ON THE CLASS OF OPTIMAL CONTROL OF 3-STEP NILPOTENT SYSTEMS

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Abstract

This paper addresses the study of non-holonomic systems that are written by means of invariant distributions satisfying the property that the Lie algebra generated by the distribution is 3-step nilpotent. A system in that class can be written as an optimal control problem with a plant that is affine in the control parameters and a cost which is given by the kinetic energy of the system. Standard techniques in optimal control theory provide necessary conditions for the extremal trajectories. The paper presents a general theory of this class of systems along with detailed calculations for some low dimension cases.

Key words

Non-holonomic system, 3-step nilpotent Lie algebra, extremal curves.

1 Introduction

Classical mechanics is a venerable and broad subject that includes rigid bodies, fluid mechanics, elasticity, electrodynamics, etc., this body of knowledge brings together in a beautiful coherent fashion several branches of mathematics, and lately has incorporated methods of geometric control theory, see for instance [Marsden, 1992] and [Bloch, 2003] .

The common wisdom about mechanics starts by the so-called Lagrangian formalism, introduced by J.L. Lagrange himself around 1790. He considered generalized coordinates q and velocities $\dot{q}$ for describing the state of a mechanical system, and by taking into consideration the covariance of these quantities, he introduced the Lagrangian of the system $\mathcal{L}(q,\dot{q})$, (kinetic energy minus the potential energy), and derived what now are called the Euler-Lagrange equations. Later on, around 1830, W.R. Hamilton explained how to derive these equations from a variational principle, (the principle of critical action). In modern language, the pairs $(q,\dot{q})$ are the elements of the tangent bundle of the configuration space, whereas the pairs (q,p), with $p=\mathcal{L}_{\dot{q}}$ are the elements of the cotangent bundle. In these variables, the Hamiltonian function is defined by $\mathcal{H}(q,p) = \langle p,\dot{q}\rangle - \mathcal{L}(q,\dot{q})$ and the Euler-Lagrange equations become the so-called Hamiltonian equations which are endowed with a extremely rich geometric structure.

In the nineteenth century the German physicist H.R. Hertz (1857-1894) coined the term holonomic, (from the Greek roots *hólos* meaning *whole* and *nomos* meaning *law*), for describing some mechanical systems subject to velocity constraints. Generally speaking a system is said to be non-holonomic with respect to a given constrained motion, if the system can evolve between any two given configurations without violating the constraints, otherwise is said to be holonomic. A prototypical non-holonomic system is the one of a sphere rolling over the plane without slipping and twisting.

There is a large amount of literature regarding holonomic and non-holonomic constraints for mechanical systems, we refer the reader to the classical book in classical mechanics [Whittaker, 1988] and also [Neimark and Fufaev, 1972], both containing plenty of interesting examples.

To be precise, assume that $q=(q_1,\dots,q_n)$ denotes the coordinates of the configuration space of the system, and that the evolution of the system obeys to an ensemble of m linear constraints on the velocities written as $\sum_{i=1}^{n}\alpha_{ij}(q)\dot{q}_i = 0, \quad j=1,\dots,m$. If it is possible to find constraints on the position only, say $\beta_1(q) = \cdots = \beta_m(q) = 0$, in such a way that, $\sum_{i=1}^{n}\frac{\partial\beta_j}{\partial q_i}\dot{q}_i = 0, \quad j=1,\dots,m$, determines the same ensemble of constraints for the system, then it is said that the constraints are *holonomic*, otherwise they are called *non-holonomic*.

In the control theory literature non-holonomic systems appear as models of mechanical systems with external forces, the constraints come up in two flavors: the ones that are obtained from the derivation of the equations of motion from the Euler-Lagrange equations (or from the Hamiltonian formalism), such constraints are not imposed on the system and it is better to take them as conservation laws rather that as genuine constraints; and those constraints that are direct consequence of the kinematics, such as the non slipping and twisting of the rolling, see for instance [Bloch, 2003].

An important class of non-holonomic control systems is the one of driftless control-affine systems, such systems are defined either by a finite family of vector fields or by a Pfaffian system, that is, by the kernel of a finite family of differential 1-forms.

