# ON DYNAMICAL RECONSTRUCTION OF AN UNKNOWN INPUT IN THE PARABOLIC OBSTACLE PROBLEM

**A. M. Kadiyev V. I. Maksimov**

*Institute of Mathematics and Mechanics,*
*Ural Branch, Russian Academy of Sciences,*
*S. Kovalevskoi str., 16, Yekaterinburg, 6200219, Russia*
*e-mail: kadiev@imm.uran.ru, maksimov@imm.uran.ru*

Abstract: A problem of dynamical reconstruction of a control for a parabolic obstacle problem is considered. A solving algorithm for this problem is presented. This algorithm is stable with respect to informational noise and computational errors. It adaptively takes into account inaccurate measurements of phase trajectories and is regularizing in the sense that the final result becomes better as the input information becomes more accurate. The algorithm suggested in the paper is based on the theory of positional control. The main element of this algorithm is a procedure of stabilizing some auxiliary functionals of the Lyapunov type.

Keywords: Control (closed-loop), Model based control,

## 1. PROBLEM STATEMENT

A linear distributed system

$$x_t(t,\eta) - \triangle_L x(t,\eta) = u(t,\eta) + F(t,\eta)$$
$$\text{a.e. on } \{(t,\eta) \in T \times \Omega : x(t,\eta) > \mu(\eta)\},$$

$$x_t(t,\eta) = \max\{u(t,\eta) + F(t,\eta) + \triangle_L \mu(\eta), 0\} \quad (1)$$
$$\text{a. e. on } \{(t,\eta) \in T \times \Omega : x(t,\eta) = \mu(\eta)\},$$

$$x(t,\eta) \geq \mu(\eta) \ \ \forall\, t \in T, \quad \text{for a. a. } \eta \in \Omega;$$
$$x = 0 \ \text{ a. e. on } T \times \Gamma,$$

$$\mu \in H_2(\Omega), \quad \mu(\sigma) \leq 0 \ \text{ a. e. on } \Gamma.$$

is considered in this paper. Here $T = [0, \vartheta]$, $\vartheta < +\infty$; $\Omega \subset \mathbb{R}^n$ is a simply connected open and bounded domain with a sufficiently smooth boundary $\Gamma$; $\triangle_L$ is the Laplace operator, i.e., $\triangle_L x(\eta) = \sum_{j=1}^n \partial^2 x(\eta)/\partial \eta_j^2$; $F(t,\eta) \in L_2(T; L_2(\Omega))$ is a given function; $u(t,\eta)$ is an unknown disturbance. System (1) describes, for example, a process of oxygen diffusion in an absorbing tissue. A similar system, introduced and investigated in (Magenes, 1977), was called "the parabolic obstacle problem" (Barbu, 1984). From the feedback control theory's standpoint, the obstacle problem was investigated, for example, in the works (Blizorukova and Maksimov, 2003; Maksimov, 2000, § 3, ch. 3).

The problem can be formulated in the following way. At discrete and sufficiently frequent instants

$$\tau_i \in T, \quad \tau_i = \tau_{i+1} + \delta, \quad i \in [1 : m-1],$$
$$\tau_0 = 0, \quad \tau_m = \vartheta,$$

the phase state $x(\tau_i, \eta) = x(\tau_i; x_0, u(\cdot)) \in H = L_2(\Omega)$ of system (1) is inaccurately measured. Hereinafter the symbol $x(\cdot; x_0, u^*(\cdot)) \in C(T; H)$ denotes a solution of (1). This unknown solution depends on a time-varying unknown control $u(\cdot) = u^*(\cdot)$. As is well known, under the assumption

$$x_0(\eta) \in H_1^0(\Omega), \quad x_0(\eta) \geq \mu(\eta) \text{ a. e. on } \Omega$$

(throughout the following this assumption is considered to be fulfilled), there exists a unique solution of problem (1) with the properties (see (Barbu, 1984, corollary 4.4)):

$$x(\cdot, x_0, u^*(\cdot)) \in L_2(T; H^2(\Omega)) \cap C(T; H_0^1(\Omega)),$$
$$x_t^0(\cdot, x_0, u^*(\cdot)) \in L_2(T; H).$$

