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STATIONARY AND NON-STATIONARY RESONANCE DYNAMICS OF THE FINITE CHAIN OF WEAKLY COUPLED PENDULA

Leonid I. Manevitch
Polymer and Composite Materials Department
N. N. Semenov Institute of Chemical Physics RAS
Russia
leonidmanevitch3@gmail.com

Valeri V. Smirnov
Polymer and Composite Materials Department
N. N. Semenov Institute of Chemical Physics RAS
Russia
vvs@polymer.chph.ras.ru

Francesco Romeo
Department of Structural and Geotechnical Engineering
Sapienza University of Rome
Italy
francesco.romeo@uniroma1.it

Abstract
We discuss new phenomena of energy localization and transition to chaos in the finite system of coupled pendula (which is a particular case of the Frenkel-Kontorova model), without any restrictions on the amplitudes of oscillations. The direct significant applications of this fundamental model comprise numerous physical systems. In the infinite and continuum limit the considered model is reduced to integrable sine-Gordon equation or certain non-integrable generalizations of it. In this limit, the chaotization is absent, and the energy localization is indicated by the existence of soliton-like solutions (kinks and breathers). As for more realistic finite models, analytical approaches are lacking, with the exception of cases limited to two and three pendula. We propose a new approach to the problem based on the recently developed Limiting Phase Trajectory (LPT) concept in combination with a semi-inverse method. The analytical predictions of the conditions providing transition to energy localization are confirmed by numerical simulation. It is shown that strongly nonlinear effects in finite chains tend to disappear in the infinite limit.

Key words
Nonlinear dynamics, Frenkel-Kontorova model, nonlinear normal mode, energy localization.

1 Introduction

The applications of finite Frenkel-Kontorova model [Frenkel and Kontorova, 1938; Frenkel and Kontorova, 1939] include dislocation motion, ferromagnetic chains, Josephson junction, paraffin crystals, DNA macromolecule etc. [Atkinson and Cabrera, 1965; Caudrey et al, 1975; Cataliotti et al, 2001; Likharev, 1986; Yakushevich, Savin, and Manevitch, 2002; Braun and Kivshar, 2004]. Because of its complexity, the absolute majority of studies are based on the infinite and continuum limit leading to sine-Gordon equation, and many physical effects were explained in terms of soliton-like solutions of this equation [Scott, 2003; Braun and Kivshar, 2004]. As for more realistic discrete finite model, mostly numerical studies were so far proposed, with the exception of systems with few degrees of freedoms [Braun and Kivshar, 2004] in which new significant nonlinear effects are cumbersome and difficult to predict, especially for strongly non-stationary dynamics. In previous studies, devoted to different fields of nonlinear dynamics, ranging from forced oscillator to wave propagation in carbon nanotubes [Manevitch, 2007; Manevitch and Smirnov, 2010; Smirnov and Manevitch, 2011; Manevitch and Romeo, 2015; Smirnov, Shepelev, and Manevitch, 2014], it was shown that adequate description of strongly non-stationary processes can be achieved by resorting to a new framework, which is based on the notion of Limiting Phase Trajectory (LPT). Similarly to Nonlinear Normal Modes (NNMs), LPTs turn out to be a fundamental notion, as they describe maximum possible energy exchange between oscillators, clusters of oscillators or different parts of the system. Their role in understanding and describing non-stationary processes is similar to the role played by NNMs in stationary dynamics. By relying on LPT-concept new significant nonlinear effects were predicted, such as self-sustained vibrations of a bi-harmonically excited Duffing oscillator [Starosvetsky and Manevitch, 2011], non-conventional synchronization in weakly coupled autogenerators [Manevitch, Kovaleva, and Pilipchuk, 2013], energy exchange and clusters mobility in finite oscillatory chains, energy localization in carbon nanotubes [Manevitch and Smirnov, 2010; Smirnov and Manevitch, 2011; Smirnov, Shepelev, and Manevitch, 2014]. All these effects were revealed in quasi-linear approximation, i.e. in the small-amplitude assumption, in which the effect of the oscillations amplitude on the oscillation frequency is negligible. However, in many important applications of nonlinear dynamics, such as target energy transfer [Aubry et al, 2001; Memboeuf and Aubry, 2005], energy harvesting, etc., strongly nonlinear behaviour, which cannot be linearized even in the main asymptotic approximation, has to be taken into account; thus, such problems are usually merely numerically tackled.