A relevant example of this class is the so-called Goursat chained form, which provides a model for a robot towing a finite number of trailers, all of them satisfying the non slipping and twisting rolling conditions. If Q denotes the configuration space of this mechanical system, the dynamical equation can be written as $\dot{q} = u_1X_1(q) + u_2X_2(q),\ q \in Q$, where u_1 and u_2 are the control parameters of the velocity of the center of mass of the leading robot, and the vector fields X_1 and X_2 satisfy the following commuting relations: $[X_1, X_i] =: X_{i+1},\ i = 2,\dots,n$, with $n = \dim Q$, (of course n depends on the number of trailers), and all other commutators are zero, for details see [Tilbury *et al.*, 1995]. A very interesting variation of this system is the kinematic model for a rolling ball towing a trailer, again satisfying the non slipping and twisting rolling conditions, this system can also be written as $\dot{q} = u_1X_1(q) + u_2X_2(q)$ and has been recently introduced in [Boizot and Gauthier, 2013]. An important feature of the former example is that the Lie algebra generated by the set of vector fields is already nilpotent, whereas for the later a nilpotent approximation can be explicitly calculated as it is done in the aforementioned reference .

The property of a system of being nilpotent presents theoretical and computational advantages for tackling various problems in control: optimal synthesis, path planning, small time controllability, stability etc. However it is a very strong condition to impose, in part for this reason techniques of nilpotent approximations for control systems have been extensively developed.

In this paper we consider the optimal control problem of a driftless control-affine system with quadratic cost, for which the Lie algebra generated by the vector fields defining the system is 3-step nilpotent, that is, Lie brackets of length greater that three vanish.

Apart from this introduction the paper contains five sections, in section 2 we present a characterization of non-holonomic control systems and the basic definitions of nilpotent Lie algebras and nilpotent approximations. In section 3 we formulate the optimal control problem and apply the Pontryagin Maximum Principle that provides necessary conditions for the optimal controls. In section 4 we specialize the general results to some low dimensional cases, in particular we discuss the so-called cross-chained form. At the end in section 5 we derive some conclusions and perspectives of future work on the study of non-holonomic nilpotent control systems.

2 Non-Holonomic Nilpotent Systems

We present in this section a characterization of non-holonomic control systems and the basic definitions of nilpotent Lie algebras and nilpotent approximations.

2.1 Non-Holonomic Control Systems

Let G be a smooth manifold, and let $\Delta = \{X_1,\dots,X_n\}$ with $n < \dim(G)$ be a distribution of smooth vector fields on G, the Lie algebra generated by the distribution is denoted as $\mathcal{G}(\Delta)$ and it consists of the Lie algebra spaned by iterations of all the Lie brackets of elements of Δ. It is said that the distribution Δ is of *full rank*,[1] if for all $g \in G$ it holds that $\mathcal{G}(\Delta)_g = T_gG$.

For $k = 1, 2, \dots$, the modules of vector fields Δ^j are defined inductively as follows: $\Delta^1 := \Delta$ and $\Delta^{k+1} := \Delta^k + [\Delta, \Delta^k]$. For each $g \in G$, the full rank condition implies the existence of an integer $\nu(g)$ such that $\Delta_g^{\nu(g)} = T_gG$, together with a flag of modules of vector fields naturally defined as

$$\Delta_g^1 \subset \Delta_g^2 \subset \cdots \subset \Delta_g^{\nu(g)} = T_gG.$$

Furthermore if $n_i(g) = \dim \Delta_g^i,\ i = 1,\dots,\nu$, then the vector $(n_1(g), n_2(g),\dots,n_\nu(g))$ is called the *growth vector* of the distribution Δ at g, and $\nu(g)$ the *nonholonomy degree* of the distribution at g. The distribution is said to be regular at g if the growth vector is constant on a neighborhood of g, it is said to be regular in G if it is regular for all g with the same degree of nonholonomy.

Assume that Δ is a full rank, regular distribution of vector fields in G, an absolutely continuous curve $g : [0, T_g] \to G$ is said to be *admissible* for the distribution Δ if satisfies $\dot{g}(t) \in \Delta(g(t)), a.e.$, which is tantamount of saying that, there is a measurable and bounded function $t \mapsto \mathbf{u} = (u_1,\dots,u_n)$ such that $g(t)$ is an admissible solution of the following control-affine system:

$$\dot{g}(t) = \sum_{i=1}^{n} u_iX_i(g(t)), \tag{1}$$

that we shall call it a *non-holonomic control system* on G; the family of all admissible curves is denoted as $\mathcal{A}$, whereas the one of admissible control parameters is

[1]This condition goes also in the literature under the names of bracket generating distribution, Lie algebra rank condition or Hörmander condition.

