The Spectra and Pseudospectra of Electric Power Systems

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***Abstract*—The application of the quadratic eigenvalue problem in electrical power systems is reviewed. The spectra and pseudospectra of an electrical power system are defined.**

I. Introduction

The development of electrical energetic in Russia has stimulated the development of mathematical models and methods for solving a wide range of problems on steady-state and transient modes of operation of electric power systems, state estimation, behavior extrapolation, reliability provision, control optimization and development of electric power systems, power production, computer systems, and control and diagnostic tools for power systems. These problems are solved with various models and methods. The development of electrical energetic entrains the development of mathematical models and methods for solving these problems. Several promising approaches have been recently developed for power dispatch control, diagnostics of electrical equipment, instant and short-term prediction (from one hour to one day) of power system operation modes; earlier they were regarded as difficult problems. This review outlines the application of the methods of the quadratic eigenvalue problem in electrical energetic [1].

II. Eigenvalue problem formulation

A central topic in matrix theory is standard eigenvalue problem, i.e., determination of nontrivial solutions of the system of algebraic equations

$$Ax = \lambda x, \qquad y^* A = \lambda y^* .$$

Here $A \in \mathbb{C}^{n\times n}$ is a known $n\times n$ matrix defined over the field of complex numbers $\mathbb{C}$. The vectors *x* and *y* are called the right and left eigenvector of the matrix *A* for the eigenvalue λ, respectively.

The quadratic eigenvalue problem is concerned with the properties of $n\times n$ matrix second-degree polynomials (square matrix pencils)

$$Q(\lambda) = \lambda^2 M + \lambda C + K . \qquad (1)$$

Here $M, C, K \in \mathbb{C}^{n\times n}$.

The quadratic eigenvalue problem consists in determining the complex scalars λ and nonzero complex vectors *x* and *y* satisfying the algebraic equations

$$Q(\lambda)x = 0, \quad y^* Q(\lambda) = 0 . \qquad (2)$$

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In recent years we solved vital control problems for electric power systems and plants. They are quadratic eigenvalue problems [1]: 1) complex electrical power system static stability; 2) stability of shaft trains of turbine units; 3) seismic stability of arch dams of hydroelectric stations; 4) extrapolation of the electric power systems behavior; 5) estimation of an electrical power system state. In what follows, we mainly study the quadratic eigenvalue problem (2).

III. Perturbations of the quadratic eigenvalue problem and the pseudospectra concept

The elements of the matrices *M*, *C*, and *K* (1) in the decomposition for real electrical power systems, as a rule, are determined directly from physical observations and, therefore, contain the errors inherent in observations. In this case, the matrix $Q(\lambda)$, which is experimentally determined, is only an approximation of the matrix $Q(\lambda)$ corresponding to exact measurements. Hence different types of distortions and deformations arise in the solution of the quadratic eigenvalue problem. Such distortions are also introduced by computational errors. On the other hand, there is no need in many practical problems for an expression to vanish, what matters is that it must be "negligibly small" compared to some predefined threshold. Hence by the solution of an equation, we mean the value of variables that convert the value of a function into a negligibly small quantity. These solutions represent a set of points, called the pseudosolutions (ε-solutions), or almost near solutions if the threshold of this negligibly small quantity is unimportant or not specified explicitly. Such problems are encountered in power systems in processing telemetric information, state estimation from "accessible measurements", parameter identification, and determination of static and global stability ranges of a power system, etc.

Pseudosolutions are stable to small perturbations of the initial data, continuous, and the "determination of almost near solutions" is a computationally correct problem [5]. Concepts based on a small margin ε are not novel. Such are, for example, the ε-subdifferential, ε-optimal solution of an optimization problem, ε-invariant systems, etc. The recent theory of pseudospectra (ε-spectra) of numerical matrices must be interpreted precisely in this sense [4].

Let a triple (λ, x, y) be the solution of the quadratic eigenvalue problem (2). Let $(\tilde{\lambda}, \tilde{x}, \tilde{y})$ be an approximation of the solution (λ, x, y). Let us examine two questions: 1) How do the perturbations experienced by the matrices *M*, *C*, and *K* distort the solution (λ, x, y) (such errors are called direct errors)? 2) If $\left(\tilde{\lambda},\, \tilde{x},\, \tilde{y}\right)$ is an approximation for the solution of the quadratic eigenvalue problem, then how can we determine the perturbations acting on the elements of the matrices *M*, *C*, and *K* under which the approximation is close to the solution of the quadratic eigenvalue problem (such errors are called the backward errors)?

The Crawford causality number $\kappa(\lambda, Q)$ is helpful in defining the conditional inequality [3] "direct errors ≤ backward errors × the Crawford number".

Using the error matrix $\Delta Q(\lambda) = \lambda^2 \Delta M + \lambda \Delta C + \Delta K$ let us find the causality number $\kappa(\lambda, Q)$ in terms of eigenvalues of $Q(\lambda)$ by formula [3]

$$\kappa(\lambda, Q) = \lim_{\varepsilon \to 0} \sup \Big\{ |\Delta\lambda| / \varepsilon |\lambda| \ \Big| \ \left[Q(\lambda + \Delta\lambda) + \Delta Q(\lambda + \Delta\lambda)\right](x + \Delta x) = 0, \ \|\Delta M\| \le \varepsilon \alpha_2,\ \|\Delta C\| \le \varepsilon \alpha_1,\ \|\Delta K\| \le \varepsilon \alpha_0 \Big\}.$$

The following theorem holds.

