COUPLING OF EIGENVALUES WITH APPLICATIONS IN PHYSICS AND MECHANICS

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Abstract
This is a review paper presenting a general theory of coupling of eigenvalues of complex matrices of an arbitrary dimension depending on real parameters. The cases of weak and strong coupling are distinguished and their geometric interpretation in two and three-dimensional spaces is given. General asymptotic formulae for eigenvalue surfaces near diabolic and exceptional points are presented demonstrating crossing and avoided crossing scenarios. A physical example on propagation of light in a homogeneous non-magnetic crystal illustrates effectiveness and accuracy of the presented theory. As applications in mechanics stability problems for a pendulum with periodically varying length and stabilization effect for a buckled elastic rod by longitudinal vibrations are considered.

Key words
Coupling of eigenvalues, stability problems, physics, mechanics.

1 Introduction

The behavior of eigenvalues of matrices and differential operators dependent on parameters is a problem of general interest having many important applications in natural and engineering sciences. In modern physics, e.g. quantum mechanics, crystal optics, physical chemistry, acoustics and mechanics, singular points of matrix spectra associated with specific effects have attracted great interest of researchers since the papers [Von Neumann and Wigner, 1929] and [Teller, 1937]. These are the points where matrices possess multiple eigenvalues. In applications the case of double eigenvalues is the most important. With a change of parameters, coupling and decoupling of eigenvalues with crossing and avoided crossing scenarios occur. In recent papers two important cases are distinguished: the diabolic points (DPs) and the exceptional points (EPs). From the mathematical point of view DP is a point where the eigenvalues coalesce (become double), while corresponding eigenvectors remain different (linearly independent); and EP is a point where both eigenvalues and eigenvectors merge forming a Jordan block. Both the DP and EP cases are interesting in applications and were observed in experiments. In early studies only real and Hermitian matrices were considered while modern physical systems require the study of complex symmetric and non-symmetric matrices.

In this paper we present the results on coupling of eigenvalues of complex matrices of arbitrary dimension smoothly depending on multiple real parameters. Two essential cases of weak and strong coupling based on a Jordan form of the system matrix are distinguished. These two cases correspond to diabolic and exceptional points, respectively. We derive general formulae describing coupling and decoupling of eigenvalues, crossing and avoided crossing of eigenvalue surfaces. We present typical (generic) pictures showing the movement of eigenvalues, the eigenvalue surfaces and their cross-sections. It is emphasized that the theory of coupling of eigenvalues of complex matrices gives not only qualitative, but also quantitative results on the behavior of eigenvalues based only on the information taken at the singular points.

Interaction of eigenvalues for real matrices depending on multiple parameters with mechanical applications is given in the book [Seyranian and Mailybaev, 2003] where significant mechanical effects related to diabolic and exceptional points were studied. These include transference of instability between eigenvalue branches, bimodal solutions in optimal structures under stability constraints, flutter and divergence instabilities in undamped nonconservative systems, effect of gyroscopic stabilization, destabilization of a nonconservative system by infinitely small damping, which were described and explained from the point of view of coupling of eigenvalues. The presented theory for complex matrices is based on the papers [Seyranian, Kirillov and Mailybaev, 2005; Kirillov, Mailybaev and Seyranian, 2005].

The paper is organized as follows. In section 2 we present general results on weak and strong coupling of eigenvalues of complex matrices depending on parameters. These two cases correspond to the study of eigenvalue behavior near diabolic and exceptional points. Section 3 is devoted to a physical example of propagation of light in a homogeneous non-magnetic crystal. Section 4 presents applications in mechanics, and the conclusion is given in section 5.