denoted as $\mathcal{U}$. It is known that the full rank condition guarantees the controllability of the system, see for instance [Jurjdevic, 1997]. A smooth varying inner product $g \mapsto \langle \cdot, \cdot \rangle_g$ on the vector spaces Δ_g can be defined by means of $\langle X_i, X_j \rangle_g = \delta_{ij}$, in such a way that the energy functional $\Lambda : \mathcal{A} \to \mathbb{R}^+$ for admissible curves is written as follows:

$$\Lambda(g, \mathbf{u}) = \int_0^{T_g} \langle \dot{g}(t), \dot{g}(t) \rangle = \int_0^{T_g} u_1^2 + \cdots + u_n^2. \quad (2)$$

In this paper we study the optimal control problem defined by (1) and (2), for the particular case when $\mathcal{G}(\Delta)$ is a 3-nilpotent and 2-solvable Lie algebra.

2.2 Nilpotent Lie Algebras and Control Systems

In order to set the notation for presenting our results, we shall digress on some aspects of structure theory of $n-$step nilpotent Lie algebras, for more details we refer the reader to [Jacobson, 1962] and [Corwin and Greenleaf, 1990].

2.3 Nilpotent and Solvable Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$, the *lower central series* is defined follows $\mathfrak{g} := \mathfrak{g}^1 \supseteq [\mathfrak{g}^2, \mathfrak{g}] \supseteq [\mathfrak{g}^3, \mathfrak{g}] \cdots$, where $\mathfrak{g}^j := [\mathfrak{g}^{j-1}, \mathfrak{g}]$ for $j = 2, 3, \ldots$ The Lie algebra is said to be *nilpotent* if there is an integer n such that $\mathfrak{g}^{n+1} = 0$, if such a n is minimal in the sense that $\mathfrak{g}^n \neq 0$ then the Lie algebra is said to be $n-$*step nilpotent*. The Jacobi identity together with an elementary induction argument clearly imply

$$[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j} \cup \text{for all } i \text{ and } j. \quad (3)$$

As a consequence, any product of k elements of $\mathfrak{g}$ is an element of $\mathfrak{g}^k$, independently of the order. Furthermore, $\mathfrak{g}$ is $n-$step nilpotent if and only if all brackets of order greater that n vanish. A typical element $x \in \mathfrak{g}^k$ is the *monomial* $[X_{i_1}[X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]]$, with $\{X_{i_1}, X_{i_2}, \ldots, X_{i_k}\} \subset \mathfrak{g}$, the degree of an such element is naturally defined as $\deg(x) = k$.

In a similar manner, the *derived series* of $\mathfrak{g}$ is defined inductively as follows $\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \cdots$, with $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{(j)} = [\mathfrak{g}^{(j-1)}, \mathfrak{g}^{(j-1)}]$ for $j = 2, 3, \ldots$. The Lie algebra is said to be *solvable* if there is an integer m such that $\mathfrak{g}^{(m)} = 0$. For then, relation (3) implies $\mathfrak{g}^{(k)} \supseteq \mathfrak{g}^{2^k}$, for all k, therefore nilpotent Lie algebras are also solvable.

In summary, it can be stated that nilpotency determines the length of non-trivial Lie monomials whereas the solvability counts for the shape of the brackets, for instance, nilpotenty three and solvability two implies that brackets longer than $[\cdot, [\cdot, \cdot]]$ are zero and $[[\cdot, \cdot], [\cdot, \cdot]]$ are not allowed.

There is a collecting process for organizing the commutators of free Lie algebras generated by a finite number of elements, that was originally presented by Philip Hall in [Hall, 1934] and has recently been utilized in applications to path planning problem and constructive controllability, see for instance [Laferriere and Sussmann, 1992].

Let $\mathfrak{g}$ be the free Lie algebra generated by the symbols $\{X_1, \ldots, X_p\}$ which are considered of being of degree one. For two given monomials m_1 and m_2 the relation $\deg[m_1, m_2] = \deg(m_1) + \deg(m_2)$ readily follows. A linear combination of monomials of degree k is said to be homogeneous of degree k, and any element of $\mathfrak{g}$ is written as linear combination of monomials.

Since there are only a finite number of monomials of a given degree, then for each n, a number of monomials, say $m_{n_1}, \ldots, m_{n_s}$, denominated *standard monomials*, which are linearly independent and have the property that each homogeneous expression of degree n is written as linear combination of $m_{n_1}, \ldots, m_{n_s}$, such a collection is defined recursively as follows:

Definition 2.1. *The standard monomials of degree one are $X_1, \ldots, X_p$. Assume that the standard monomials of degree $n-1$, are defined, and that they are $\prec$ $-$ordered in such a way that $u \prec v$ provided deg$(u) <$ deg(v). If deg$(x) = i$, deg$(v) = j$ and deg$[x, v] = i + j$, then $[x, v]$ is a standard monomial if and only if satisfies:*

1. *x and v are standard monomials with $x \prec v$.*
2. *If $v = [y, z]$ then $y \preceq x$ and $y \prec z$.*

An element of the free Lie algebra $\mathfrak{g}$ is said to be in standard form if it is written as linear combination of standard monomials.