The main goal of this paper is to study the non-stationary dynamics of an essentially nonlinear system, the Frenkel-Kontorova lattice, undergoing arbitrary oscillation amplitudes. At first, we consider the spectrum of the NNMs in the framework of an asymptotic semi-inverse method. Then, by analysing the resonant interaction of NNMs, we show the instability of the modes with lowest wave number (uniform modes) as well as the transition to energy localization in some domain of the lattice. The corresponding threshold values of the system parameters are eventually identified.

2 The Model

Let us consider the periodic system of weakly coupled particles in the field of the local (on site) potential. As it was mentioned in the introductory section, the best known examples of such systems are the Frenkel-Kontorova and Klein-Gordon models, in particular, the model φ^4. In the linear approximation these models are described by the same equation, the linear discrete Klein-Gordon equation, the properties of which are well studied in the continuum limit. However, for relatively small systems, discreteness plays an important role and the traditional approach relies on the application of the linear normal modes technique. In the quasi-linear case, at first glance, the nonlinear normal modes [Pilipchuk, 2011; Pilipchuk, 1996] could also be used with their stability being usually studied in the linear approximation. However, for finite systems, this analysis turns out to be quite cumbersome and does not give information on the dynamic regimes resulting from loss of stability. The only aspect captured by the linear analysis is the existence of the maximum growth mode.

Let us write the energy of the chain consisting of N coupled pendula in the form:

$$H = \sum_{j=1}^{N} [\frac{1}{2}(\frac{dq_j}{dt})^2 + \frac{\beta}{2}(q_{j+1}-q_j)^2 + (1-\cos q_j)] \quad (1)$$

under periodic boundary conditions: $q_{N+1} = q_1$.

The corresponding equation of motion

$$\begin{aligned} &\frac{d^2 q_j}{dt^2} - \beta\Delta_2 q_j + \sin q_j = 0 \\ &\Delta_2 q_j = q_{j+1} - 2q_j + q_{j-1}, \end{aligned} \quad (2)$$

may be represented in the terms of complex variables

$$\Psi_j = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{\omega}}\frac{dq_j}{dt} + i\sqrt{\omega}q_j), \quad (3)$$

where ω is a frequency of the oscillations, which will be later defined.

Taking into account definition (3) of the functions Ψ_j one can rewrite equation (2) as

$$\begin{aligned} &i\frac{d\Psi_j}{dt} + \frac{\omega}{2}\left(\Psi_j + \Psi_j^*\right) - \frac{\beta}{2\omega}\Delta_2\left(\Psi_j - \Psi_j^*\right) \\ &+ \frac{1}{2\omega}\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{1}{2\omega}\right)^k\left(\Psi_j - \Psi_j^*\right)^{2k+1} = 0 \end{aligned} \quad (4)$$

Let us represent the solution of equation (4) as follows:

$$\Psi_j(t) = \psi_j e^{i\omega t}, \quad (5)$$

where ψ_j is a slow-changing function of time t. An alternative interpretation is that the function ψ_j is a function of "slow time", the scale of which is defined by a small parameter entering the problem, such as a weak enough coupling between pendula. In such a case, the derivation with respect to time can be written as

$$\frac{d}{dt} = \frac{\partial}{\partial\tau_0} + \frac{\partial}{\partial\tau_1} + \ldots, \quad (6)$$

where τ_1 is a "slow" time.

Substituting expression (5) into equation (4) and taking into account expansion (6), the resonant (secular) equation for the functions ψ_j is obtained:

$$\begin{aligned} &i\frac{\partial\psi_j}{\partial\tau_1} - \frac{\beta}{2\omega}\Delta_2\psi_j + \\ &\qquad \frac{1}{\sqrt{2\omega}}\frac{\psi_j}{|\psi_j|}J_1\left(\sqrt{\frac{2}{\omega}}|\psi_j|\right) - \frac{\omega}{2}\psi_j = 0, \end{aligned} \quad (7)$$

where J_1 is the Bessel function of the first kind.