Results of measurements $\xi_i^h = \xi^h(\tau_i) \in H$ satisfy the inequality

$$|\xi_i^h - x(\tau_i)|_H \leq h, \quad i \in [0 : m-1], \qquad (2)$$

where $h \in (0,1)$ is the level of informational noise. It is required to design an algorithm that allows us to reconstruct (synchronously with the process) some unknown input $u^*(\cdot) \in L_2(T; U)$ generating the unknown output $x(\cdot)$, i.e., it is required to find an input $u^*(\cdot)$ such that the solution $x(\cdot; x_0, u^*(\cdot))$ corresponding to this disturbance coincides with $x(\cdot)$. This is the meaningful statement of the problem.

The problem described above is embedded into the class of inverse problems and, in more general context, into the class of ill-posed problems. Such problems in a posteriori statement were studied by many authors. In (Kryazhimskii and Osipov, 1983) a method of dynamical reconstruction of an unknown input was suggested for a finite dimensional dynamical system affine in control for the case when a control under reconstruction at every time instant is an element of a priori given convex bounded and closed set $P$, i.e., $u(t) \in P$ for $t \in T$. Later the method was extended to systems described by differential equations of various types (see (Maksimov, 2000; Osipov and Kryazhimskii, 1995; Osipov *et al.*, 2003)). The method is based on the theory of positional control (see (Krasovskii and Subbotin, 1974)) and on the smoothing functional and discrepancy methods (see (Tikhonov and Arsenin, 1979)), which are well known in the theory of ill-posed problems.

In the present work, the method of dynamical regularization is applied in order to reconstruct a right-hand side in the parabolic obstacle problem.

## 2. THE APPROACH TO SOLVING THE PROBLEM

In this section the method used for solving the problem considered in the paper is described. This method follows (Maksimov, 2000; Kryazhimskii and Osipov, 1983; Osipov and Kryazhimskii, 1995; Osipov *et al.*, 2003).

Let $U(x(\cdot))$ be the set of all inputs $u(\cdot) \in L_2(T; U)$ that are compatible with $x(\cdot)$, i.e.,

$$U(x(\cdot)) = \{u(\cdot) \in L_2(T; U) : x(\cdot; x_0, u(\cdot)) = x(\cdot)\},$$

$\Xi_T$ be the set of measurements, i.e., the set of all piecewise constant functions $\xi(\cdot) : T \to Z$, $\Xi(x(\cdot), h)$ be the set of all $h$-accurate measurements, i.e., the set of all functions $\xi^h(\cdot) \in \Xi_T$ satisfying (2).

Let us introduce an auxiliary system $M$ (a model). A trajectory of this model denoted by

$$w^h(\cdot) = w^h(\cdot; w_0^h, v^h(\cdot)) \in C(T; H).$$

depends on a control. An initial state $w_0^h$ of the model is chosen by using the value $\xi_0^h$ of measurement at the initial time moment in accordance with some rule $\mathcal{W}_h$ fixed in advance:

$$w_0^h = \mathcal{W}_h(\xi_0^h) \in X_0 \subset H. \qquad (3)$$

Here $X_0$ is the given set of all initial states of the model. In particular, if the initial state $x_0$ is known then it is natural to suppose $X_0 = \{x_0\}$.

Model control rules are identified with pairs $S_h = (\Delta_h, \mathcal{U}_h)$, where

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h} \qquad (4)$$

is a partition of the interval $T$ into half-intervals $[\tau_{h,i}, \tau_{h,i+1})$, $\tau_{h,i+1} = \tau_{h,i} + \delta$, $\delta = \delta(h)$, $\tau_{h,0} = 0$, $\tau_{h,m_h} = \vartheta$; $\mathcal{U}_h$ is a function taking every triple $(\tau_i, \xi_i^h, w^h(\tau_i))$ to an element

$$v^h_{\tau_i,\tau_{i+1}}(\cdot) = \mathcal{U}_h(\tau_i, \xi_i^h, w^h(\tau_i)) \in L_2([\tau_i, \tau_{i+1}]; U), \qquad (5)$$

