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SLOW DRIFT IN A SLOW-FAST HAMILTONIAN SYSTEM

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Abstract

We study the drift of slow variables induced by chaotic motion of the fast variables in a Hamiltonian system with two different time-scales. We assume that the fast system with frozen slow variables has a pair of hyperbolic periodic orbits connected by two transversal heteroclinic trajectories. We define the class of accessible paths and show every accessible path is shadowed by the slow component of a trajectory of the full system.

For any periodic trajectory of the fast subsystem with the frozen slow variables we define an action. For a family of periodic orbits, the action is a scalar function of the slow variables and can itself be considered as a Hamiltonian function. An accessible path consists of segments of the corresponding trajectories.

Key words

Slow-fast Hamiltonian system

1 Introduction

We consider a slow-fast Hamiltonian system described by a smooth Hamiltonian function $H(p, q, v, u; \varepsilon)$, where (p, q) and $(\varepsilon v, u)$ are pairs of canonically conjugated variables. Equations of this type often arise after rescaling a part of the variables in a standard Hamiltonian system. The corresponding equations of motion have the form

$$\begin{aligned} \dot q &= \frac{\partial H}{\partial p}\,, & \dot p &= -\frac{\partial H}{\partial q}\,, \\ \dot u &= \varepsilon\frac{\partial H}{\partial v}\,, & \dot v &= -\varepsilon\frac{\partial H}{\partial u}\,. \end{aligned} \tag{1}$$

The variable (p, q) are fast and (v, u) are slow. We assume that the system has $m + d$ degrees of freedom, where m is the number of fast degrees of freedom and d is the number of slow ones.

At $\varepsilon = 0$ equation (1) takes the form

$$\begin{aligned} \dot q &= \frac{\partial H}{\partial p}\,, & \dot p &= -\frac{\partial H}{\partial q}\,, \\ \dot u &= 0\,, & \dot v &= 0\,. \end{aligned} \tag{2}$$

The values of (v, u) remain constant in time and the system can be interpreted as a family of Hamiltonian systems with m degrees of freedom which depends on $2d$ parameters. We call it a *frozen fast system*.

If $m = 1$, the frozen fast system typically represents an oscillator and averaging is used to eliminate the dependence on a fast phase. Then the slow dynamics is approximated by an autonomous system with d degrees of freedom over very long time intervals (see e.g. [Arnold, 1978; Bogolyubov et al, 1961]). More precisely, the slow component of a full trajectory stays in an ε neighbourhood of a trajectory of the following system. Suppose the fast space is foliated into periodic orbits $L(v, u, h)$, where h stands for the total energy of the system. An action of the periodic orbit is defined by the integral

$$J(v, u, h) := \oint_{L(v,u,h)} p\, dq\,. \tag{3}$$

The function J defines a slow Hamiltonian system in the (v, u) variables:

$$u' = \frac{1}{T}\frac{\partial J}{\partial v}\,, \qquad v' = -\frac{1}{T}\frac{\partial J}{\partial u}\,, \tag{4}$$

where $'$ stands for the derivative with respect to the slow time $\tau = \varepsilon t$ and $T = T(v, u, h)$ is the period of L. System (4) is Hamiltonian with the non-standard symplectic form $T\, dv \wedge du$. Alternatively the equations can be interpreted as a result of a time scaling in a standard Hamiltonian system.

If $m \geq 1$, the averaging method predicts that the slow component of the dynamics is described in the leading order by the vector field

$$u' = \left\langle \frac{\partial H}{\partial v} \right\rangle , \qquad v' = -\left\langle \frac{\partial H}{\partial u} \right\rangle \tag{5}$$

obtained by taking an average of the slow component of (1) over the space of fast variables. Of course, in the case of $m = 1$ this system coincides with (4).

The validity of this prediction strongly depends on the dynamics of the frozen fast system. In particular it was verified for the frozen fast system which oscillates with a constant vector of frequencies [Lochak and Meunier, 1988; Neishtadt, 1976]. The averaging can be also used if the frozen system is uniformly hyperbolic [Anosov, 1960] or, more generally, if the frozen system is ergodic and the time averages converge sufficiently fast to space averages [Kifer, 2005].

The averaging method is based on the assumption that the time average over a trajectory converges to the space average. In an ergodic system almost every trajectory has this property. On the other hand, it is well known that if $m > 1$ there are trajectories which do not posses this property. The most remarkable example is a periodic trajectory. Therefore the slow component of a trajectory whose fast component stays near a periodic orbit of the frozen system should strongly deviate from the averaged dynamics described by (5).

Hyperbolic periodic orbits of the frozen fast system typically form families parametrised by (v, u, h). This family forms a normally hyperbolic invariant manifold. It is well known that normally hyperbolic manifolds persist under perturbations [Fenichel, 1971]. Consequently the full system has a $(2d + 2)$-dimensional invariant manifold provided ε is sufficiently small. Since the variables (v, u) are slow, the restriction of the full dynamics onto this manifold has a single fast degree of freedom. Therefore it is similar to the case $m = 1$ described above.

In this paper we assume that the frozen system has a compact invariant set bearing chaotic dynamics of horseshoe type created by transversal heteroclinics between two saddle periodic orbits. This situation typically arises when a periodic orbit has a transversal homoclinic. In this invariant set hyperbolic periodic orbits are dense and every two periodic orbits are connected by a heteroclinic orbit. We select a finite subset of periodic orbits with relatively short periods. We construct trajectories of the full system which switch between neighbourhoods of the periodic orbits in a prescribed way. We show that the slow component of such trajectories drifts in a way quite similar to trajectories of a random Hamiltonian dynamical system with d degrees of freedom.