The origin of the frequency ω can now be clarified by noticing that any function

$$\psi_j(\tau_1) = \sqrt{X} e^{i\left(\omega' \tau_1 - \kappa j\right)} \tag{8}$$

is the solution of equation (7) with dispersion ratio

$$\omega' = \frac{2\beta \sin^2 \kappa/2}{\omega} + \frac{1}{\sqrt{2\omega X}} J_1(\sqrt{\frac{2X}{\omega}}) - \frac{\omega}{2} \tag{9}$$

If the wave number κ is equal to zero, the frequency ω' is defined by the expression

$$\omega' = \frac{1}{\sqrt{2\omega X}} J_1(\sqrt{\frac{2X}{\omega}}) - \frac{\omega}{2}. \tag{10}$$

If the right hand side of equation (10) is equal to zero, the function $\psi_j = const$, i.e. the solution (3), describes uniform oscillations, with frequency ω, of the chain of pendula. According the definition of the functions Ψ_j (see eq. (3)) their modules X can be expressed in the term of the oscillation amplitude Q:

$$X = \frac{\omega Q^2}{2}$$

and the frequency may be written as follows:

$$\omega = \sqrt{\frac{2}{Q} J_1(Q)}. \tag{11}$$

The comparison between expression (11) and the one providing with the exact value of the oscillation frequency of single pendulum, i.e.:

$$\omega = \frac{\pi}{2K(\sin{(Q/2)})}, \tag{12}$$

where K is the complete elliptic integral of the first kind, is shown in Fig. 1. One can see, that their values are in good agreement in a wide interval of oscillation amplitudes.

So, taking into account relation (11), the dispersion ratio for equation (7) may be written as follows

$$\omega' = \frac{\beta \sin^2(\kappa/2)}{2\omega} \tag{13}$$

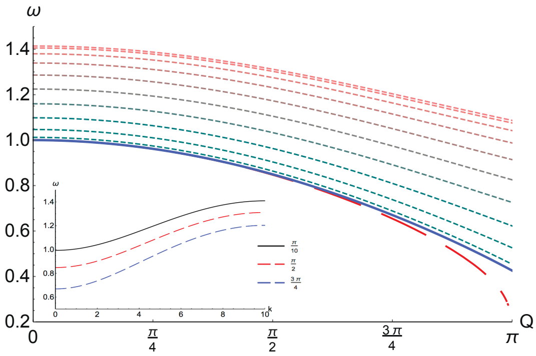

Figure 1. (Color online) Comparison of resonant frequencies that are computed according to equations (12) and (11) (black and blue curve, respectively). Dashed curves show the eigenfrequencies obtained from the dispersion ratio (13) for the chain with 20 pendula and wave numbers $k = 1, \ldots 10$. The inset shows the dispersion relations (the frequency vs mode's number) for some values of the oscillation amplitude. The coupling parameter is $\beta = 0.1$.

The preceding results allow to conclude that the mono-frequency vibrations (normal modes — NMs) of the chain of pendula obey the linearized version of equation (7) and they represent quasilinear waves with dispersion law (13). The feature of this dispersion ratio is the spectrum crowding near the left edge ($\omega\prime \sim \kappa^2$). The nonlinearity of the system appears when the chain motion corresponds to a combination of NMs. In such a case the NMs interaction leads to the non-zero contribution of the last terms in eq. (7).