2 Coupling of Eigenvalues

Let us consider the eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u} \tag{1}$$

for a general $m \times m$ complex matrix $\mathbf{A}$ smoothly depending on a vector of n real parameters $\mathbf{p} = (p_1, \ldots, p_n)$. Assume that, at $\mathbf{p} = \mathbf{p}_0$, the eigenvalue coupling occurs, i.e., the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has an eigenvalue λ_0 of multiplicity 2 as a root of the characteristic equation $\det(\mathbf{A}_0 - \lambda_0\mathbf{I}) = 0$; $\mathbf{I}$ is the identity matrix. This double eigenvalue can have one or two linearly independent eigenvectors $\mathbf{u}$, which determine the geometric multiplicity. The eigenvalue problem adjoint to (1) is

$$\mathbf{A}^*\mathbf{v} = \eta\mathbf{v}, \tag{2}$$

where $\mathbf{A}^* = \overline{\mathbf{A}}^T$ is the adjoint matrix operator (Hermitian transpose). The eigenvalues λ and η of problems (1) and (2) are complex conjugate: $\eta = \overline{\lambda}$.

Double eigenvalues appear at sets in parameter space, whose codimensions depend on the matrix type and the degeneracy (EP or DP). In this paper we analyze general (nonsymmetric) complex matrices. The EP degeneracy is the most typical for this type of matrices. In comparison with EP, the DP degeneracy is a rare phenomenon in systems described by general complex matrices. However, some nongeneric situations may be interesting from the physical point of view. As an example, we mention complex non-Hermitian perturbations of symmetric two-parameter real matrices, when the eigenvalue surfaces have coffee-filter singularity.

Let us consider a smooth perturbation of parameters in the form $\mathbf{p} = \mathbf{p}(\varepsilon)$, where $\mathbf{p}(0) = \mathbf{p}_0$ and ε is a small real number. For the perturbed matrix $\mathbf{A} = \mathbf{A}(\mathbf{p}(\varepsilon))$, we have

$$\begin{gathered}\mathbf{A} = \mathbf{A}_0 + \varepsilon\mathbf{A}_1 + \tfrac{1}{2}\varepsilon^2\mathbf{A}_2 + o(\varepsilon^2),\\ \mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0), \quad \mathbf{A}_1 = \sum_{i=1}^{n} \frac{\partial \mathbf{A}}{\partial p_i}\frac{dp_i}{d\varepsilon},\\ \mathbf{A}_2 = \sum_{i=1}^{n} \frac{\partial \mathbf{A}}{\partial p_i}\frac{d^2 p_i}{d\varepsilon^2} + \sum_{i,j=1}^{n} \frac{\partial^2 \mathbf{A}}{\partial p_i \partial p_j}\frac{dp_i}{d\varepsilon}\frac{dp_j}{d\varepsilon}.\end{gathered} \tag{3}$$

The double eigenvalue λ_0 generally splits into a pair of simple eigenvalues under the perturbation. Asymptotic formulae for these eigenvalues and corresponding eigenvectors contain integer or fractional powers of ε.

2.1 Weak Coupling of Eigenvalues

Let us consider the coupling of eigenvalues in the case of λ_0 with two linearly independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. This coupling point is known as a diabolic point. Let us denote by $\mathbf{v}_1$ and $\mathbf{v}_2$ two eigenvectors of the complex conjugate eigenvalue $\eta = \overline{\lambda}$ for the adjoint eigenvalue problem (2) satisfying the normalization conditions

$$\begin{gathered}(\mathbf{u}_1, \mathbf{v}_1) = (\mathbf{u}_2, \mathbf{v}_2) = 1,\\ (\mathbf{u}_1, \mathbf{v}_2) = (\mathbf{u}_2, \mathbf{v}_1) = 0,\end{gathered} \tag{4}$$

where $(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} u_i \overline{v}_i$ denotes the Hermitian inner product. Conditions (4) define the unique vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for given $\mathbf{u}_1$ and $\mathbf{u}_2$.