where $\tau_i = \tau_{h,i}$, $w^h(\tau_i) = w^h(\tau_i; w_0^h, v^h(\cdot))$, $\xi_i^h = \xi^h(\tau_i)$, $\xi^h(\cdot) \in \Xi(x(\cdot), h)$, the symbol $v_{a,b}(\cdot)$ denotes a contracted measure of the function $v(\cdot)$ into the half-interval $[a, b)$. Thus, the quadruple $(M, \mathcal{W}_h, \Delta_h, \mathcal{U}_h)$ for every $h \in (0,1)$ determines some algorithm $D_h$ on the space of measurements $\xi(\cdot) \in \Xi(x_r(\cdot), h)$ ($D_h : \Xi_T \mapsto U_T$) forming an output $v^h(\cdot) = D_h\xi(\cdot)$ according to feedback principle (3)–(5). The algorithm $D_h$ is identified with the quadruple $(M, \mathcal{W}_h, \Delta_h, \mathcal{U}_h)$. The result of work of the algorithm on the interval $T$ is a piecewise constant control $v^h(\cdot)$ of the form

$$v^h(t) = v_i^h, \quad t \in [\tau_i, \tau_{i+1}).$$

Let the following condition be fulfilled.

*Condition 1.* The set $U_*(x(\cdot))$ of inputs from $U(x_r(\cdot))$ with minimal $L_2(T; U)$-norm is one-element, i.e., $U_*(x(\cdot)) = \{u_*(\cdot; x(\cdot))\}$.

Thus,

$$u_*(\cdot; x(\cdot)) = \arg\min\{|u(\cdot)|_{L_2(T;U)} : u(\cdot) \in U(x(\cdot))\}.$$

A family $D_h$, $h \in (0,1)$ of operators from $\Xi_T$ to $U_T$ is called *regularizing* if

$$\lim_{h \to 0} \sup\{|D_h\xi^h(\cdot) - u_*(\cdot; x(\cdot))|_{L_2(T;U)} :$$
$$\xi^h(\cdot) \in \Xi(x(\cdot), h)\} = 0.$$

The goal of the present work is to construct a regularizing family

$$D_h = (M, \mathcal{W}_h, \Delta_h, \mathcal{U}_h), \qquad h \in (0,1) \qquad (6)$$

of modeling algorithms of the form (2)–(5).

After a model and its initial state (3) are chosen, the work of the algorithm $D_h$ corresponds to the following outline. First, before the start time $t_0 = 0$, a disturbance $h$ and a partition $\Delta = \Delta_h = \{\tau_i\}_{i=0}^m$, $(\tau_i = \tau_{h,i})$ of the interval $T$ are fixed. At the $i$-th step carried out during the time interval $[\tau_i, \tau_{i+1})$, the following sequence of actions takes place. An output $x(\tau_i)$ is inaccurately measured, i.e., a value $\xi_i^h \in H$ with properties (2) is obtained. Then the model control is determined by (5). After that the next part of the model trajectory $w^h(t)$, $t \in (\tau_i, \tau_{i+1}]$ is formed in addition to $w^h(t)$, $t \in [t_0, \tau_i]$ (memory correction). The procedure stops at the moment $\vartheta$.

Construction of the family of algorithms $D_h$ is based on Theorem 1 formulated below. Let us fix a functional $\Lambda^0(\cdot,\cdot)$ on the Cartesian product $C(T;H) \times C(T;H)$.

*Definition 1.* (Maksimov, 2000) A family $D_h$, $h \in (0,1)$ of positional modeling algorithms is said to be $\Lambda^0$-stable if there exist functions $k_1(\cdot)$, $k_2(\cdot)$, $k_3(\cdot)$: $[0,+\infty) \to [0,+\infty)$ such that $k_1(h) \to 1$, $k_2(h) \to 0$, $k_3(h) \to 0$ as $h \to 0$, and, for every measurement result $\xi^h(\cdot) \in \Xi(x(\cdot), h)$, the inequalities

$$|v^h(\cdot)|_{L_2(T;U)} \le k_1(h)|u_*(\cdot; x(\cdot))|_{L_2(T;U)} + k_2(h), \qquad (7)$$

$$\Lambda^0(x(\cdot), w^h(\cdot)) \le k_3(h) \qquad (8)$$

hold.

