CONTROL OVER THE TRAINING PERFORMANCE OF QUANTUM STATE TOMOGRAPHY WITH RESERVOIR COMPUTING NETWORKS

Sergey Borisenok [1,2]

[1] Department of Electrical and Electronics Engineering
Faculty of Engineering
Abdullah Gül University
Kayseri, Türkiye
sergey.borisenok@agu.edu.tr

[2] Feza Gürsey Center for Physics and Mathematics
Boğaziçi University
Istanbul, Türkiye
borisenok@gmail.com

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Abstract

The evaluation of unknown states for a given quantum system is one of the key problems in quantum information processing. The most efficient method of state characterization is quantum state tomography (QST), where the full-density matrices are reconstructed from the experimental measurements or numerical simulations performed on quantum states. The improvement of the computational performance in quantum state tomography and its related problems is a challenging task for modern theoretical physics. The general scheme of computing deals with the input information that goes into a quantum reservoir through a recurrent evolution. After the evolution, the final output is obtained as the linear combination of the readout elements.

In our approach, the quantum reservoir is modeled with the Lindbladian equation. The control over performance is made by the coherent coupling parameter between the input quantum state and the reservoir. The control feedback algorithm is represented with the set of Kolesnikov's target attractor algorithm to drive certain parameters of quantum state tomography, particularly, the outputs for the density matrix. Here we formulate the target attractor feedback in a discrete form to improve the training performance of QST and then develop a basic example of the state tomography for the quantum system of spin 1/2. We conclude by mentioning the basic features of our algorithm and its possible development.

Key words

Quantum informatics, reservoir computing networks, quantum state tomography, non-linear feedback algorithms, target attractor.

1 Introduction

Cybernetic methods have a long history of their applications to different quantum systems [Fradkov, 2007; Girolami, Schmidt, et al., 2015]. Adaptive control [Uzhva, Granichin, 2021; Granichin, Uzhva, et al., 2022] and machine learning [Knyazev, Pershin, et al. 2023] algorithms serve as efficient tools for control over the states of complex networks. An inspiring perspective is hidden also in the implementation of small non-linear neuron populations to control over performance of quantum machines [Borisenok, 2022].

The concept of reservoir computing (RC) originates in algorithms based on stable learning at a real-time lower computational cost, such as echo state networks and liquid state machines [Suzuki, Gao, et al., 2022]. The main part of RC is a so-called 'reservoir': a high-dimensional dynamical system forms a neural network with fixed random connections, such architecture allows to avoid the overhead of controlling a large number of connections [Schrauwen, Verstraeten, et al., 2007]. The reservoir gets the temporal input data, adjusts the weights of the readout signal by training, and approximates the target output signal [Mujal, Martinez-Pena, et al., 2021].

Different physical realizations of reservoirs are based on electronic circuits, photonic systems, spintronic systems, mechanical machines (soft and compliant robots), and even biological networks (in-vitro cultured cells) [Tanaka, Yamane, et al., 2019].

Quantum reservoir computing systems have features that could not be simulated on conventional classical computers [Fujii, Nakajima, 2021]. The evaluation of unknown states for a given quantum system is one of the key problems of quantum information processing.

In the typical architecture of a quantum reservoir network, the information about the input system enters the quantum reservoir, which goes through a recurrent evolution. After the evolution, the final output is taken as the linear combination of the readout elements. The weights in the system are of two types: the weights representing the coupling of the reservoir with the input and output layers, and the bidirectional recurrent weights connecting the reservoir nodes [Ghosh, Nakajima, et al., 2021].

The most efficient method of state characterization is quantum state tomography (QST), where the density matrices are reconstructed from experimental measurements or numerical simulations performed on quantum states [Ghosh, Nakajima, et al., 2021]. The improvement of the computational performance in quantum state tomography and its related problems is a challenging task for modern theoretical physics. The general scheme of computing deals with the input information that goes into a quantum reservoir through a recurrent evolution. After the evolution, the final output is obtained as the linear combination of the readout elements [Ghosh, Opala, et al., 2019].

Alternative approaches to QST use the universal compilation method for the training process [Hai, Ho, 2023], tensor network representations of mixed states through locally purified density operators [Guo, Yang, 2024], method of the high-performance lightweight in training a truncated density matrix [Hsieh, Chen, et al., 2024], constructing confidence regions [De Gois, Kleinmann, 2024], the tailor-made protocol [Binosi, Garberoglio, et al., 2024], attention-based neural network architecture [Palmieri, Müller-Rigat, et al., 2024], and others.

