AXIS-SYMMETRIC FRACTIONAL DIFFUSION-WAVE PROBLEM: PART I-ANALYSIS

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Abstract: This is part I of a two part paper on an axis-symmetric fractional diffusion-wave problem. In this part we focus on the response of the system subjected to external excitation. We define the problem in terms of Riemann-Liouville fractional derivatives and use modal analysis approach to reduce the continuum problem to a countable infinite degrees-of-freedom problem for which solution could be found in closed form. Here we use Grünwald-Letnikov approximation to find a numerical solution to the problem. This will allow us to solve axis-symmetric fractional optimal control problems which could not be solved in closed form. We validate the scheme by comparing the numerical results with the analytical solutions. The formulation and the approach presented here extends our earlier work on fractional diffusion in 2-dimension to axis symmetric case.

Key words

Fractional Calculus, Fractional derivatives, Fractional diffusion-wave equation, Axis symmetric, diffusion-wave equation, Grünwald-Letnikov scheme, Bessel function.

1 Introduction

Fractional derivatives are generalizations of ordinary differentiations and integrations to non-integer orders. Some of the definitions of fractional derivatives proposed include Riemann-Liouville, Grünwald-Letnikov, Weyl, Caputo, Riesz and Marchaud derivatives ([Oldham and Spanier, 1974], MillerRoss93, [Samko, Kilbas and Marichev, 1993], [Podlubny, 1999]). Fractional derivatives arise in many physical problems, for example frequency dependent damping behavior of materials, relaxation functions for viscoelastic materials, fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ control of dynamical systems, motion of a plate in a Newtonian fluid, and phenomena in electromagnetics and acoustics ([Podlubny, 1999]).

In this paper we focus on an axis-symmetric Fractional Diffusion Wave Equations (FDWEs). The FDWEs are obtained by replacing the integer order (time and/or space) derivatives in ordinary diffusion/wave equations with the fractional derivatives. The FDWEs have been considered by the several investigators in recent years. Oldham and Spanier [1974] considered a fractional diffusion equation that contained first-order derivative in space and half-order derivative in time. Wyss [1986] and Schneider and Wyss [1989] presented the solutions of the time-fractional diffusion and wave equations in terms of Fox functions. Giona, Cerbelli and Roman [1992] presented fractional diffusion equations describing the relaxation phenomena in complex viscoelastic materials. Giona and Roman [1992a] presented fractional diffusion equations for transport phenomena in random media. Giona and Roman [1992b] and Roman and Giona [1992] used fractional diffusion equations to describe one and three dimensional cases of anomalous diffusions on fractals without external forces. Their work extends the expression of Oldham and Spanier [1972].

Mainardi ([1996a], [1996b], [1997]) used a Laplace transform method to obtain fundamental solutions for a FDWE and the solutions for fractional relaxation-oscillations. Mainardi [1997] and Mainardi and Paradisi [1997] showed that as the order of fractional derivative in a FDWE increases from 0 to 2, the process changes from slow diffusion to classical diffusion to diffusion-wave to classical wave processes. Agrawal [2000] presented fundamental solutions for an FDWE where the diffusion equation contained a fourth order space derivative and a fractional order time derivative. Zou, Ren and Qiu [2004] considered a fractional diffusion equation of higher dimension to describe anomalous diffusion processes involving external force fields by using Giona and Roman's heuristic argument. Hilfer [2000] proposed the closed form solution of a fractional diffusion problem in terms of H-functions.

Agrawal ([2001], [2002]) used modal/integral transform methods to find solutions of FDWEs defined in finite domains. The method is general and can be used to find closed form solutions to many problems in vibration analysis of fractional systems where modal/integral transform methods could be applied. Given that vibration of continuous system is a vast field, these papers open multitude of possibilities and new problems in the field of fractional diffusion-wave. Modal/integral transform methods have recently been used to solve fractional generalization of Navier-Stokes equations in El-Shahed and Salem [El-Shahed and Salem, 2004] and a time fractional radial diffusion in a sphere [Povstenko, 2007]. Agrawal [2003] presented stochastic analysis of FDWEs defined in 1-dimension. It should be pointed out that very little work has been done in the area of stochastic analysis of fractional order engineering systems. Since the approach presented here also uses modal/integral transform methods, it could be extended to all fractional stochastic problems in multi-dimensions where the transform methods could be applied.