The trajectories constructed in this paper strongly deviate from the averaged dynamics. We think this mechanism is responsible for the largest possible rates of deviation.

A similar construction is used in [Gelfreich and Turaev, 2007] for studying drift of the energy in a Hamiltonian system which depends on time explicitly and slowly. In particular, it was shown in [Gelfreich and Turaev, 2007] that switching between fast periodic orbits indeed provides the fastest rate of energy growth in several situations.

2 Drift of slow variables

In this section we state our main result. The full details of the proof can be found in [Brännström and Gelfreich, 2007].

The total energy is preserved, so we study the dynamics on a single energy level. Without any loss in generality we may consider the dynamics in the zero energy level

$$\mathcal{M}_\varepsilon = \{ H(p, q, v, u; \varepsilon) = 0 \} .$$

First we state our assumptions on the dynamics of the frozen fast system. Let $D \subset \mathbb{R}^{2d}$ be a bounded domain. We assume

[A1] the frozen system has two smooth families of hyperbolic periodic orbits $L_c(v, u) \subset \mathcal{M}_0$ defined for all $(v, u) \in D$, $c \in \{ a, b \}$.

[A2] the frozen system has two smooth families of transversal heteroclinic orbits: $\forall (v, u) \in D$

$$\Gamma_{ab}(v, u) \subset W^u(L_a(v, u)) \cap W^s(L_b(v, u)) ,$$
$$\Gamma_{ba}(v, u) \subset W^u(L_b(v, u)) \cap W^s(L_a(v, u)) .$$

We note that under these assumptions the frozen system has a family of uniformly hyperbolic invariant transitive sets $\Lambda_{(v,u)}$, also known as Smale horseshoes [Shilnikov et al., 1998]. For every $(v, u) \in D$, this set contains a countable number of saddle periodic orbits, which are dense in $\Lambda_{(v,u)}$. Moreover, every two periodic orbits in $\Lambda_{(v,u)}$ are connected by a transversal heteroclinic orbit, which also belongs to $\Lambda_{(v,u)}$. It is well known that the dynamics on the Smale horseshoe can be described using the language of Symbolic Dynamics. We define

$$\Lambda := \bigcup_{(v,u) \in D} \Lambda_{(v,u)} .$$

Consider a finite family of periodic orbits in Λ. Let $J_k : D \to \mathbb{R}$ denote the corresponding actions and T_k be their periods, $k = 1, \dots, n$.

Let Φ_k^τ be the Hamiltonian flow with Hamiltonian function J_k and the symplectic form $\Omega_k = T_k dv \wedge du$. It is defined by the Hamiltonian equations (4) with $J = J_k$ and $T = T_k$. For every point $z = (v, u) \in D$ we define

$$\sigma_k(z) = \sup\{ \tau : \Phi_k^{\tau'}(z) \in D \text{ for all } \tau' \in (0, \tau) \} ,$$

which is the time required to leave the domain D. If the trajectory is defined for all $\tau > 0$ we set $\sigma_k(z) = +\infty$. Obviously, $\sigma_k(z) > 0$ for any $z \in D$ and $k = 1, \dots, n$ due to openness of D.

We say that $\Gamma : [0, T] \to D$ is *an accessible path* if Γ is a piecewise smooth curve composed from a finite number of forward trajectories of the Hamiltonian systems generated by J_k.

More formally, Γ is an accessible path if there are $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ such that the sequence of points $z_i := \Gamma(\tau_i)$ breaks the curve Γ into trajectories, i.e., for every i, $0 \le i < N$, there is k_i, $1 \le k_i \le n$, such that

$$\Gamma(\tau) = \Phi_{k_i}^{\tau-\tau_i}(z_i)$$

for $\tau \in [\tau_i, \tau_{i+1}]$. Of course, the curve Γ is well defined only if

$$0 < \tau_{i+1} - \tau_i < \sigma_{k_i}(z_i)$$

which ensures that the trajectories do not leave the domain D.

Theorem 1. *If D is a bounded domain in $\mathbb{R}^{2d}$, the frozen fast system satisfies assumptions* [A1] *and* [A2], *$\{J_k\}_{k=1}^n$ is a set of actions corresponding to a finite set of frozen periodic orbits in Λ, and Γ is an accessible path, then there is a constant $C_0 > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ there is a trajectory of the full system* (1) *such that its slow component $z(t)$ satisfies*

$$\|z(t) - \Gamma(\varepsilon t)\| < C_0 \varepsilon$$

provided $0 \le t \le \varepsilon^{-1} T$.

For any $z_0, z_1 \in D$, we say that z_1 is *accessible* from z_0 via the system $\{J_k\}$ if there is an accessible path such that $\Gamma(0) = z_0$ and $\Gamma(T) = z_1$.

In the case $d = 1$ the accessibility property has a simple geometrical meaning since trajectories of the Hamiltonian systems generated by J_k are level lines of the functions J_k. In this case the theorem provides trajectories which follow segments of the level lines. The main obstacle for the drift in the slow space is provided by level lines common for all J_k.

Consider actions generated by two periodic orbits, a and b. Those level lines of $J_{a,b}$, which are inside D, are closed curves. The non-singular level lines form rings (or disks), D_a and D_b. Let $V = D_a \cap D_b \subset D$. If J_a and J_b do not have common level lines, then any point $z_1 \in V$ is accessible from any point $z_0 \in V$.

Under the same assumptions. Let us take any finite family of open sets $V_i \subset V$, which do not depend on ε. Then for all sufficiently small ε, there is a trajectory which visits all the sets V_i.

We note that this theorem has an obvious generalisation to the case when the frozen systems have a family of uniformly hyperbolic invariant subsets such that that the dynamics on the latter is conjugated to a suspension of a transitive subshift of finite type.

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