During the last years, QST devices started to get the experimental implementation for qubits [Ivanova-Rohling, Rohling, et al., 2023; Aasen A. S., Di Giovanni, et al., 2024; Hu, Wei, et al., 2024], photons [Czerwinski, 2024; Fuenzalida, Kysela, et al., 2024], nitrogen-vacancy centers in diamond [Zhang, Hegde, et al., 2023].

In our case, the quantum reservoir is modeled with the Lindbladian equation. The control over performance is made by the coherent coupling parameter between the input quantum state and the reservoir [Pechen, Borisenok, 2015; Pechen, Borisenok, et al, 2022]. Usually, the control during the quantum state tomography is performed as an open-loop (feedforward) scheme [Ghosh, Nakajima, et al., 2021; Ghosh, Opala, et al., 2021], but here we discuss the closed-loop (feedback) algorithm.

In Section 2 we present briefly the main ideas of QST. Then in Section 3, the control feedback algorithm is represented with the discrete analog of Kolesnikov's Target Attractor approach [Kolesnikov, 2013] to improve the training performance of QST, In Section 4 we develop a basic example of the state tomography for the quantum system of spin 1/2. In Section 5 we conclude by mentioning the basic features of our algorithm and its possible development.

2 Quantum state tomography

Quantum state tomography is a method to reconstruct the density matrix of a given quantum system [Banaszek, Cramer, et al., 2013; Toninelli, Ndagano, et al., 2019; Czerwinski, 2022]. In finite D-dimensional Hilbert space, the density matrix is described by $D^2 - 1$ independent real-valued parameters [Ghosh, Oppala, et al., 2021]. The training states in the reservoir need to be linearly independent, by the set of D^2 randomly generated quantum states can be sufficient [Ghosh, Oppala, et al., 2021].

The reservoir dynamics are described by the equation [Ghosh, Nakajima, et al., 2021]:

$$i\hbar \frac{d\rho}{dt} = \left[\hat{H}_R, \rho\right] + \frac{i\gamma}{2}\hat{L}(\rho) + \hat{T}_{\rm int}(\rho) \,, \tag{1}$$

where $\hat{H}_R$ is the reservoir Hamiltonian, $\hat{L}(\rho)$ is a Lindbladian operator describing the dissipation in the system; and $\hat{T}_{\rm int}(\rho)$ is the operator activating the coupling between the input modes and the reservoir.

The process of quantum state tomography can be performed in the following steps [Ghosh, Opala, et al., 2021]:

1. Initially the reservoir stays in the vacuum state or the excited state only with the uniform field.
2. A coupling between the input modes and the reservoir is activated through a cascade coupling as a set of the Heaviside delta-functions or through a coherent coupling, see [Ghosh, Opala, et al., 2019; Ghosh, Nakajima, et al., 2021; Ghosh, Opala, et al., 2021] for details.
3. The vector $\mathbf{n}$ consisting of the occupation numbers n_j of each readout mode is measured.
4. The desired output is evaluated as:

$$\mathbf{Y}^{\rm out} = \mathbf{M}^{\rm out}\mathbf{n} + \mathbf{m} \,. \tag{2}$$

In (2), the output weight matrix $\mathbf{M}^{\rm out}$ and the constant vector $\mathbf{m}$ are determined through the training process.

For the training stage, we chose: $\mathbf{Y}^{out} = \rho_{\rm in}$ (here the density matrix is arranged in a column vector format). Then in the tomographic process, we expect that: $\rho_{\rm in} = \mathbf{M}^{\rm out}\mathbf{n} + \mathbf{m}$.

In reality, the vector representation of the density matrix reconstructed in the process of tomography is given by:

$$\rho_{\rm in}^{\rm tomo} = \mathbf{M}^{\rm out}\mathbf{n} + \mathbf{m} \,. \tag{3}$$

To evaluate the error of QST, the fidelity is defined as [Ghosh, Opala, et al., 2021; Ghosh, Krisnanda, et al., 2021]:

$$F = \left(\mathrm{Tr} \left[\sqrt{\sqrt{\rho_{\mathrm{in}}}\, \rho_{\mathrm{in}}^{\mathrm{tomo}} \sqrt{\rho_{\mathrm{in}}}} \right] \right)^2 . \tag{4}$$

For the multiple inputs $j = 1, ..., N$, the fidelity (4) should be computed for each input separately: F_j, and then the average fidelity is calculated:

$$\bar{F} = \frac{1}{N} \sum_{j=1}^{N} F_j . \tag{5}$$

In the case of ideal error-free quantum tomography, the fidelity (4)-(5) must be equal to 1: $F = 1$, otherwise: $F < 1$ [Ghosh, Opala, et al., 2021].