Here $v^h(\cdot) = D_h\xi^h(\cdot)$ and $w^h(\cdot)$ is the model motion generated by the algorithm $D_h$ for the measurement result $\xi^h(\cdot)$.

The following theorem is true.

*Theorem 1.* (Maksimov, 2000) Let a) a family $D_h$ of positional modeling algorithms be $\Lambda^0$-stable, b) for every $h_k > 0$ ($h_k \to 0+$ as $k \to \infty$), $\xi^{h_k}(\cdot) \in \Xi(x(\cdot), h_k)$, $w^{h_k}(\cdot) = w^{h_k}(\cdot; w_0^{h_k}, v^{h_k}(\cdot))$, $v^{h_k}(\cdot) = D_{h_k}\xi^{h_k}(\cdot)$, the convergences

$$v^{h_k}(\cdot) \to v(\cdot) \quad \text{weakly in} \quad L_2(T;U),$$

$$\Lambda^0(x(\cdot), w^{h_k}(\cdot)) \to 0 \quad \text{as} \quad k \to \infty$$

imply the inclusion $v(\cdot) \in U(x(\cdot))$. Then the family $D_h$, $h \in (0,1)$ is regularizing.

## 3. SOLVING ALGORITHM

In this paper, it is considered the case where a control is a norm-square integrable function, i.e., $u_*(\cdot; x(\cdot)) \in L_2(T;U)$. Let the model $M$ be described by

$$w_t^h(t,\eta) - \triangle_L w^h(t,\eta) = v^h(t,\eta) + F(t,\eta)$$
$$\text{a. e. on } \{(t,\eta) \in T \times \Omega : w^h(t,\eta) > \mu(\eta)\},$$

$$w_t^h(t,\eta) = \max\{v^h(t,\eta) + F(t,\eta) + \triangle_L \mu(\eta), 0\} \qquad (9)$$
$$\text{a. e. on } \{(t,\eta) \in T \times \Omega : w^h(t,\eta) = \mu(\eta)\},$$

$$w^h(t,\eta) \ge \mu(\eta) \ \ \forall\, t \in T, \quad \text{for a. a. } \eta \in \Omega;$$

$$w^h = 0 \ \text{ a. e. on } \Gamma \times T,$$

i.e., a copy of system (1) is taken as a model. Let a number $a > 0$ be chosen such that

$$x_0 \in X_0 = \{x \in H_1^0(\Omega) : |x|_H^2 + \varphi(x) \le a < +\infty\}.$$

Here $\varphi(y)$ is the indicator function of the set $K = \{y \in L_2(\Omega) : y(\eta) \ge \mu(\eta) \text{ for a.a. } \eta \in \Omega\}$, i.e., $\varphi(y) = 0$ if $y \in K$, $\varphi(y) = +\infty$ otherwise.

A family of partitions $\Delta_h$ of the interval $T$, and a function $\alpha(h) : R^+ \to R^+$ satisfying the following conditions:

$$h\delta^{-1}(h) \le C, \quad \delta(h)\alpha^{-2}(h) \le C,$$

$$\alpha(h) \to 0, \quad \delta(h) \to 0, \qquad (10)$$

$$(h + \delta(h))\alpha^{-1}(h) \to 0, \quad \text{as} \quad h \to 0+$$

are taken. Here $C > 0$ is a constant that does not depend on $h$.

The family $\mathcal{W}_h$ of $t_0$-algorithms is defined by rule (3), where

$$w_0^h \in B(\xi_0^h) = \{x \in X_0 : |\xi_0^h - x|_H \le 2h\}. \qquad (11)$$

From inequality (2) and the inclusion $x_0 \in X_0$, it follows that $B(\xi_0^h) \ne \emptyset$.

Let the model control law $S_h = (\Delta_h, \mathcal{U}_h)$ be defined by rules (4), (5), where

$$\mathcal{U}_h(\tau_i, \xi_i^h, w^h(\tau_i)) = v_i^h$$

$$= \arg\min\{l(\alpha, v, s_i) : v \in U\},$$

$$l(\alpha, v, s_i) = 2(s_i, v) + \alpha(h)|v|_U^2, \quad s_i = w^h(\tau_i) - \xi_i^h,$$

i.e.,

$$v_i^h = -\alpha^{-1}s_i. \qquad (12)$$

Let a functional $\Lambda^0$ have the form:

$$\Lambda^0(x(\cdot), w^h(\cdot)) = |x(\cdot) - w^h(\cdot)|_{C(T;H)}.$$

The following theorem is true.