For nonzero small ε, the two eigenvalues λ_+ and λ_- resulting from the bifurcation of λ_0 and the corresponding eigenvectors $\mathbf{u}_\pm$ are given by

$$\begin{gathered}\lambda_\pm = \lambda_0 + \mu_\pm\varepsilon + o(\varepsilon),\\ \mathbf{u}_\pm = \alpha_\pm\mathbf{u}_1 + \beta_\pm\mathbf{u}_2 + o(1).\end{gathered} \tag{5}$$

The coefficients $\mu_\pm$, $\alpha_\pm$, and $\beta_\pm$ are found from the 2×2 eigenvalue problem, see e.g. [Seyranian and Mailybaev, 2003]

$$\begin{pmatrix} (\mathbf{A}_1\mathbf{u}_1, \mathbf{v}_1) & (\mathbf{A}_1\mathbf{u}_2, \mathbf{v}_1) \\ (\mathbf{A}_1\mathbf{u}_1, \mathbf{v}_2) & (\mathbf{A}_1\mathbf{u}_2, \mathbf{v}_2) \end{pmatrix} \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} = \mu_\pm \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix}. \tag{6}$$

Solving the characteristic equation for (6), we find

$$\begin{gathered}\mu_\pm = \frac{(\mathbf{A}_1\mathbf{u}_1, \mathbf{v}_1) + (\mathbf{A}_1\mathbf{u}_2, \mathbf{v}_2)}{2} \pm \\ \sqrt{D^2 + (\mathbf{A}_1\mathbf{u}_1, \mathbf{v}_2)(\mathbf{A}_1\mathbf{u}_2, \mathbf{v}_1)},\\ D = \tfrac{1}{2}((\mathbf{A}_1\mathbf{u}_1, \mathbf{v}_1) - (\mathbf{A}_1\mathbf{u}_2, \mathbf{v}_2)).\end{gathered} \tag{7}$$

We note that for Hermitian matrices $\mathbf{A}$ one can take $\mathbf{v}_1 = \mathbf{u}_1$ and $\mathbf{v}_2 = \mathbf{u}_2$ in (6), where the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are chosen satisfying the conditions $(\mathbf{u}_1, \mathbf{u}_1) = (\mathbf{u}_2, \mathbf{u}_2) = 1$ and $(\mathbf{u}_1, \mathbf{u}_2) = 0$, and obtain the well-known formula.

As the parameter vector passes the coupling point $\mathbf{p}_0$ along the curve $\mathbf{p}(\varepsilon)$ in parameter space, the eigenvalues λ_+ and λ_- change smoothly and cross each other at λ_0, see Figure 1a. At the same time, the corresponding eigenvectors $\mathbf{u}_+$ and $\mathbf{u}_-$ remain different (linearly independent) at all values of ε including the point $\mathbf{p}_0$.

We call this interaction *weak coupling*. By means of eigenvectors, the eigenvalues $\lambda_\pm$ are well distinguished during the weak coupling.

We emphasize that despite the eigenvalues $\lambda_\pm$ and the eigenvectors $\mathbf{u}_\pm$ depend smoothly on a single parameter ε, they are non-differentiable functions of multiple parameters at $\mathbf{p}_0$ in the sense of Frechét.

2.2 Strong Coupling of Eigenvalues

Let us consider coupling of eigenvalues at $\mathbf{p}_0$ with a double eigenvalue λ_0 possessing a single eigenvector $\mathbf{u}_0$. This case corresponds to the exceptional point. The second vector of the invariant subspace corresponding to λ_0 is called an associated vector $\mathbf{u}_1$ (also called a generalized eigenvector; it is determined by the equation

$$\mathbf{A}_0\mathbf{u}_1 = \lambda_0\mathbf{u}_1 + \mathbf{u}_0. \tag{8}$$

An eigenvector $\mathbf{v}_0$ and an associated vector $\mathbf{v}_1$ of the matrix $\mathbf{A}^*$ are determined by

$$\begin{aligned} &\mathbf{A}_0^*\mathbf{v}_0 = \overline{\lambda}_0\mathbf{v}_0, \quad \mathbf{A}_0^*\mathbf{v}_1 = \overline{\lambda}_0\mathbf{v}_1 + \mathbf{v}_0, \\ &(\mathbf{u}_1, \mathbf{v}_0) = 1, \quad (\mathbf{u}_1, \mathbf{v}_1) = 0, \end{aligned} \tag{9}$$

where the last two equations are the normalization conditions determining $\mathbf{v}_0$ and $\mathbf{v}_1$ uniquely for a given $\mathbf{u}_1$.