Equation (7) corresponds to the Hamilton function in the form:

$$\begin{aligned} H_a = & \sum_{j=1}^{N} \frac{\beta}{2\omega} |\psi_{j+1} - \psi_j|^2 - \\ & \sum_{j=1}^{N} \left(J_0(\sqrt{\frac{2}{\omega}} |\psi_j|) + \frac{1}{2} \omega |\psi_j|^2 \right) \end{aligned} \tag{14}$$

The additional integral of motion is the "excitation" number:

$$X = \frac{1}{N} \sum_{j=1}^{N} |\psi_j|^2. \tag{15}$$

Let us consider the nonlinear NMs interaction. It is known that if the chain's length N is large enough, two low-frequency NMs interaction leads to the "coherent" motion of the pendula corresponding to some parts of the chain [Manevitch and Smirnov, 2010; Smirnov and Manevitch, 2011]. These parts of the chain, previously named as "effective particles", are here more suitably defined as "coherent domains". The proper "coordinates" of the domains can also be introduced to describe the specific dynamics of the chain. Let us consider the combination of the NMs near the left edge of the spectrum. Under periodic boundary conditions the first NMs correspond to the wave numbers $\kappa_0 = 0$ (uniform motion) and $\kappa_1 = 2\pi/N$ (one node mode), the latter being twice degenerated. It is obvious that any linear combination of the NMs leads to a non-uniform distribution of the pendulum displacements along the chain. The group of pendula with similar displacements represent the above mentioned "coherent domains" . The periodic change of the predominant displacement of the pendula from one coherent domain to another is the consequence of the eigenfrequency difference in the exactly linear chain, in analogy with the classic beating process in a system of two weakly coupled oscillators. In the nonlinear system this process is not ordinary and may depend on the system parameters and vibration amplitudes.

Let us introduce the "domain coordinates" as follows:

$$\begin{aligned}\chi_1 &= \frac{1}{\sqrt{2N}}\sum_{j=1}^{N}\psi_j\left[1+(\cos\kappa_1 j+\sin\kappa_1 j)\right]\\ \chi_2 &= \frac{1}{\sqrt{2N}}\sum_{j=1}^{N}\psi_j\left[1-(\cos\kappa_1 j+\sin\kappa_1 j)\right].\end{aligned} \quad (16)$$

The inverse transformation leading to the complex amplitude of the particles is written as follows:

$$\begin{aligned}\psi_j =& \frac{1}{\sqrt{2N}}\left[(\chi_1+\chi_2)+\right.\\ &\left.(\chi_1-\chi_2)(\cos\kappa_1 j+\sin\kappa_1 j)\right].\end{aligned} \quad (17)$$

One can see that the amplitude vectors $(\chi_1, \chi_2) = (1, 0)$ and $(\chi_1, \chi_2) = (0, 1)$ correspond to the maximum pendulum displacements in the first and second domains, respectively. (It is easy to check out that the domain coordinates χ_j coincide with the coordinates of the pendula in the case $N = 2$.)

Substituting the relation (17) into equation (14), the equations of motion for the domain coordinates in terms of complex amplitudes may be obtained immediately by the Hamiltonian variation with respect to the domain coordinates χ_1, χ_2. However, the resulting equations are quite lengthy and they do not allow to analyse clearly the chain dynamics. Therefore, a simplified formulation is sought. First of all, one can see that transformation (17) preserves the integral of excitation number (15):

$$X = |\chi_1|^2 + |\chi_2|^2. \quad (18)$$

Using the integral (18), the dimension of the system's phase space may be reduced by introducing the relative amplitudes of the domain coordinates χ_j [Manevitch and Smirnov, 2010]:

$$\chi_1 = \sqrt{X}\cos\theta e^{i\delta_1}, \quad \chi_2 = \sqrt{X}\sin\theta e^{i\delta_2}. \quad (19)$$

In such a case the parameter X specifies the total excitation of the system, while the "angle" θ shows the relative excitation of the coherent domains. In fact, the energy of the system does not depend on the absolute values of the phases δ_1 and δ_2, but it depends on their difference $\Delta = \delta_1 - \delta_2$ only:

$$\begin{aligned}H(\theta,\Delta) =& \frac{\beta X\sin^2(\kappa_1/2)}{\omega}(1-\cos\Delta\sin 2\theta) -\\ & X\frac{\omega}{2} - \sum_{j=1}^{N} J_0(\xi_j),\\ \xi_j =& [\frac{2X}{\omega N}(1+\cos 2\theta\,(\cos(\kappa_1 j)+\sin(\kappa_1 j)) +\\ & \sin(\kappa_1 j)\cos(\kappa_1 j)\,(1-\cos\Delta\sin 2\theta)]^{1/2}\end{aligned} \quad (20)$$

The Hamiltonian (20) allows to analyse the phase portrait of the system and to define the bifurcations of the phase trajectories under different excitation levels, as specified by the excitation number X.