*Theorem 2.* The family of positional modeling algorithms $D_h$ (6) of the form (3)–(5), (9), (11), (12) satisfies the conditions of Theorem 1 and is regularizing.

**PROOF.** In order to show that the family $D_h$ (6) of the form (3)–(5), (10), (11) is $\Lambda^0$-stable, it is convenient to estimate the variation of the value

$$\varepsilon_h(t) = |w^h(t) - x(t)|_H^2 + \alpha(h) \int_{t_0}^{t} \{|v^h(\tau)|_H^2 - |u_*(\tau)|_H^2\} \, d\tau$$

for every $t \in T$.

Notice that system (1) is equivalent to the inclusion (see [2, pp. 138–140])

$$\begin{gathered} x_t(t,\eta) - \triangle_L x(t,\eta) + \beta(x(t,\eta) - \mu(\eta)) \\ - F(t,\eta) \ni u(t,\eta), \quad (t,\eta) \in T \times \Omega, \\ x(t,\eta)|_\Gamma = 0, \quad t \in T, \\ x(t_0,\eta) = x_0(\eta), \quad \eta \in \Omega, \\ \beta(r) = 0 \quad \text{if } r > 0, \\ \beta(0) = (-\infty, 0], \quad \beta(r) = \emptyset \quad \text{for } r < 0. \end{gathered} \tag{13}$$

In this case, taking into account inclusions (9), (12) and monotonicity of the mapping $\beta$, the inequality

$$1/2 d|\mu^h(t)|_H^2/dt \le (u_*(t) - v^h(t), \mu^h(t)) \tag{14}$$

holds for almost all $\delta_i = [\tau_i, \tau_{i+1})$. Here $\mu^h(t) = x(t) - w^h(t)$. It is readily seen that the estimate

$$\begin{aligned} (w^h(\tau_i) - \xi_i^h, \mu^h(t)) \le k_* \Big( h + \int_{\tau_i}^{t} \{|w_\tau^h(\tau)|_H \\ + |x_\tau(\tau)|_H\} \, d\tau \Big) |\mu^h(t)|_H - |\mu^h(t)|_H^2 \end{aligned} \tag{15}$$

and the inequality

$$\begin{aligned} (u_*(t) - v^h(t), \mu^h(t)) \le (u_*(t) - v^h(t), \xi_i^h - w^h(\tau_i)) \\ + c_1\{|u_*(t)| + |v^h(t)|\} \\ \times \Big( h + \int_{\tau_i}^{t} \{|w_\tau^h(\tau)|_H + |x_\tau(\tau)|_H\} \, d\tau \Big) \end{aligned}$$

are fulfilled for $t \in \delta_i$. In this case, for a.a. $t \in \delta_i$

$$\begin{aligned} 1/2 d|\mu^h(t)|_H^2/dt \le (u_*(t) - v^h(t), \xi_i^h - w^h(\tau_i)) \\ + \rho_i(t,h,\delta), \end{aligned} \tag{16}$$

where

$$\rho_i(t,h,\delta) = c_2\{|u_*(t)|_U + |v^h(t)|_U\} \times \Big( h + \int_{\tau_i}^{t} \{|w_\tau^h(\tau)|_H + |x_{r\tau}(\tau)|_H\} \, d\tau \Big).$$

Let a number $h_* > 0$ be chosen in such a way that $d(h) > Q = |u_*(\cdot)|_{L_\infty(T;U)}$ for $h \in (0, h_*)$.