Theorem 2.1. (M. Hall, 1950) *The standard monomials form a basis for the free Lie algebra $\mathfrak{g}$ generated by $X_1, \ldots, X_p$.*

Applying this process to a 3-step nilpotent free Lie algebra generated by $\{X_1, X_2, \ldots\}$ one has that a basis is given by

$$\begin{aligned} [X_{i_1}, X_{i_2}] &= X_{i_1 i_2}, \quad i_1 < i_2, \\ [X_{i_{12}}, X_{i_1 i_2}] &= X_{i_{12}, i_1 i_2}, \ i_1 < i_2 \leq i_{12}; \\ [X_{i_2}, X_{i_1 i_{12}}] &= X_{i_2, i_1 i_{12}}, \ i_1 \leq i_2 < i_{12}, \end{aligned}$$

and the remaining elements are again $X_{i_2, i_1} = -X_{i_1, i_2}$ and $[X_{i_1}, X_{i_2 i_{12}}] = X_{i_{12}, i_1 i_2} - X_{i_2, i_1 i_{12}}$, $i_1 < i_2 < i_{12}$ with $i_1, i_2, i_{12} = 1, \ldots, n$.

Example 2.1. *The basis for a 3-step nilpotent free Lie algebra generated by seven symbols $\Delta = \{X_1, X_2, X_{12}, X_{112}, X_{212}, X_6, X_7\}$.*

The standard monomials of degree 2, are denoted as $X_{ij} = [X_i, X_j], i < j$. We have then the seven elements of Δ, and the 21 elements of length two,

$$\begin{aligned}
\Delta_{12} &= \{X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{17}\},\\
\Delta_{23} &= \{X_{23}, X_{24}, X_{25}, X_{26}, X_{27}\},\\
\Delta_{34} &= \{X_{34}, X_{35}, X_{36}, X_{37}\},\\
\Delta_{45} &= \{X_{45}, X_{46}, X_{47}\},\\
\Delta_{56} &= \{X_{56}, X_{57}\},\\
\Delta_{67} &= \{X_{67}\},
\end{aligned}$$

and if we denote as $\Delta_{ijk} = \{[X_i, X_{jk} \mid X_{jk} \in \Delta_{jk}]\}$, *we further have the following 112 elements of length three*

$$\begin{aligned}
\Delta_{112} \cup \Delta_{212} \cup \Delta_{312} \cup \Delta_{412} \cup \Delta_{512} \cup \Delta_{612} \cup \Delta_{712}\\
\cup\Delta_{223} \cup \Delta_{323} \cup \Delta_{423} \cup \Delta_{523} \cup \Delta_{623} \cup \Delta_{723}\\
\cup\Delta_{334} \cup \Delta_{434} \cup \Delta_{534} \cup \Delta_{634} \cup \Delta_{734}\\
\cup\Delta_{445} \cup \Delta_{545} \cup \Delta_{645} \cup \Delta_{645}\\
\cup\Delta_{556} \cup \Delta_{656} \cup \Delta_{756}\\
\cup\Delta_{667} \cup \Delta_{767}
\end{aligned}$$

2.4 Nilpotent Approximations

A nilpotent approximation of a distribution of vector fields is another family of vector fields with the same generic properties that further has the property of generating a nilpotent Lie algebra. The definition of nilpotent approximations is based on the notion of *order of smooth functions and vector fields*, see for instance [Vendittelli *et al.*, 2004].

Let G be a smooth manifold, and let $\Delta = \{X_1, \dots, X_n\} \subsetneq TG$ be a regular and full rank distribution of vector fields. A smooth function $f : G \to \mathbb{R}$ is said to be of *order* $\geq k$ at a point $g \in G$, if all its Lie derivatives, (with respect to vectors X_is) of order $\leq k-1$ vanish at g, if is of order $\geq k$ but is not of order $\geq k+1$ at g, then it is said to be of order k at g.