The exact solution of a fractional diffusion equation with an absorbent term and a linear external force appears in [Schot, et al, 2007]. Exact solutions of generalized nonlinear fractional diffusion equations with external force and absorption are presented in [Liang, et al 2007].

This brief review of formulations and methods for FDWEs is by no means complete. Other formulations, methods, and solutions could be found, among others, in [West, Bologna and Grigolini 2003] and the references there in.

In this paper, we present a numerical scheme for an axis-symmetric fractional diffusion-wave problem. We define the problem in terms of Riemann-Liouville fractional derivatives and use modal/integral transform method presented in Agrawal ([2001], [2002]) to reduce the continuum problem to a countable infinite degrees-of-freedom problem for which solution could be found in closed form. Problems similar to the one considered here have also been solved using similar techniques in [El-Shahed and Salem, 2004] and [Povstenko, 2007]. However, the formulation here differs with Agrawal ([2001], [2002]),

[El-Shahed and Salem, 2004] and [Povstenko, 2007] in two respects. First, Agrawal ([2001], [2002]) considers one dimensional problems whereas this paper considers an axis-symmetric problem. Second, Agrawal ([2001], [2002]), [El-Shahed and Salem, 2004] and [Povstenko, 2007] find only closed form solutions, whereas this paper also finds numerical solutions using Grünwald-Letnikov approximation. In the sequel, we will present a formulation for an axis-symmetric fractional optimal control problem for which a closed form solutions could not be found. Our numerical scheme used here will allow us to find numerical solutions to the problem.

In the next section, we present the formulation and the analytical solution to an axis-symmetric FDWE.

2 Axis-symmetric FDWE and its analytical solution

In this section, we present an axis-symmetric FDWE in terms of the Riemann-Liouville Fractional Derivative (RLFD), and provide its analytical solution. However, much of it can also be applied to formulations defined in terms of other fractional derivatives. We begin with the RLFD which is defined as [Podlubny, 1999],

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \tag{1} $$

where $f(t)$ is a function, α, $(n-1<\alpha<n)$, is the order of the derivative, t is the time variable, and n is an integer. In case α is an integer, the fractional derivative is replaced with an ordinary derivative. Furthermore, if f is dependent on two or more variables, then the ordinary derivative in Eq. (1)is replaced with a partial derivative.

The axis-symmetric fractional diffusion-wave problem can now be defined as follows: Find the response of the system

$$ \frac{\partial^{\alpha} w}{\partial t^{\alpha}} = \beta \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right) + u\left(r,t\right), \tag{2} $$

subjected to the following boundary and initial conditions,

$$ w\left(R,t\right) = 0, \quad t>0, \tag{3} $$

and

$$ w\left(r,0\right) = \dot{w}\left(r,0\right) = 0, \tag{4} $$

where r is the radial space coordinate, β is a constant which depends on the physical properties of the system, $u(r,t)$ is the external source term, and R is the boundary of the domain of r. The second condition in Eq. (4) is considered only if $\alpha>1$. In the case of heat transfer, $u(r,t)$ represents the rate of heat generation, and in the case of membrane vibration, it represents the external forcing function.