This procedure was well discussed for the coupled nonlinear oscillators [Manevitch, 2007] as well as for the nonlinear chains [Manevitch and Smirnov, 2010; Smirnov and Manevitch, 2011]. These studies concerned with small-amplitude oscillations in long enough nonlinear chains. The typical phase portraits for the different dynamical regimes in the nonlinear chain are shown in Fig. 2. The dispersion ratio (11) and the asymptotic equation (5) are not restricted by any assumptions about the smallness of the oscillation amplitudes. Therefore the bifurcations at any given amplitude as a function of the chain parameters can be found; this result is obtained by considering the coupling of the pendula as the crucial parameter of the asymptotic procedure. Using the dispersion ratio (11) in addition to the Hamiltonian (20), one can estimate the threshold values of coupling parameter ε.

As it was mentioned above, a linear combination of the NMs leads to the slow redistribution of the energy along the chain if the dynamical regime is far from the instability bifurcation (see Fig. 2(a)). The stationary points ($\theta = \pi/4, \Delta = 0$) and ($\theta = \pi/4, \Delta = \pi$) correspond to the stable NMs, while the trajectories surrounding these points are associated with combinations of NMs. The Limiting Phase Trajectory (LPT) separates the attractive area of NMs and it is closed at the values $\theta = 0$ and $\theta = \pi/2$. These states correspond to the domain "vectors" ($\chi_1 = \sqrt{X}, \chi_2 = 0$) and ($\chi_1 = 0, \chi_2 = \sqrt{X}$), respectively. One should

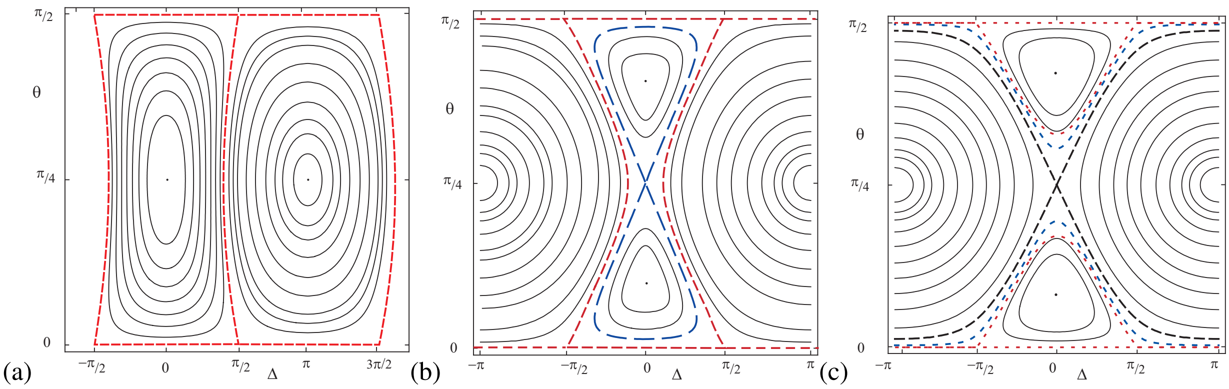

Figure 2. (Color online). Typical phase portraits of the system (20). The states corresponding to domains χ_1 and χ_2 link with $\theta = 0$ and $\theta = \pi/2$, respectively. (a) Before the instability bifurcation. Intensive energy exchange associates with the phase trajectory (LPT, red dashed curve) that separates the stationary points $(\theta = \pi/4, \Delta = 0)$ and $(\theta = \pi/4, \Delta = \pi)$. The latter correspond to the NMs. (b) Before the localization bifurcation. Two new nonlinear normal modes at $\Delta = 0$ are surrounded by the separatrix (blue long dashed curve) that crosses the unstable NM. (c) After the localization bifurcation. The separatrixes (black dashed curves) surround the stable NMs at $\Delta = \pm\pi$. No trajectory starting outside the separatrix can cross the line $\theta = \pi/4$.

note that the NMs themselves correspond to the domain "vectors" with $(\chi_1 = \chi_2)$ and $(\chi_1 = -\chi_2)$.