Applying (12) gives

$$\begin{aligned} \varepsilon_h(t) \le \varepsilon_h(\tau_i) + c_2 \int_{\tau_i}^{t} \{[|u_*(\tau)|_U + |v^h(\tau)|_U]\} \, d\tau \\ \times \Big( h + \int_{\tau_i}^{t} \{|w_\tau^h(\tau)|_H + |x_\tau(\tau)|_H\} \, d\tau \Big) \end{aligned} \tag{17}$$

$$\le \varepsilon_h(\tau_i) + c_2 \Big( h^2 + 3\delta \int_{\tau_i}^{t} \{|u_*(\tau)|_U^2 |v^h(\tau)|_U^2\} \Big) \, d\tau + 4\delta^2 \int_{\tau_i}^{t} \{|w_\tau^h(\tau)|_H^2 + |x_\tau(\tau)|_H^2\} d\tau$$

for $t \in \delta_i$. Hence, the inequality

$$\begin{aligned} \varepsilon_h(t) \le \varepsilon_h(0) + c_3 h(1 + h/\delta) \\ + c_4 \delta \Big( 1 + \int_{0}^{t} \{|u_*(\tau)|_U^2 + |v^h(\tau)|_U^2\} d\tau \Big) \end{aligned} \tag{18}$$

holds for $t \in T$. From (18) and the inclusion $u_*(\cdot) \in L_2(T;U)$ it follows that

$$\begin{aligned} \varepsilon_h(t) \le \varepsilon_h(0) + c_3 h(1 + h/\delta) + c_5 \delta \\ + \alpha |u_*(\cdot)|_{L_2(T;U)}^2 + c_4 \delta^2 \sum_{j=0}^{i(t)} |v_j^h|_U^2, \end{aligned} \tag{19}$$

where $i(t)$ denotes the integer part of $t$. From (12) it follows that

$$|v_i^h|_U^2 \le 2(\nu_i^h + h^2)\alpha^{-2} \le c_6(\mu_i^h + h^2)\alpha^{-2}, \tag{20}$$

where $\nu_i^h = \nu^h(\tau_i) = x(\tau_i) - w^h(\tau_i)$. Combining (19) and (20) and using the relations

$$\varepsilon_h(0) \le 4h^2, \quad h\delta^{-1}(h) \le C,$$
$$h\alpha^{-1}(h) \to 0 \quad \text{as} \quad h \to 0,$$

we get

$$\begin{aligned} \mu_i^h \le \varepsilon_h(0) + c_3 h(1 + h/\delta) + c_5\delta + \alpha |u_*(\cdot)|_{L_2(T;U)}^2 \\ + c_4 c_6 \sum_{j=0}^{i-1} \delta^2 (\mu_j^h + h^2)\alpha^{-2} \le c_7(h + \delta + \alpha) \\ + c_8 \delta^2 \alpha^{-2} \sum_{j=0}^{i-1} \mu_j^h. \end{aligned}$$

From the Gronwall inequality and the inequality $\delta(h)\alpha^{-2}(h) \le C$, it follows that the estimate

$$\mu_i^h \le c_7(h + \delta + \alpha) \exp\{c_8 \vartheta \delta / \alpha^2\} \le c_9(h + \delta + \alpha)$$

is true. Summing the right-hand and left-hand sides of inequality (20) over $i$ yields

$$\begin{aligned} \delta^2 \sum_{j=0}^{m_h - 1} |v_j^h|^2 \le c_6 \delta^2 \sum_{j=0}^{m_h - 1} (\mu_i^h + h^2)\alpha^{-2} \\ \le c_{10} \delta \alpha^{-2} (\alpha + h + \delta). \end{aligned} \tag{21}$$

Using (10), (19), and (21), it is easy to get

$$\varepsilon_h(t) \le c_{11}(h + \delta + \delta^2\alpha^{-2} + h\delta\alpha^{-2}) \le c_{12}(h + \delta). \quad (22)$$

Hence,

$$|v^h(\cdot)|^2_{L_2(T;U)} \le |u_*(\cdot)|^2_{L_2(T;U)} + c_{12}(h + \delta)\alpha^{-1}.$$

Therefore, inequality (7) holds when

$$k_1(h) = 1, \qquad k_2(h) = \left(c_{12}(h + \delta(h))\alpha^{-1}(h)\right)^{1/2}.$$

Inequality (8) follows from (22) if $k_3(h) = c_{13}(h + \delta(h) + \alpha(h))$, where $c_{13}$ is a constant, which can be obtained in an explicit form.

The theorem is proved.

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