Using the method of separation of variables, it can be demonstrated that the eigenfunctions $\phi_j(r)$ for this problem are (see, e.g. [Kreyszig, 2006]),

$$ \phi_j(r) = J_0\left(\mu_j \frac{r}{R}\right), \quad j=1,2,\cdots,\infty \tag{5} $$

where $J_0(*)$ is the zero-order Bessel function of the first kind, and μ_j, $j=1,2,\cdots,\infty$, are the positive roots of the equation

$$ J_0(\mu) = 0. \tag{6} $$

To find the solution of the problem defined by Eqs. (2) to (4), assume that $w(r,t)$ can be given as

$$ w\left(r,t\right) = \sum_{j=1}^{\infty} q_j\left(t\right) J_0\left(\mu_j \frac{r}{R}\right). \tag{7} $$

Using Eqs. (2), (3), (4), and (7), and the orthogonality conditions

$$\int_0^1 xJ_0(\mu_i x)\, J_0(\mu_j x)\, dx = \begin{cases} 0 & , \; i \neq j \\ \frac{J_1^2(\mu_j)}{2} & , \; i = j \end{cases} . \tag{8}$$

we obtain

$$\frac{d^\alpha q_k(t)}{dt^\alpha} = -\beta \left(\frac{\mu_k}{R}\right)^2 q_k(t) + \bar{f}_k(t) \tag{9}$$

and

$$q_k(0) = \dot{q}_k(0) = 0 \tag{10}$$

where $J_1(*)$ is the first-order Bessel function of the first kind, and $\bar{f}_k(*)$ is given as

$$\bar{f}_k(t) = \frac{2}{R^2 J_1^2(\mu_k)} \int_0^R rJ_0\left(\mu_k \frac{r}{R}\right) u(r,t)\, dr. \tag{11}$$

Once again, the second condition in Eq. (10) is considered when $\alpha > 1$.

By applying the Laplace transform to Eq. (9), using Eq. (10), and then taking inverse Laplace transform, we get

$$q_k(t) = \int_0^t Q_k(t-\tau)\, \bar{f}_k(\tau)\, d\tau \tag{12}$$

where

$$Q_k(t) = L^{-1}\left\{\frac{1}{s^\alpha + \beta\left(\frac{\mu_k}{R}\right)^2}\right\}, \tag{13}$$

is the fractional Green's function, which can be written in closed form as [Podlubny, 1999],

$$Q_k(t) = t^{\alpha-1} E_{\alpha,\alpha}\left(-\beta\left(\frac{\mu_k}{R}\right)^2 t^\alpha\right). \tag{14}$$

Here L^{-1} is the inverse Laplace transform operator, and $E_{\alpha,\beta}$ is the two-parameter Mittag-Leffler function (see [Podlubny, 1999]).

Substituting Eq. (12) into Eq. (7), we obtain the closed form solution of the axis-symmetric fractional diffusion wave equation defined by Eqs. (2) to (4) as

$$w(r,t) = \sum_{k=1}^{\infty} J_0\left(\mu_k \frac{r}{R}\right) \int_0^t Q_k(t-\tau)\, \bar{f}_k(\tau)\, d\tau. \tag{15}$$

Thus, $w(r,t)$ can be obtained provided $u(r,t)$ is known. It will be shown in the second part of this paper that in the case of fractional optimal control $u(r,t)$ is not known *a priori*, and it is solved along with other variables. For this case, a numerical scheme is necessary.

In the next section, we present a numerical algorithm to solve the fractional differential equations defined by Eqs. (9) and (10).

3 Numerical Algorithm

The numerical algorithm presented here is based on an algorithm given in [Podlubny, 1999]), and it relies on the Grünwald-Letnikov approximation of the fractional derivative. For simplicity in the discussion to follow, we drop the subscript k from Eqs. (9) and (10), and rewrite them as

$$\frac{d^\alpha q(t)}{dt^\alpha} = -cq(t) + f(t) \tag{16}$$

and

$$q(0) = \dot{q}(0) = 0 \tag{17}$$

where $c = \beta\left(\mu_k/R\right)^2$.