Bifurcation of λ_0 into two eigenvalues $\lambda_\pm$ and the corresponding eigenvectors $\mathbf{u}_\pm$ are described by, see [Seyranian and Mailybaev, 2003]

$$\begin{aligned} &\lambda_\pm = \lambda_0 \pm \sqrt{\mu_1\varepsilon} + \mu_2\varepsilon + o(\varepsilon), \\ &\mathbf{u}_\pm = \mathbf{u}_0 \pm \mathbf{u}_1\sqrt{\mu_1\varepsilon} \\ &+(\mu_1\mathbf{u}_0 + \mu_2\mathbf{u}_1 - \mathbf{G}^{-1}\mathbf{A}_1\mathbf{u}_0)\varepsilon + o(\varepsilon), \end{aligned} \tag{10}$$

where $\mathbf{G} = \mathbf{A}_0 - \lambda_0\mathbf{I} + \mathbf{u}_1\mathbf{v}_1^*$. The coefficients μ_1 and μ_2 are

$$\begin{aligned} &\mu_1 = (\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_0), \\ &\mu_2 = \big((\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_1) + (\mathbf{A}_1\mathbf{u}_1, \mathbf{v}_0)\big)/2. \end{aligned} \tag{11}$$

With a change of ε from negative to positive values, the two eigenvalues $\lambda_\pm$ approach, collide with infinite speed (derivative with respect to ε tends to infinity) at λ_0, and diverge in the perpendicular direction, see Figure 1b. The eigenvectors interact too. At $\varepsilon = 0$, they merge to $\mathbf{u}_0$ up to a scalar complex factor. At nonzero ε, the eigenvectors $\mathbf{u}_\pm$ differ from $\mathbf{u}_0$ by the leading term $\pm\mathbf{u}_1\sqrt{\mu_1\varepsilon}$. This term takes the purely imaginary factor i as ε changes the sign, for example altering from negative to positive values.

We call such a coupling of eigenvalues as *strong*. An exciting feature of the strong coupling is that the two eigenvalues cannot be distinguished after the interaction. Indeed, there is no natural rule telling how the eigenvalues before coupling correspond to those after the coupling.

3 Applications in Physics

As a physical example, we consider propagation of light in a homogeneous non-magnetic crystal in the general case when the crystal possesses natural optical activity (chirality) and dichroism (absorption) in addition to biaxial birefringence, see [Berry and Dennis, 2003] for the general formulation. The optical properties of the crystal are characterized by the inverse dielectric tensor $\boldsymbol{\eta}$. The vectors of electric field $\mathbf{E}$ and displacement $\mathbf{D}$ are related as $\mathbf{E} = \boldsymbol{\eta}\mathbf{D}$ [Landau, Lifshitz and Pitaevskii, 1984]. The tensor $\boldsymbol{\eta}$ is described by a non-Hermitian complex matrix. The electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ in the crystal are determined by Maxwell's equations

$$\mathrm{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \ \ \mathrm{rot}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}, \tag{12}$$

where t is time and c is the speed of light in vacuum.

A monochromatic plane wave of frequency ω that propagates in a direction specified by a real unit vector $\mathbf{s} = (s_1, s_2, s_3)$ has the form

$$\begin{aligned} &\mathbf{D}(\mathbf{r}, t) = \mathbf{D}(\mathbf{s}) \exp i\omega\Big(\tfrac{n(\mathbf{s})}{c}\mathbf{s}^T\mathbf{r} - t\Big) \\ &\mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mathbf{s}) \exp i\omega\Big(\tfrac{n(\mathbf{s})}{c}\mathbf{s}^T\mathbf{r} - t\Big), \end{aligned} \tag{13}$$

where $n(\mathbf{s})$ is a refractive index, and $\mathbf{r} = (x_1, x_2, x_3)$ is the real vector of spatial coordinates. Substituting the wave (13) into Maxwell's equations (12), we find

$$\mathbf{H} = n[\mathbf{s}, \boldsymbol{\eta}\mathbf{D}], \ \ \mathbf{D} = -n[\mathbf{s}, \mathbf{H}], \tag{14}$$

where square brackets indicate cross product of vectors [Landau, Lifshitz and Pitaevskii, 1984]. Then we obtain an eigenvalue problem for the complex non-Hermitian matrix $\mathbf{A}(\mathbf{s})$ dependent on the vector of parameters $\mathbf{s} = (s_1, s_2, s_3)$:

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}, \ \ \mathbf{A}(\mathbf{s}) = (\mathbf{I} - \mathbf{s}\mathbf{s}^T)\boldsymbol{\eta}(\mathbf{s}), \tag{15}$$

where $\lambda = n^{-2}$, $\mathbf{u} = \mathbf{D}$, and $\mathbf{I}$ is the identity matrix. Multiplying the matrix $\mathbf{A}$ by the vector s from the left we conclude that $\mathbf{s}^T\mathbf{A} = 0$, i.e., the vector s is the left eigenvector with the eigenvalue $\lambda = 0$. Zero eigenvalue always exists, because $\det(\mathbf{I} - \mathbf{s}\mathbf{s}^T) \equiv 0$, if $\|\mathbf{s}\| = 1$.

The matrix $\mathbf{A}(\mathbf{s})$ defined by equation (15) is a product of the matrix $\mathbf{I} - \mathbf{s}\mathbf{s}^T$ and the inverse dielectric tensor $\boldsymbol{\eta}(\mathbf{s})$. The symmetric part of $\boldsymbol{\eta}$ constitutes the

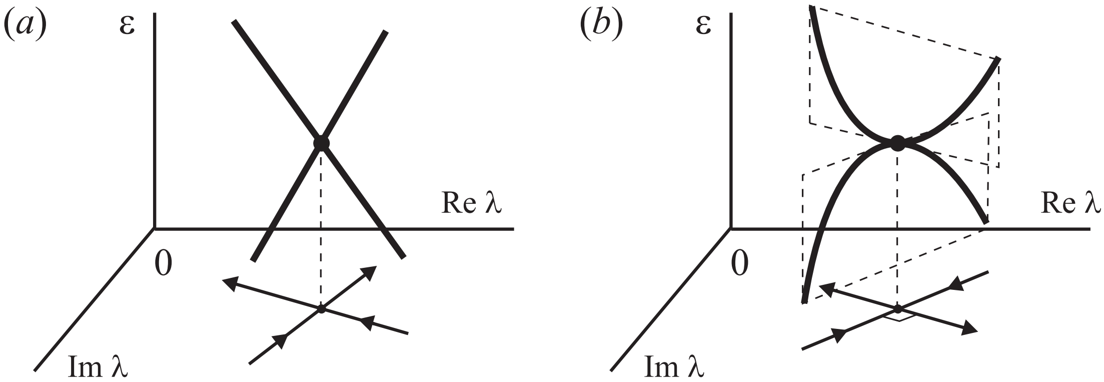

Figure 1. Eigenvalue coupling: (a) weak, (b) strong.

anisotropy tensor describing the birefringence of the crystal. It is represented by the complex symmetric matrix $\mathbf{U}$, which is independent of the vector of parameters $\mathbf{s}$. The antisymmetric part of $\boldsymbol{\eta}$ is determined by the optical activity vector $\mathbf{g}(\mathbf{s}) = (g_1, g_2, g_3)$, describing the chirality (optical activity) of the crystal. It is represented by the skew-symmetric matrix

$$\mathbf{G} = i\begin{pmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{pmatrix}. \tag{16}$$

The vector $\mathbf{g}$ is given by the expression $\mathbf{g}(\mathbf{s}) = \boldsymbol{\gamma}\mathbf{s}$, where $\boldsymbol{\gamma}$ is the optical activity tensor represented by a symmetric complex matrix. Thus, the matrix $\mathbf{G}(\mathbf{s})$ depends linearly on the parameters s_1, s_2, s_3.