3 Control over the performance of quantum state tomography

The absolute majority of theoretical [Thew, Nemoto, et al., 2002; Choi, 2022] and experimental [Oren, Mutzafi, et al., 2017; Torlai, Mazzola, et al., 2018; Striker, Meth, et al., 2022] algorithms for QST follow open-loop patterns, some methods also cover inherent noise in the quantum systems [Schmied, 2016; Gupta, Xia, et al., 2021]

To design a control over the quantum tomography performance, we use here the concept of creating an artificial target attractor in the phase space locking the system dynamics in the neighborhood of the target subspace. Such a method has been proposed By Kolesnikov in the form of tracking feedback control [Kolesnikov, 2013].

In this paper, we focus on the improvement of training performance to reproduce the desired output (2).

3.1 Target attractor control over the QST training performance

In the continuous formulation of target attractor feedback, one defines a goal function ψ to form an artificial target attractor in the dynamical system as:

$$\psi(t) = v(t) - v_t(t) . \tag{6}$$

In (6), $v(t)$ represents a target variable, while $v_t(t)$ stays for the target function, i.e. we make a control tracking for the variable v. Then one demands the exponential convergence of the goal function:

$$T \frac{d\psi}{dt} = -\psi . \tag{7}$$

Here T is an arbitrary positive constant, corresponding to the typical time scale of the control. The solution to Eq.(6) has an exponential decay:

$$\psi(t) = e^{-t/T} \psi(0) . \tag{8}$$

We formulate here a matrix discrete analog of Kolesnikov's algorithm (6) as:

$$\psi_k = \mathbf{Y}_k^{\mathrm{out}} - \mathbf{Y}_t^{\mathrm{out}} , \tag{9}$$

with the target vector outcome:

$$\mathbf{Y}_t^{\mathrm{out}} = \mathbf{M}_t \mathbf{n} + \mathbf{m} . \tag{10}$$

This outcome is a sample for the process of network training. The target outcome contains the target matrix $\mathbf{M}_t$. We demand the exponential convergence of the control procedure, like in (8), as the discrete step k increases:

$$\psi_k = e^{-\gamma k} \psi_0 \; ; \; \gamma = \mathrm{const} > 0 . \tag{11}$$

The training procedure for the discrete Kolesnikov algorithm looks at the k-th steps as:

$$\mathbf{Y}_k^{\mathrm{out}} = \mathbf{Y}_t^{\mathrm{out}} + e^{-\gamma k} (\mathbf{Y}_0^{\mathrm{out}} - \mathbf{Y}_t^{\mathrm{out}}) . \tag{12}$$

Presenting $\mathbf{Y}_k^{\mathrm{out}}$ in the form:

$$\mathbf{Y}_k^{\mathrm{out}} = \mathbf{M}_k \mathbf{n}_k + \mathbf{m} , \tag{13}$$

we get by (12):

$$\mathbf{M}_k \mathbf{n}_k = \mathbf{M}_t \mathbf{n} + e^{-\gamma k} (\mathbf{M}_0 \mathbf{n}_0 - \mathbf{M}_t \mathbf{n}) . \tag{14}$$

Asymptotically, by (13) and (14) we can evaluate:

$$\begin{aligned} \mathbf{Y}_k^{\mathrm{out}} &\to \mathbf{Y}_t^{\mathrm{out}} \ \text{ as } \ k \to \infty ; \\ \mathbf{M}_k \mathbf{n}_k &\to \mathbf{M}_t \mathbf{n} \ \text{ as } \ k \to \infty ; \\ \mathbf{m} &= \ \text{const for each } \ k . \end{aligned} \tag{15}$$

Thus, the discrete algorithm is converging exponentially fast to the target vector $\mathbf{Y}_t^{\mathrm{out}}$ for any given set of $\mathbf{n}$ and initial conditions.