The algorithm can now be described as follows:

1. Divide the time interval into several subintervals of equal size h (also called the step size).
2. Approximate $d^\alpha q/dt^\alpha$ at node i using the Grünwald-Letnikov formula as [Podlubny, 1999]),

$$\frac{d^\alpha q}{dt^\alpha} = \frac{1}{h^\alpha} \sum_{j=0}^{i} w_j^{(\alpha)} q^{(i-j)}, \tag{18}$$

   where $q^{(j)}$ is the numerically computed value of q at node j, and $w_j^{(\alpha)}$ are the coefficients defined as [Podlubny, 1999]),

$$w_0^{(\alpha)} = 1;\; w_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right) w_{j-1}^{(\alpha)},\; j = 1, \cdots \tag{19}$$

3. Approximate Eq. (16) at node i, and solve for $q^{(i)}$ to obtain

$$q^{(i)} = \frac{1}{1+ch^\alpha}\left(h^\alpha f(t_i) - \sum_{j=1}^{i-1} w_j^{(\alpha)} q^{(i-j)}\right),\; i = i_\alpha, \cdots \tag{20}$$

   where $i_\alpha = 1$ if $i_\alpha \leq 1$, and $i_\alpha = 2$ if $i_\alpha > 1$. In case α is greater than 1, $q^{(1)}$ is determined using Eq. (17) and taking linear approximation between $q^{(0)}$ and $q^{(1)}$ (see, [Podlubny, 1999])).
4. Use Eq. (20) to find $q^{(i)}$ at all nodes.

Thus, the numerical solution of Eqs. (16) and (17) is obtained. To obtain the solution of Eqs. (2) to (4), we replace ∞ in Eq. (7) with an integer M (i.e. we truncate the series), solve Eqs. (9) and (10) for k from 1 to M, and substitute the results in Eq. (7). Numerical studies show that only a small M is sufficient to find accurate results.

4 Numerical Example

In this section, we present some simulation results for the Diffusion-Wave problem defined by Eqs. (2) to (4) for $t > 0$, $0 < \alpha \leq 2$, $r \in [0, R]$. For simulation purpose, we take $R = \beta = u(r,t) = 1$, and vary M and h. We compute $\bar{f}_k(t)$ using Eq. (11) and solve Eqs. (9) and (10) for $k = 1, 2, \cdots, M$ using the algorithm discussed in Sec. 3. Finally, we use Eq. (7) to find the response. We also find the analytical solution using Eq. (15) for comparison purpose. For computation purpose, this series is truncated after M terms. The results of this study are as follows:

Figures 1 and 2 show the analytical and numerical results for $w(r,t)$ for $\alpha = 1$ and $\alpha = 2$, respectively. In this study, we take $r = 0.5$, $M = 5$, and $h = 0.01$. Figure 1 shows that the diffusion reaches a steady state in a very short time, and Figure 2 shows the undamped vibrational characteristics of the system. Note that in both cases, the analytical and numerical results are very close.

Rest of the figures show numerical results for various M, h, and α. Figure 3 shows $w(r,t)$ at $r = 0.5$ for $\alpha = 0.5$ and $M = 5$, 10, and 20. All three curves are very close indicating that only few terms are necessary to compute response of the system. Note that the diffusion process is slow. Figure 4 shows $w(r,t)$ at $r = 0.5$ for $\alpha = 1.0$, $M = 5$ and $h = 1$, 0.1, 0.01, and 0.001. Note that the solutions converge as the step size is reduced, which indicates that the algorithm may be stable.

Figure 5 shows $w(r,t)$ for $\alpha = 0.5$, 0.7, 0.9 and 1, and Figure 6 shows the same for $\alpha = 1.5$, 1.7, 1.9 and 2. In both cases, the following values are used: $r = 0.5$, $M = 5$ and $h = 0.01$. Both figures show that as α approaches an integer value, the solution for the integer order system is recovered. These two figures also show that as α changes from 0.5 to 2, the response changes from sub-diffusion to diffusion to diffusion-wave to wave solution.