A vector field X is said to be of *order* $\geq k$ at a point $g \in G$ if for every j and every function f of order j at g, the function $X(f)$ has order $\geq k+j$ at g, again if X is of order $\geq k$ but is not of order $\geq k+1$ at g, then it is said to be of order k at g. From this definition of order it is clear that the X_is are of order -1, the brackets $[X_i, X_j]$ are of order -2, etc.

A family of vector fields $\widetilde{\Delta} = \{\widetilde{X}_1, \dots, \widetilde{X}_n\}$ is said to be a *nilpotent approximation* of Δ at g if

1. The vector fields $X_i - \widetilde{X}_i$ are of positive order at g
2. The Lie algebra $\mathcal{G}(\widetilde{\Delta})$ is κ-step nilpotent, with $\kappa > \nu(g)$, that is Lie brackets of length greater that κ vanish.

The explicit computation of nilpotent approximations for a given distribution is rather technical and is based on the existence of the so-called privileged coordinates. There are in the literature various algorithmic processes for finding nilpotent approximations, see for instance [Vendittelli *et al.*, 2004], however, those processes are far from the purposes of this paper. From now on we shall consider regular full rank distributions that generate 3-step nilpotent Lie algebras, or distributions that are 3-step nilpotent approximations of non-nilpotent ones

3 The Optimal Control Problem

Following the technique of completing the Philip Hall basis for finitely generated Lie algebras, it has been shown in [Monroy-Pérez and Anzaldo-Meneses, 2011] that a 3-step nilpotent, 2-solvable Lie algebra $\mathfrak{g}$ generated by a set Δ of n symbols has dimension at most

$$\eta := \underbrace{n}_{\Delta^1} + \underbrace{\frac{(n-1)n}{2}}_{\Delta^2} + \underbrace{\frac{(n-1)n(n+1)}{3}}_{\Delta^3}. \tag{4}$$

The associated Lie group and corresponding group law can be obtained by standard BCH techniques. It has also been shown in the aforementioned reference that good models for this situation are the Lie the subgroups of $\mathbb{R}^n \times \mathfrak{so}_n \times \mathbb{R}^{\mathrm{D}}$ where $\mathrm{D} = (n-1)n(n+1)/3$; in this case the group law can be written as follows:

$$g \odot h = (\alpha + \beta, a + b + \alpha \wedge \beta, \breve{a} + \breve{b} + \Gamma), \tag{5}$$

where

$$\begin{aligned}
\alpha \wedge \beta &= \frac{1}{2}(\alpha \otimes \beta^T - \beta \otimes \alpha^T), \quad \text{and}\\
\Gamma &= -\frac{1}{2}\varphi(a, \alpha, b, \beta) - \frac{1}{12}\zeta(\alpha \wedge \beta, \alpha, \beta),
\end{aligned}$$

for certain smooth functions φ and ζ.

For the remaining of the paper G shall be taken as the simply connected Lie group $\mathbb{R}^n \times \mathfrak{so}_n \times \mathbb{R}^{\mathrm{D}}$ of dimensión η with group law (5), whose Lie algebra $\mathfrak{g}$ is the 3-step nilpotent, 2-solvable Lie algebra generated by a given distribution $\Delta = \{X_1, \dots, X_n\} \subsetneq TG$ of left invariant vector fields on G.

As explained in section 2, the left invariant distribution Δ determines on the Lie group G an optimal control problem, namely: for certain given initial conditions find, among the solutions of (1), the one that minimizes the functional (2).

3.1 Hamiltonian Formalism

We approach the aforementioned optimal control problem by means of the Hamiltonian formalism on the cotangent bundle T^*G and the necessary condition for optimality given by the Pontryagin Maximum Principle of optimal control theory, see for instance [Agrachev and Sachkov, 2004] and [Jurdjevic, 1997]. We summarize in this paragraph the basic notation for the symplectic structure of the cotangent bundle and the Hamiltonian formalism.

The differential of the left translation $g \mapsto L_g$ on G, yields the trivialization of the cotangent bundle $T^*G \simeq G \times \mathfrak{g}^*$, in such a way that Hamiltonian functions corresponding to left invariant vector fields on G are linear functions on $\mathfrak{g}^*$. Each left invariant vector field X on G defines a Hamiltonian function H_X on T^*G as $H_X(g,p) = p(X(e))$.

The double bundle $T(T^*G)$ is identified with $(G \times \mathfrak{g}) \times (\mathfrak{g}^* \times \mathfrak{g}^*)$, and with this realization, any tangent vector $((g,X),(p,Y^*)) \in T(T^*G)$, is simply represented by means of the pair $(X,Y^*) \in \mathfrak{g} \times \mathfrak{g}^*$.