The first bifurcation of the phase portrait is associated with the instability of uniform mode (stationary point at $\theta = \pi/4, \Delta = 0$). This bifurcation leads to the creation of two new stationary states that correspond to a stationary weakly non-uniform distribution of the oscillation energy (Fig. 2(b)); any trajectory lying inside the separatrix cannot cross the ideal line $\theta = \pi/4$. Therefore, any energy excess contained in one of the domains will remain in it. Nevertheless, the slow energy exchange between two domain is preserved for the motion occurring along the LPT.

The instability threshold is obtained from the condition

$$\frac{\partial^2 H(\theta, \Delta)}{\partial \theta^2} \Big|_{(\theta=\pi/4, \Delta=0)} = 0 \qquad (21)$$

Solving equation (21) with respect to coupling parameter β, one can obtain the following threshold

$$\beta_{ins} = \frac{J_2(Q)}{2\sin^2(\kappa_1/2)}. \qquad (22)$$

The localization bifurcation occurs when the energy of unstable stationary point $(\theta = \pi/4, \Delta = 0)$ becomes equal to the energy of "domain state" $(\theta = \pi/2, \Delta = \pm\pi/2)$ or $(\theta = 0, \Delta = \pm\pi/2)$. Under this condition the solution of the corrisponding equation leads to the localization threshold in the form:

$$\beta_{loc} = 2\frac{\frac{1}{N}\sum_{j=1}^{N} J_0\left(\frac{Q}{\sqrt{2}} f_j\right) - J_0(Q)}{Q^2 \sin^2(\kappa_1/2)} \qquad (23)$$
$$f_j = 1 + \cos(\kappa_1 j) + \sin(\kappa_1 j)$$

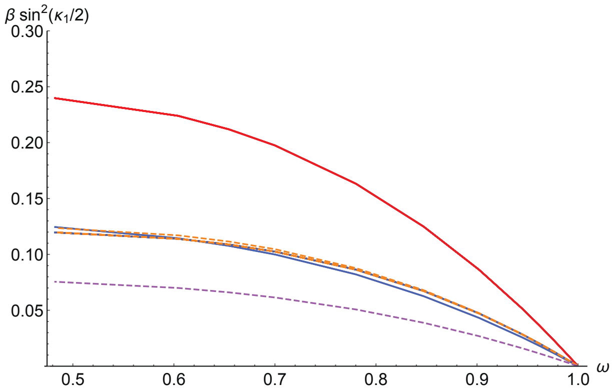

Figure 3. (Color online) The instability (red) and localization (blue, pink and orange dashed curves) "effective" thresholds for the chains with various lengths (see text). The dashed pink curve shows the "effective" localization threshold for the chain with 3 pendula.

3 Discussion

The instability threshold (22) as well as the localization one (23) are both inversely proportional to the square of $\sin(\kappa_1/2)$. Taking into account that $\kappa_1 \sim 1/N$, it can be inferred that the critical coupling values grow as the chain length increases. However, it is clear that the real parameter, which determines the resonant conditions, is the value of the gap between uniform and first non-uniform modes. This value is defined by the "effective coupling constant" given by $\beta \sin^2(\pi/N)$ [Smirnov and Manevitch, 2011]. For a pair of coupled pendula, the gap becomes $\beta_{ins} \sin^2 \kappa_1/2$ and it does not depend on the length of the chain, in agreement with [Manevitch and Romeo, 2015]. Moreover, again for $N = 2$, it should be noticed that also the localization threshold, given by (23), exactly coincide with the threshold obtained in [Manevitch and Romeo, 2015]. Fig. 3 shows the "effective" threshold values for the instability and localization bifurcations.

Apart from the case of a chain with three pendula, the localization thresholds for other chains with different length are all extremely close. We suppose that such an exception is associated with the ambiguity of the division of three pendula into two clusters.

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