In the present formulation, the problem was studied in [Berry and Dennis, 2003]. Below we present a specific numerical example in case of a non-diagonal matrix $\boldsymbol{\gamma}$, for which the structure of singularities was not fully investigated. Unlike [Berry and Dennis, 2003], where the reduction to two dimensions was carried out, we work with the three-dimensional form of problem (15). Our intention here is to give guidelines for using our theory by means of the relatively simple 3×3 matrix family, keeping in mind that the main area of applications would be higher dimensional problems.

For numerical example, we choose the inverse dielectric tensor in the form

$$\boldsymbol{\eta} = \begin{pmatrix} 3\,0\,0 \\ 0\,1\,0 \\ 0\,0\,2 \end{pmatrix} + i\begin{pmatrix} 0\,1\,2 \\ 1\,0\,0 \\ 2\,0\,0 \end{pmatrix} + i\begin{pmatrix} 0 & -s_1 & 0 \\ s_1 & 0 & -s_3 \\ 0 & s_3 & 0 \end{pmatrix} \tag{17}$$

where $s_3 = \sqrt{1 - s_1^2 - s_2^2}$. The crystal defined by (17) is dichroic and optically active with the non-diagonal matrix $\boldsymbol{\gamma}$. When $s_1 = 0$ and $s_2 = 0$ the spectrum of the matrix $\mathbf{A}$ consists of the double eigenvalue $\lambda_0 = 2$ and the simple zero eigenvalue. The double eigenvalue possesses the eigenvectors $\mathbf{u}_0$, $\mathbf{v}_0$, and associated vectors $\mathbf{u}_1$, $\mathbf{v}_1$. Calculating the derivatives of the matrix $\mathbf{A}(s_1, s_2)$ at the point $\mathbf{s}_0 = (0, 0, 1)$ and substituting it together with the vectors of Jordan chains $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{v}_0$, $\mathbf{v}_1$ yields the vectors $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$, $\mathbf{r}$ as

$$\mathbf{f} = (0, 4), \;\; \mathbf{g} = (-4, 0), \;\; \mathbf{h} = (0, 0), \;\; \mathbf{r} = (-4, 0). \tag{18}$$

With these vectors we find from the approximations of the eigensurfaces $\mathrm{Re}\lambda(s_1, s_2)$ and $\mathrm{Im}\lambda(s_1, s_2)$ in the vicinity of the point $\mathbf{s}_0 = (0, 0, 1)$:

$$\begin{aligned} \mathrm{Re}\lambda_\pm &= 2 \pm \sqrt{2s_2 + 2\sqrt{s_1^2 + s_2^2}}, \\ \mathrm{Im}\lambda_\pm &= -2s_1 \pm \sqrt{-2s_2 + 2\sqrt{s_1^2 + s_2^2}}. \end{aligned} \tag{19}$$

Calculation of the exact solution of the characteristic equation for the matrix $\mathbf{A}$ with the inverse dielectric tensor $\boldsymbol{\eta}$ defined by equation (17) shows a good agreement of the approximations (19) with the numerical solution, see Figure 2. One can see that the both surfaces of real and imaginary parts have a Whitney umbrella singularity at the coupling point; the surfaces self-intersect along different rays, which together constitute a straight line when projected on parameter plane. Other physical examples related to strong and weak coupling of eigenvalues are presented in [Seyranian, Kirillov and Mailybaev, 2005; Kirillov, Mailybaev and Seyranian, 2005; Mailybaev, Kirillov and Seyranian, 2005].

4 Mechanical Applications

In this section we consider applications in mechanics. First we study oscillations and stability problems for a pendulum with periodically varying length. It is a model of child's swing. Then we consider stabilization effect for an elastic rod, compressed by a longitudinal force greater than the critical Euler's value, by longitudinal vibrations applied to the rod end. This is called Chelomei's problem.