The canonical symplectic form on $T^*G \simeq G \times \mathfrak{g}^*$, allows to write, for each Hamiltonian function H on T^*G, the corresponding Hamiltonian vector field $\vec{H}(g,p) = (X(g,p), Y^*(g,p))$ as follows

$$\begin{aligned} X(g,p) &= \frac{\partial H}{\partial p}(g,p), \\ Y^*(g,p) &= -dL_g^*\left(\frac{\partial H}{\partial g}(g,p)\right) - (\mathrm{ad})^* X(p), \end{aligned}$$

or equivalently, integral curves $t \mapsto (g(t), p(t))$ of the Hamiltonian vector field $\vec{H}$, satisfy the Hamilton equations,

$$\begin{aligned} \frac{dg}{dt} &= dL_g\left(\frac{\partial H}{\partial p}\right), \quad \text{and} \\ \frac{dp}{dt} &= -dL_g^*\left(\frac{\partial H}{\partial g}\right) - \left((\mathrm{ad})^* \frac{\partial H}{\partial p}\right). \end{aligned}$$

For details on the representation of tangent and cotangent bundles of Lie groups, and the integral curves of the Hamiltonian lifting of left invariant vector fields, we refer the reader to V. Jurdjevic's book [Jurdjevic, 1997].

3.2 Pontryagin Maximum Principle

Following the notation explained in example 2.1, X_{ij} denotes the Lie bracket $[X_i, X_j]$ whereas X_{ijk} denotes $[X_i,[X_j,X_k]]$. The Hamiltonian functions associated to the left invariant vector fields X_i, X_{ij} and X_{ijk} shall be denoted as H_i, H_{ij} and H_{ijk}, respectively. These Hamiltonians span the dual Lie algebra $\mathfrak{g}^*$, endowed with the Poisson brackets that clearly satisfy the following commuting relations,

$$\{H_i, H_j\} = H_{ij} \text{ and } \{H_i, H_{jk}\} = H_{ijk},$$

evidently the Lie algebra $\mathfrak{g}^*$ is also 3-step nilpotent and 2-solvable.

Each admissible control $\mathbf{u} = (u_1, \ldots, u_n) \in \mathcal{U}$, determines a control dependent Hamiltonian function

$$\mathcal{H}_{\mathbf{u}}^{\lambda_0} = -\frac{\lambda_0}{2}\left[\, u_1^2 + \cdots + u_n^2\right] + u_1 H_1 + \cdots + u_n H_n.$$

Integral curves $t \mapsto \xi(t) = (g(t), p(t))$ of the corresponding Hamiltonian vector field $\vec{\mathcal{H}}_{\mathbf{u}}^{\lambda_0}$ are called extremal curves, the ones for $\lambda_0 \neq 0$, are called normal, whereas the ones for $\lambda_0 = 0$ are called abnormal. The necessary conditions for $\Lambda-$optimal trajectories, i.e., solutions of (1) that minimize (2), read as follows:

Theorem 3.1. *(Pontryagin Maximum Principle) If a trajectory $t \mapsto (g, \widehat{\mathbf{u}})$ is $\Lambda-$optimal then it is the projection of an extremal curve $t \mapsto \xi = (g,p)$, satisfying:*

i $\mathcal{H}_{\widehat{\mathbf{u}}}^{\lambda_0}(\xi) \geq \mathcal{H}_{\mathbf{v}}^{\lambda_0}(\xi)$, *for all* $\mathbf{v} \in \mathcal{U}$
ii $\mathcal{H}_{\widehat{\mathbf{u}}}^{0}(\xi)$ *is not identically zero.*

Remark 3.1. *Abnormal extremals, are trajectories independent of the cost functional, they play a very important role in the geometric analysis of the so-called optimal synthesis, these extremals deserve a careful treatment and shall not be discussed here. For the remaining of the paper we shall consider that all $\Lambda-$optimal trajectories are projections of normal extremals only.*

A direct application of the Pontryagin Maximum Principle yields the necessary condition for optimality. Observe that the dual variable can be expressed in terms of the dual basis as $(h, \omega, \check{\mathrm{H}}) \in \mathbb{R}^n \times \mathfrak{so}_n \times \mathbb{R}^{\mathrm{D}}$, where $h = (H_1, \ldots, H_n)^{\mathrm{T}}$, $\omega = (H_{ij})_{i<j}$ and $\check{\mathrm{H}} = (\Upsilon_1, \ldots, \Upsilon_n)$ with Υ_i the skew-symmetric matrix dual to the matrix $A_k = (X_{ij\ k})_{ij}$ which result from the length three brackets organized according to the Philip Hall process, for details see [Monroy-Pérez and Anzaldo-Meneses, 2011].

