,0) - \frac{1}{2} \sum_{2k\pi\in[-t,0]} \operatorname{sign} g(0, t+2k\pi), \quad (52)$$

and, therefore,

$$|g(0,t) - g(t,0)| \leq \frac{t}{4\pi}. \quad (53)$$

Since $\rho(g) \geq M$ there exist points $x \in \mathbf{R}/2\pi\mathbf{Z}$, where either $g(x,0) \geq M-1$ or $g(x,0) \leq -(M-1)$. Assume for definiteness that $g(x,0) \geq M-1$. Then, in view of (53), $\operatorname{sign} g(0, t+2k\pi) = +1$ for $t \in [0,T]$ provided that $\frac{T}{4\pi} \leq M-1$. For instance, this is the case if M is large and $T = O(\sqrt{M})$.

In view of (51) this means that

$$g(z-t,t) = g(z,0) - \frac{t}{2\pi} \operatorname{sign} g(z,0) + O(1), \quad (54)$$

where $|O(1)| \leq 1$, and $g(z,0) \geq M-1$. This implies that

$$\sup_z g(z,t) = \sup_z g(z,0) - \frac{t}{2\pi} + O(1), \quad (55)$$

since $\operatorname{sign} g(z,0) = +1$ if $g(z,0) \geq M-1$. Since $\rho(t) = 2\pi \sup_z g(z,t)$ we obtain the approximate equality

$$(\rho(0) - \rho(t))/t = 1 + O(1/t), \quad (56)$$

provided that the length T of the time interval is less than $4\pi(M-1)$.

By partition of any sufficiently long interval of time $[0,T]$ into many equal intervals of length $\leq 4\pi(M-1)$ we come to the following:

Theorem 6. *Consider evolution $\rho(t) = \rho(g_t)$ of ρ under control (51). Let*

$$M = \min\{\rho(0), \rho(T)\}. \quad (57)$$

Suppose that $M \to +\infty$, $T \to +\infty$. Then, we have

$$(\rho(0) - \rho(T))/T = 1 + O(1/T + 1/M). \quad (58)$$

Under any other admissible control,

$$(\rho(0) - \rho(T))/T \leq 1 + O(1/T + 1/M). \quad (59)$$

The preceding arguments prove statement (58), statement (59) follows from Theorem 4.

12 Attractors of the dry-friction flow

Methods used in Section 9 allows to reveal the basic properties of the dry-friction flow. In particular, it is possible to derive the asymptotic optimality of the dry-friction flow directly from equations (43)–(45). Indeed, it follows from equations (43), (44) that if $\sup_x g(x,0) > 1/2$, then $\sup_{t\in[0,2\pi]} \phi(t) = \sup_x g(x,0) - 1/2$. From equation (45) it follows that

13 Singular arcs

These are by definition the time-intervals, where in the controlled motion

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) g(x,t) = \delta(x)u(t), \quad u = -\operatorname{sign} g(0,t) \tag{60}$$

the control is not uniquely defined, i.e. $g(0,t) \equiv 0$. To construct a motion of this kind we use the spectral decompositions $g(x,t) = \sum g_\mu(x)e^{i\mu t}$, and $u(t) = \sum u_\mu e^{i\mu t}$. This almost periodic function should be bounded: $|u| \le 1$ Then, the functions g_μ should satisfy

$$i\mu g_\mu - \frac{\partial}{\partial x} g_\mu = -\delta u_\mu, \text{ and } g_\mu(0) = 0. \tag{61}$$

The first equation (61) guarantees that 0 is the point of discontinuity of g_μ, so that the second equation (61) should be treated cautiously. In fact, the discussion of Section 9 shows that we have to take $\frac{1}{2}(g_\mu(0+) + g_\mu(0-))$ for $g_\mu(0)$. Indeed, according to the first equation (61), the function g_μ is piecewise differentiable with jumps at $x = 0$. Therefore, 2π-periodic function g_μ should have the form

$$g_\mu(x) = C_\mu e^{i\mu x} \text{ for } x \in [0, 2\pi), \tag{62}$$

where the constant $C_\mu = (1 - e^{2\pi i\mu})^{-1}u_\mu$. The condition $g_\mu(0) = 0$ gives

$$1 + e^{2\pi i\mu} = 0,$$

which implies that μ should have the form $\mu = \frac{1}{2}\nu$, where ν is an odd integer, and $C_\mu = u_\mu/2$. Therefore, the control $u(t) = \sum u_\mu e^{i\mu t}$ is not just almost periodic but 4π-periodic. Moreover,

$$\begin{aligned} g(x,t) &= \sum g_\mu(x)e^{i\mu t} = \tfrac{1}{2}\sum u_\mu e^{i\mu(t+x)} \\ &= \tfrac{1}{2}u(t+x) \text{ for } x \in [0, 2\pi). \end{aligned}$$

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References

L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko. *The mathematical theory of optimal processes*. Interscience, 1962.

R.E. Kalman. *On the general theory of control systems*, Proc. First IFAC Congr., Moscow, **1** (1960), 481–492.

V.I. Arnold. *Mathematical methods of classical mechanics*. Springer-Verlag, 1989.

A.A. Agrachev and R. V. Gamkrelidze. *The geometry of maximum principle*, Proc. Steklov Inst. Math., **273** (2011), 1–22.

A. Varchenko. *Multidimensional hypergeometric functions in conformal field theory, algebraic K-theory, algebraic geometry*, Proc. ICM-90, vol. I, 281–300.

A.F. Filippov. *Differential equations with discontinuous righthand sides*. Springer, 1988.

R.J. DiPerna and P.L. Lions. *Ordinary differential equations, transport theory and Sobolev spaces*, Inventiones mathematicae, **98** (1989), 511–547.

I.A. Bogaevskii. *Discontinuous gradient differential equations and trajectories in calculus of variations*, Sbornik: Mathem., **97** (2006), 12, 1723-1751.

A.K. Fedorov and A.I. Ovseevich. *Asymptotic control theory for a system of linear oscillators*, Moscow Mathematical Journal, Vol. 16, No. 3, July–September 2016, 561–598

A.I. Ovseevich and A.K. Fedorov. *Asymptotically optimal feedback control for a system of linear oscillators*, Doklady Mathematics, **88** (2013), 2, 613–617.

J.L. Lions. *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Review, **30** (1988), 1, 1–68.

E.V. Goncharova and A.I. Ovseevich. *Comparative analysis of the asymptotic dynamics of reachable sets to linear systems*, Journal of Computer and Systems Sciences International, **46** (2007), 4, 505–513.

A.I. Ovseevich. *Singularities of attainable sets*, Russian Journal of Mathematical Physics, **5** (1997), 3, 389–398.

R. Schneider. *Convex bodies: the Brunn-Minkowski theory*. Cambridge University Press, 1993.

Andre Weil. *Elliptic Functions according to Eisenstein and Kronecker*, Springer Verlag, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 88 (1976)