4.1 Stability of a Pendulum with Variable Length

Oscillations of a pendulum with variable length is among classical problems of mechanics. Usually, the

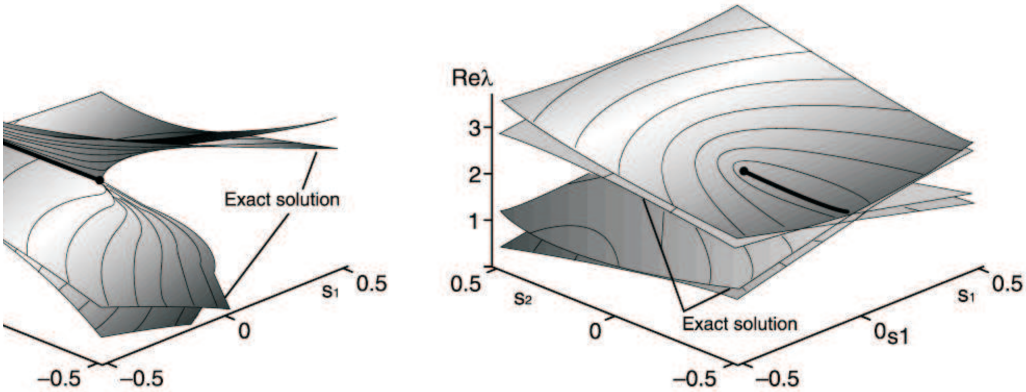

Figure 2. Eigensurfaces of a crystal and their approximations

pendulum with periodically varying length is associated with a child's swing. As probably everyone can remember, to swing a swing one must crouch when passing through the middle vertical position and straighten up at the extreme positions, i.e. perform oscillations with a frequency which is approximately twice the natural frequency of the swing. Despite popularity of the swing, in the literature on oscillations and stability there are not many analytical and numerical results on behavior of the pendulum with periodically varying length dependent on parameters. In this paper the stability of the lower vertical position of the pendulum with damping and arbitrary periodic excitation function is investigated.

Equation for motion of the swing can be derived with the use of angular momentum alteration theorem. Taking into account also linear damping forces we obtain

$$\frac{d}{dt}\left(ml^2\frac{d\theta}{dt}\right)+\gamma l^2\frac{d\theta}{dt}+mgl\sin\theta=0, \qquad (20)$$

where m is the mass, l is the length, θ is the angle of the pendulum deviation from the vertical position, g is the acceleration due to gravity, and t is the time, Fig. 3.

It is assumed that the length of the pendulum changes according to the periodic law

$$l=l_0+a\varphi(\Omega t), \qquad \int_0^{2\pi}\varphi(\tau)d\tau=0, \qquad (21)$$

where l_0 is the mean pendulum length, a and Ω are the amplitude and frequency of the excitation, $\varphi(\tau)$ is the smooth 2π-periodic function with zero mean value.

We introduce the following dimensionless parameters and variables

$$\tau=\Omega t,\ \epsilon=\frac{a}{l_0},\ \Omega_0=\sqrt{\frac{g}{l_0}},\ \omega=\frac{\Omega_0}{\Omega},\ \beta=\frac{\gamma}{m\Omega_0}. \qquad (22)$$

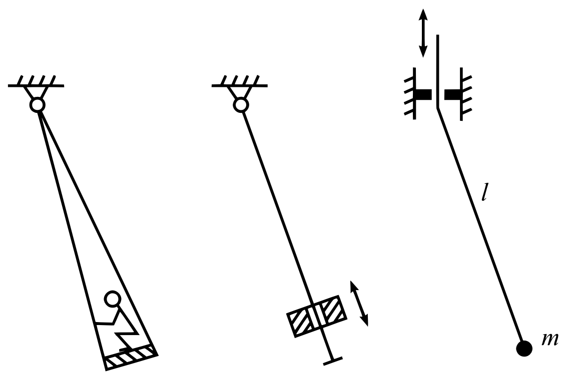

Figure 3. Schemes of the pendulum with periodically varying length.