Theorem 3.2. *If $(g, \widehat{\mathbf{u}})$ is a $\Lambda-$optimal then it is the projection of an extremal curve $(h, \omega, \Upsilon_1, \ldots, \Upsilon_n)$ along which $\widehat{\mathbf{u}} = (H_1, \ldots, H_n)$ and*

$$\dot{h} = \omega\, h, \tag{6}$$

$$\dot{\omega} = \sum_{i=1}^{n} H_i \Upsilon_i \tag{7}$$

$$\dot{\Upsilon}_i = 0, \quad i = 1 \ldots, n. \tag{8}$$

Proof. The maximality condition implies that along extremals we have $\widehat{\mathbf{u}} = (H_1, \ldots, H_n)$, for then the system Hamiltonian becomes quadratic

$$\mathcal{H} = \frac{1}{2}(H_1^2 + \cdots + H_n^2). \tag{9}$$

A straightforward differentiation (Poisson bracketing with $\mathcal{H}$), yields

$$\dot{H}_i = \{H_i, \mathcal{H}\} = \sum_{j\neq i} H_i H_{ij}, \tag{10}$$

similarly

$$\dot{H}_{ij} = \{H_{ij}, \mathcal{H}\} = -\sum_{k=1}^{n} H_k H_{ijk}, \tag{11}$$

and finally

$$\dot{H}_{ijk} = \{H_{ijk}, \mathcal{H}\} = 0, \tag{12}$$

as required.

Remark 3.2. *Equation (12) implies that all the length three Poisson brackets are constant along extremals, that is, we have already* D *integrals of motion.*

The coordinates $g = (\alpha, a, \breve{a})$ in G, can be chosen in such a way that the left invariant vector fields are written as $X_i = \partial\alpha_i + \cdots$. In such a case, we have $\dot{\alpha}_i = u_i = H_i$, and therefore

$$\frac{d}{dt}\left(H_{ij} + \sum_{k=1}^{n} \alpha_k\, H_{kij}\right) = 0.$$

We introduce the skew-symmetric constant matrix c with elements

$$c_{ij} = H_{ij} + \sum_{k=1}^{n} \alpha_k\, H_{kij}.$$

therefore, from (10)

$$\ddot{\alpha}_i - \sum_{j=1}^{n} c_{ij}\dot{\alpha}_j = -\sum_{j,k=1}^{n} \alpha_k\, H_{kij}\,\dot{\alpha}_j,$$

that is,

$$\ddot{\alpha}_i + \sum_{j,k=1}^{n} \alpha_k H_{kij}\dot{\alpha}_j = 0 \tag{13}$$

Since the H_{kij} are constant, these equations are given in terms of the $\alpha_i{\prime}s$ only. And together with the non-holonomic constraints (1), after plugging the optimal controls given by theorem (3.2), determine completely the optimal curves.

4 Some Low Dimensional Cases

We discuss here two examples for illustrating the general results of the above sections.

4.1 Cartan Lie Algebra $\mathfrak{n}$

This case corresponds to $n = 2$ and $\eta = 5$, and is provided by the rank 2 distribution $\Delta = \{X_1, X_2\}$ for which the only non-zero Lie brackets are X_{12}, X_{112}, X_{212}. The Lie algebra $\mathfrak{n} = \text{span}\{X_1, X_2, X_{12}, X_{112}, X_{212}\}$, is known as the Cartan Lie algebra.

If H_i denotes the left invariant Hamiltonian associated with the vector field X_i, then we have the non-trivial Poisson brackets H_{12}, H_{112}, H_{212}.
The corresponding system Hamiltonian writes as follows

$$\mathcal{H} = \frac{\lambda_0}{2}(u_1^2 + u_2^2) + u_1 H_1 + u_2 H_2 \tag{14}$$

For the normal case ($\lambda_0 = 1$), the maximality condition of the Maximum Principle, readily yield $u_1 = H_1$ and $u_2 = H_2$, therefore the system Hamiltonian is quadratic $\mathcal{H} = H_1^2 + H_2^2$, and the adjoint system can be directly written as follows

$$\dot{H}_1 = \frac{1}{2}\{H_1, \mathcal{H}\} = H_2 H_{12} \tag{15}$$
$$\dot{H}_2 = \frac{1}{2}\{H_2, \mathcal{H}\} = -H_1 H_{12} \tag{16}$$
$$\dot{H}_{12} = \frac{1}{2}\{H_{12}, \mathcal{H}\} = -H_1 H_{112} - H_2 H_{212} \tag{17}$$
$$\dot{H}_{112} = \dot{H}_{212} = 0. \tag{18}$$