In practice, we do not know the target $\mathbf{Y}_t^{\mathrm{out}}$. So, we need to evaluate it for each k-th step as $\mathbf{Y}_{t,k}^{\mathrm{out}}$ from the series $\mathbf{Y}_t^{\mathrm{out}}$. Then by (12):

$$\mathbf{Y}_{t,k}^{\mathrm{out}} = \frac{\mathbf{Y}_k^{\mathrm{out}} - e^{-\gamma k} \mathbf{Y}_0^{\mathrm{out}}}{1 - e^{-\gamma k}} . \tag{16}$$

Similarly, we restore the matrix $\mathbf{M}_k$ by (13):

$$\mathbf{M}_k \mathbf{n}_k = \mathbf{Y}_k^{\mathrm{out}} - \mathbf{m} . \tag{17}$$

Eqs (16)-(17) are the main result of our discrete analog for Kolesnikov's approach (6)-(8). Now we can reformulate it in the finalized algorithmic form.

3.2 Finalization of the control algorithm

The finalized form of our discrete feedback Kolesnikov-type algorithm becomes:

Step 1. Define the vector $\mathbf{m}$.

Step 2. For the 0-th step, get the occupation number vector $\mathbf{n}_0$ and define the outcome density matrix:

$$\mathbf{Y}_0^{\mathrm{out}} = \rho_{\mathrm{in},0} \ , \tag{18}$$

arranged in a column vector format, and the matrix

$$\mathbf{M}_0\mathbf{n}_0 = \mathbf{Y}_0^{\mathrm{out}} - \mathbf{Y}_t^{\mathrm{out}} \ . \tag{19}$$

Step 3. For the next k-th step, correct the occupation number vector $\mathbf{n}_k$ if necessary, get the outcome $\mathbf{Y}_k^{\mathrm{out}}$, and compute:

$$\mathbf{Y}_{t,k}^{\mathrm{out}} = \frac{\mathbf{Y}_k^{\mathrm{out}} - e^{-\gamma k}\mathbf{Y}_0^{\mathrm{out}}}{1 - e^{-\gamma k}} \ , \tag{20}$$

and

$$\mathbf{M}_k\mathbf{n}_k = \mathbf{Y}_k^{\mathrm{out}} - \mathbf{m} \ . \tag{21}$$

Step 4. Repeat Step 3 for the next $(k+1)$-th algorithmic cycle, until the required calculation precision is achieved.

Step 5. Finally, for the last K-th step, convert $\mathbf{Y}_{t,K}^{\mathrm{out}}$ back into the matrix form:

$$\rho_{\mathrm{in},K} = \mathbf{Y}_{t,K}^{\mathrm{out}} \ , \tag{22}$$

and finalize $\mathbf{M}$.

Thus, each vector and matrix variable will be computed step-by-step in the frame of an exponentially converging Kolesnikov-type algorithm.

4 Basic example: Tomography for the spin 1/2

The system of a single spin 1/2 without decay has a 2x2 density matrix as [Blum, 1996]:

$$\rho = \frac{1}{2}\left(\mathbf{I} + \mathbf{b}\cdot\sigma\right) \ , \tag{23}$$

where $\mathbf{I}$ is the identity matrix, and the vector σ is based on the Pauli matrices σ_i, where $i = x, y, z$.

The Bloch vector $\mathbf{b}$ determines whether the state of the spin 1/2 is pure ($|\mathbf{b}| = 1$, i.e. $\rho^2 = \rho$) or mixed ($|\mathbf{b}| < 1$). A totally mixed state:

$$\rho_{\mathrm{mix}} = \frac{1}{2}\mathbf{I} \tag{24}$$

corresponds to $b_i = 0$ for all i.

The constant vector $\mathbf{m}$ must be independent on the Bloch vector $\mathbf{b}$, thus:

$$\mathbf{m} = \frac{1}{2}\mathbf{I} \ . \tag{25}$$

Let's take the state (24) as the target $\mathbf{Y}_t^{\mathrm{out}}$:

$$\mathbf{Y}_t^{\mathrm{out}} = \frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0.500\\0.000\\0.000\\0.500\end{pmatrix} \ . \tag{26}$$

Now we check whether we can reproduce (26) with our algorithm.