Then, equation (20) can be written in the following form

$$\ddot{\theta}+\left(\frac{2\epsilon\dot{\varphi}(\tau)}{1+\epsilon\varphi(\tau)}+\beta\omega\right)\dot{\theta}+\frac{\omega^2\sin\theta}{1+\epsilon\varphi(\tau)}=0 \qquad (23)$$

Here the dot denotes differentiation with respect to new time τ. Behavior of the system governed by equation (23) will be studied in the following sections via analytical and numerical techniques depending on three dimensionless problem parameters: the excitation amplitude ϵ, the damping coefficient β, and the frequency ω under the assumption $\epsilon \ll 1$, $\beta \ll 1$. It is convenient to change the variable by the substitution $q=\theta(1+\epsilon\varphi(\tau))$. With this substitution we obtain a nonlinear equation for q which is useful for stability study of the vertical position of the pendulum as well as analysis of small oscillations. Equation (23) is used for stability and dynamics study of the pendulum with variable length. It is shown that the instability (parametric resonance) regions are semi-cones in three-dimensional parameter space with singularities at the DP, see also [Seyranian, 2004; Belyakov, Seyranian and Luongo, 2009].

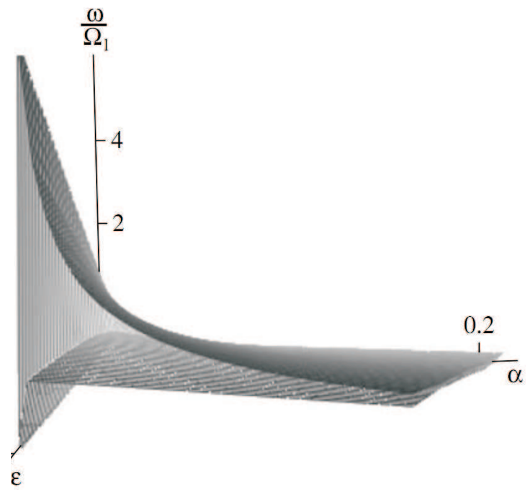

Figure 4. Stabilization region in Chelomei's problem

4.2 Chelomei's problem

The possibility to increase the stability of elastic systems by means of vibrations was originally pointed out in [Chelomei, 1956]. In particular, he arrived at the conclusion that an elastic rod compressed by a longitudinal force greater than the critical Euler's value can be stabilized by high-frequency longitudinal vibrations applied to the rod end. In this study, formulas for the upper and lower critical frequencies of rod stabilization are derived and analyzed. It is shown that, in contrast to the case of high-frequency stabilization of an inverted pendulum with a vibrating suspension point, the rod is stabilized at excitation frequencies of the order of the natural frequency of transverse oscillations belonging to a certain region.

We consider a straight elastic rod of constant cross section, loaded by a periodic longitudinal force $P(t) = P_0 + P_t\phi(\omega t)$ applied to its end. The equation of transverse oscillations of the rod can be written as

$$m\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} + P(t)\frac{\partial^2 u}{\partial x^2} + EJ\frac{\partial^4 u}{\partial x^4} = 0, \quad (24)$$

where x is the coordinate along the rod axis; t is the time; $u(x,t)$ is the rod deflection function; m is the mass per unit length; EJ is the flexural rigidity; γ is the damping coefficient; P_t and ω are the excitation amplitude and frequency of the longitudinal vibration, respectively. It is convenient to introduce nondimensional parameters $\varepsilon = P_t/P_1$ and $\alpha = P_0/P_1 - 1$ where P_1 is the first Euler buckling load. The simply supported ends of the rod are considered. Fig. 4 presents the stabilization region in Chelomei's problem. The singularities related to the points DP and EP are discussed, see also [Seyranian and Seyranian, 2008; Seyranian and Seyranian, 2006].

5 Conclusion

We have discussed coupling of eigenvalues of systems smoothly depending on multiple real parameters. Diabolic and exceptional points have been mathematically described and general formulae for coupling of eigenvalues at these points have been derived. This theory has a very broad field of applications since any physical or mechanical system contains parameters.

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