Figures 7, 8 and 9 show the whole field response $w(r,t)$ for $\alpha = 0.5$, 1.5 and 2, respectively. For these simulations, we took $M = 5$ and $h = 0.01$. These figures also show the changing behavior of $w(r,t)$ as α changes from 0.5 to 2. Note that $w(r,t)$ reaches a steady state for $\alpha = 0.5$ and 1.5, but it continue to oscillate for $\alpha = 2.0$. Further, in all three cases, $\partial w/\partial r$ is 0 at $r = 0$, as expected.

5 Conclusions

An axis-symmetric fractional diffusion-wave problem was defined in terms of the Riemann-Liouville fractional derivative, and a modal/integral transform method was presented to reduce the continuum problem to a countable infinite degrees-of-freedom problem. A Laplace transform based technique was used to find closed form solutions. The Grünwald-Letnikov approximation was used to develop an algorithm for numerical solution of the problem. Results show that both the analytical and the numerical results agree well. As the time step size is reduced, the solutions converge, and only few terms in the series are necessary to find a solution close to the exact solution. The response of the system changes from subdiffusion to diffusion to diffusion-wave to wave solutions as α changes from 0.5 to 2. The numerical algorithm developed here will allow us to find numerical solutions for axis-symmetric fractional optimal control problems.

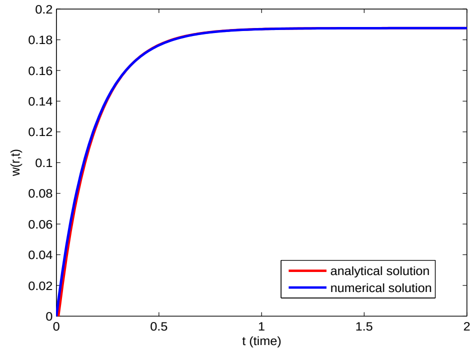

Figure 1. Comparison of the analytical and the numerical solution of $w(r,t)$ for $\alpha = 1$, $r = 0.5$, $M = 5$ and $h = 0.01$

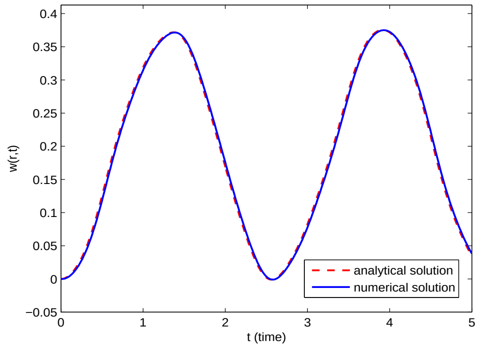

Figure 2. Comparison of the analytical and the numerical solution of $w(r,t)$ for $\alpha = 2$, $r = 0.5$, $M = 5$ and $h = 0.01$

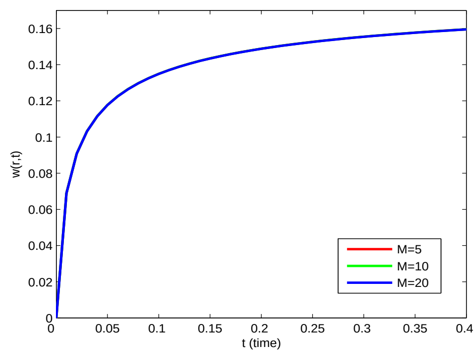

Figure 3. The solution of $w(r,t)$ for $\alpha = 0.5$, $r = 0.5$. and $M = 5$, 10, 20

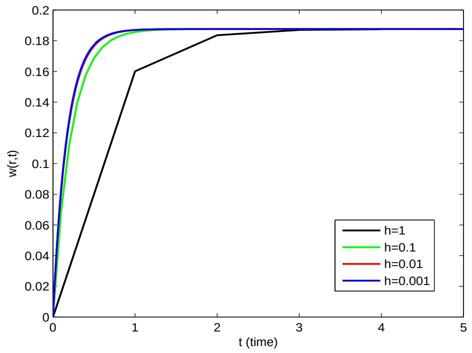

Figure 4. The solution of $w(r,t)$ for $\alpha = 1$, $r = 0.5$, $M = 5$. and h = 1, 0.1, 0.01, 0.001