Let's choose the arbitrary initial state based on the Pauli x-matrix as the 0-th outcome:

$$\rho_{\mathrm{in},0} = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \ , \tag{27}$$

which at the vector presentation corresponds to:

$$\mathbf{Y}_0^{\mathrm{out}} = \begin{pmatrix}0.000\\0.500\\0.500\\0.000\end{pmatrix} \ . \tag{28}$$

By that, we start from a totally wrong initial state.

To check the stability of our algorithm under the noisy perturbation of the vector $\mathbf{Y}_k^{\mathrm{out}}$, we add the white noise emulated by a random number generator at the level of 0.01γ taking numerically $\gamma = 1$.

By Step 3 of the algorithm, we obtain:

$$\mathbf{Y}_1^{\mathrm{out}} = \begin{pmatrix}0.488\\0.040\\0.017\\0.499\end{pmatrix} \ ; \ \mathbf{Y}_{t,1}^{\mathrm{out}} = \begin{pmatrix}0.386\\0.025\\0.011\\0.395\end{pmatrix} \ ;$$

$$\mathbf{Y}_2^{\mathrm{out}} = \begin{pmatrix}0.489\\0.024\\0.012\\0.494\end{pmatrix} \ ; \ \mathbf{Y}_{t,2}^{\mathrm{out}} = \begin{pmatrix}0.566\\0.023\\0.010\\0.572\end{pmatrix} \ ;$$

$$\mathbf{Y}_3^{\mathrm{out}} = \begin{pmatrix}0.457\\0.036\\0.049\\0.461\end{pmatrix} \ ; \ \mathbf{Y}_{t,3}^{\mathrm{out}} = \begin{pmatrix}0.481\\0.038\\0.051\\0.486\end{pmatrix} \ ;$$

$$\mathbf{Y}_4^{\mathrm{out}} = \begin{pmatrix}0.466\\0.009\\0.021\\0.485\end{pmatrix} \ ; \ \mathbf{Y}_{t,4}^{\mathrm{out}} = \begin{pmatrix}0.475\\0.009\\0.021\\0.494\end{pmatrix} \ ,$$

and so on.

Thus, practically starting from the 3-rd iteration, the restored density matrix up to the noise precision is:

$$\rho_{\mathrm{in},4} = \begin{pmatrix}0.475 & 0.009\\0.021 & 0.494\end{pmatrix}$$

versus exact:

$$\rho_{\text{in},4} = \begin{pmatrix} 0.500 & 0.000 \\ 0.000 & 0.500 \end{pmatrix}.$$

The same algorithm is applied to M. For the fermionic case, let's consider the constant occupation numbers:

$$\mathbf{n}_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{29}$$

Then by (21):

$$\mathbf{M}_0\mathbf{n}_0 = \begin{pmatrix} -0.500 \\ +0.500 \\ +0.500 \\ -0.500 \end{pmatrix},$$

and

$$\mathbf{M}_1\mathbf{n}_1 = \begin{pmatrix} -0.114 \\ +0.025 \\ +0.011 \\ -0.105 \end{pmatrix};\ \mathbf{M}_2\mathbf{n}_2 = \begin{pmatrix} +0.066 \\ +0.023 \\ +0.010 \\ +0.072 \end{pmatrix};$$

$$\mathbf{M}_3\mathbf{n}_3 = \begin{pmatrix} -0.019 \\ +0.038 \\ +0.051 \\ -0.014 \end{pmatrix};\ \mathbf{M}_4\mathbf{n}_4 = \begin{pmatrix} -0.025 \\ +0.009 \\ +0.021 \\ -0.006 \end{pmatrix},$$

versus the exact:

$$\mathbf{M}_t\mathbf{n} = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix}.$$

A similar approach can be applied to two spin-1/2 particles [Johnston, 2024] and to spin-1/2 XXZ chains [Sato, Shiroishi, 2007].

5 Conclusions

The feedback control algorithm over the performance of quantum state tomography proposed here demonstrates many advantages:
- The invented algorithm works for the optimization of the computational sources and decreases the computational cost in the real-time numerical process.
- The algorithm is robust in the sense that its dependence on the initial conditions decays exponentially.
- The algorithm is stable under a relatively small noise perturbation.
- The algorithm is valid for different types of interaction between the input modes and the reservoir: cascade coupling, and coherent coupling.

The algorithm also can be easily extended to different control goals: preparation, estimation, and reconstruction of quantum states, quantum computing, compressing quantum circuits, and others.

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