Theorem 1 [3]. If a matrix pencil $Q(\lambda)$ has simple and finite eigenvalues λ, then

$$\kappa(\lambda, Q) = \frac{|\lambda|^2 \alpha_2 + |\lambda| \alpha_1 + \alpha_0}{|\lambda| \cdot \left|y^* Q'(\lambda) x\right|} \cdot \|x\| \cdot \|y\|.$$

The following theorem define the pseudospectra of the matrix $A \in \mathbb{C}^{n \times n}$.

Theorem 2 [4]. Let $\|\cdot\|$ be the matrix norm induced by a vector norm. Then the following definitions of the pseudospectra of $\mathrm{eig}_\varepsilon(A)$ are equivalent:

$$\mathrm{eig}_\varepsilon(A) = \left\{ z \in \mathbb{C} \,\middle|\, \left\| (zI_n - A)^{-1} \right\| \ge \varepsilon^{-1} \right\},$$

$$\mathrm{eig}_\varepsilon(A) = \left\{ z \in \mathbb{C} \ \middle| \ \exists \Delta A,\ \|\Delta A\| \le \varepsilon,\ z \in \Lambda(A + \Delta A) \right\} ,$$

$$\mathrm{eig}_\varepsilon(A) = \left\{ z \in \mathbb{C} \,\middle|\, \exists v \in \mathbb{C}^n,\ \|v\| = 1,\ \|(A - zI_n)v\| \le \varepsilon \right\}.$$

If $\|\cdot\|$ is the Hermite norm and $\sigma_{\min}(\cdot)$ is the minimal singular number, then the following condition is equivalent to the previous conditions:

$$\mathrm{eig}_\varepsilon(A) = \left\{ z \in \mathbb{C} \,\middle|\, \sigma_{\min}(zI_n - A) \le \varepsilon \right\}.$$

Now we state the definition of the pseudospectra of a quadratic pencil $Q(\lambda)$ [3].

Definition 1. The pseudospectrum of a quadratic matrix pencil $Q(\lambda)$ is defined to be the set

$$\mathrm{eig}_\varepsilon(Q(\lambda)) = \Big\{ \lambda \in \mathbb{C} \ \Big| \ \exists x \ne 0,\ \left(Q(\lambda) + \Delta Q(\lambda)\right) x = 0, \ \|\Delta M\| \le \varepsilon \alpha_2,\ \|\Delta C\| \le \varepsilon \alpha_1,\ \|\Delta K\| \le \varepsilon \alpha_0 \Big\}. \quad (3)$$

The pseudospectrum $\mathrm{eig}_\varepsilon(Q(\lambda))$ (3) satisfies the relation

$$\mathrm{eig}_\varepsilon(Q(\lambda)) = \left\{ \lambda \in \mathbb{C} \ \middle| \ \left\|Q^{-1}(\lambda)\right\| \ge \varepsilon^{-1} \left(|\lambda|^2 \alpha_2 + |\lambda| \alpha_1 + \alpha_0 \right)^{-1} \right\}$$

The quadratic eigenvalue problem as well as the reformulated problem in terms of the concept of a pseudospectra have a bright future in electrical energetics. Electrical power systems experience the continuous action of signal and parametric disturbances (large load scatter and fluctuations, short circuiting, unauthorized disconnection of equipment, natural cataclysms, etc) inducing changes in the power station operation mode. The response of a power system to external perturbations is exhibited as variations in operation parameters (changes in the modulus and voltage phases, over currents and currents in elements, unstable rotation speed of synchronous and induction machines, etc.). The composition and magnitude of these variations depend on the topology of the power system circuits, its parameters (resistance and conduction, controls, characteristics of control devices, etc.) and many other factors, which are innumerable for large power systems.

The parameters that are most sensitive to external disturbances are called sensors in electrical energetic. Sensors arise as a result of heterogeneities in a power system. Analysis of a power system and, primarily, its stability and robustness is rather complicated mostly due to the multidimensionality and multi-factor nature of analysis. But the modern computation methods developed for the eigenvalue problem are helpful in this analysis. Most of the power problems (Problems 1, 2, 4, and 5) based on the quadratic eigenvalue problem can be solved by the mathematical packages developed at the "Unified Electrical Network Center" (Moscow, Russia) for automatic technological control of electrical networks.

Methods of solution of Problem 3 were applied to design an air screen for the existing high dams. The authors of this paper are coauthors of the design of Chirkeisk and Miatlinsk Hydro Electrical Projects. The results of these projects revealed that the reliability of existing dams in regions, where the rated seismicity has been increased today, can be enhanced by a special damping device—air screen that modifies the natural frequencies of a dam along with the water bulk it holds by changing the density and hydro elasticity of water.

A method based on the solution of the quadratic eigenvalue problem for Problem 3 has also been used for optimization of the constructions of dams by guaranteeing the rated seismicity of dams with minimal ferroconcrete volume.

References

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