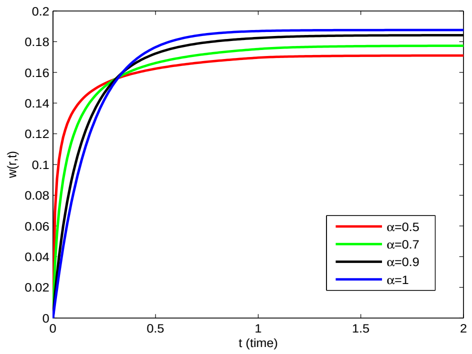

Figure 5. The solution of $w(r,t)$ for $r = 0.5$, $h = 0.01$, $M = 5$. and α = 0.5, 0.7, 0.9, 1

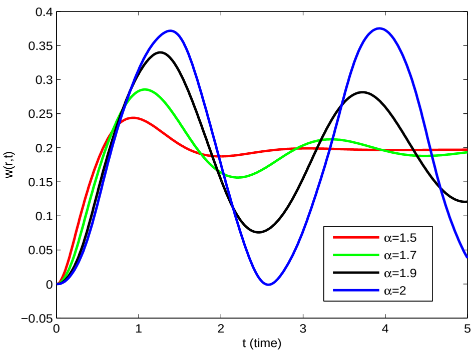

Figure 6. The solution of $w(r,t)$ for $r = 0.5$, $h = 0.01$, $M = 5$. and α = 1.5, 1.7, 1.9, 2

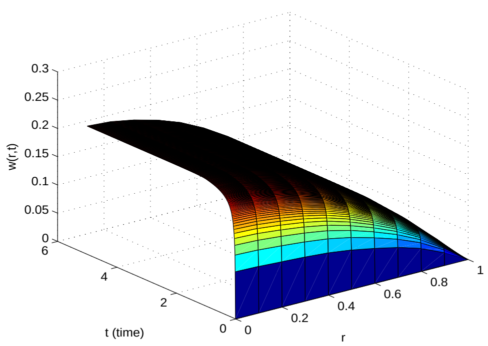

Figure 7. Three dimensional figure of $w(r,t)$ for $\alpha = 0.5$, $h = 0.01$ and $M = 5$.

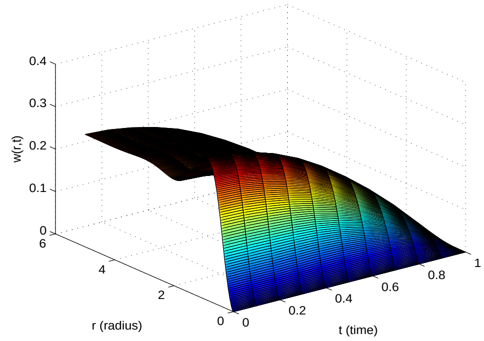

Figure 8. Three dimensional figure of $w(r,t)$ for $\alpha = 1.5$, $h = 0.01$ and $M = 5$.

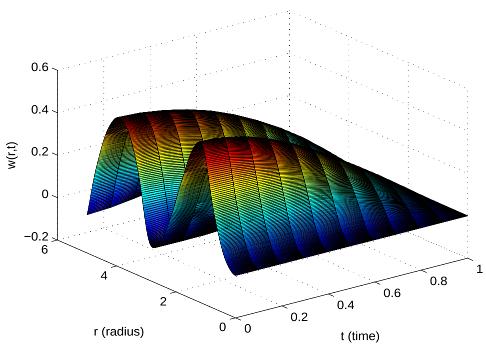

Figure 9. Three dimensional figure of $w(r,t)$ for $\alpha = 2$, $h = 0.01$ and $M = 5$.

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