H_{112} and H_{212} are central elements, and multiplying third equation by H_{12}, we get

$$\frac{1}{2}\frac{d}{dt}\left(H_{12}^2\right) = \frac{d}{dt}(H_2 H_{112} - H_1 H_{212}),$$

therefore we obtain the constant of integration

$$c_2 := \frac{1}{2}H_{12}^2 - H_2 H_{112} + H_1 H_{212}. \tag{19}$$

Further derivation of (17) yields

$$\ddot{H}_{12} = c_2 H_{12} - \frac{1}{2}H_{12}^3,$$

in consequence

$$\dot{H}_{12}\ddot{H}_{12} = \left[\frac{1}{2}\frac{d}{dt}(\dot{H}_{12})^2\right] = c_2 H_{12}\dot{H}_{12} - \frac{1}{2}H_{12}^3\dot{H}_{12}$$
$$= c_2\left[\frac{1}{2}\frac{d}{dt}(H_{12})^2\right] - \frac{1}{2}\left[\frac{1}{4}\frac{d}{dt}(H_{12})^4\right]$$

we obtain then another constant of integration

$$c_3 := \frac{1}{4}H_{12}^4 - c_2 H_{12}^2 + (H_1 H_{112} + H_2 H_{212})^2$$

Lemma 4.1. *The elements of set $\{\mathcal{H}, H_{112}, H_{212}, c_2\}$ are independent first integrals in involution, whereas $\mathcal{K} := H_{112}^2 + H_{212}^2$ and c_3 are neither independent nor in involution.*

Proof. A straightforward calculation shows that $\{c_2, H_1\} = \{c_2, H_2\} = 0$, therefore $\{c_2, \mathcal{H}\} = 0$. On the contrary we have, $\{c_3, H_1\} = -2H_2H_{12}\mathcal{K}$ and $\{c_3, H_2\} = 2H_1H_{12}\mathcal{K}$. Consequently $\{c_3, \mathcal{H}\} = 0$, but $c_3 = 2\mathcal{H}\mathcal{K} - c^2$ as can be easily shown. □

Thus the trajectories in cotangent space are given by the intersection of the cylinder $H_1^2 + H_2^2 = 1$, with the parabolic cylinder $\frac{1}{2}H_{12} - H_2H_{112} + H_1H_{212} = c_2$. A simple way to represent this curves is to note that they can be visualized as curves on the sphere $(H_1 + H_{212})^2 + (H_2 - H_{212})^2 + H_{12}^2 = \mathcal{H} + 2\,c_2 + H_{112}^2 + H_{212}^2$.

4.2 The Ball and Plate Problem and Its Nilpotent Approximation

The ball and plate problem has bee studied within the framework of geometric optimal control theory on Lie groups in [Jurdjevic, 1993] and [Pop, Aron and Petrisor, 2011] The non-holonomic control system consisting of a ball rolling on a plane without twisting and slipping and driven by the another plane can be modelled in the Lie group $G = \mathbb{R}^2 \times \mathrm{SO}_3$: the first coordinate of a state $(\vec{p}, M) \in G$ yields the center of the ball, whereas the second provides the evolution of a moving frame attached to its center.

It is known that the Lie algebra $\mathfrak{so}_3$ of skew-symmetric matrices is isomorphic to $\mathbb{R}^3$ with the standard cross-product, as follows

$$(x, y, z) \leftrightarrow -x(\mathbf{e}_2 \wedge \mathbf{e}_3) + y(\mathbf{e}_1 \wedge \mathbf{e}_3) + -z(\mathbf{e}_2 \wedge \mathbf{e}_3),$$

where $\mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_1 \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i$, and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical base of $\mathbb{R}^3$.

If $\{\mathbf{f}_1, \mathbf{f}_2\}$ denotes the canonical basis of $\mathbb{R}^2$, then the 5-dimensional Lie algebra $\mathbb{R}^2 \times \mathfrak{so}_3$, with Lie bracket $[(\vec{p}, M), (\vec{q}, N)] = (0, [M, N])$ is isomorphic to $\mathbb{R}^5$ with basis $A_1 = (0, 0, \mathbf{e}_1)$, $A_2 = (0, 0, \mathbf{e}_2)$, $A_3 = (0, 0, \mathbf{e}_3)$, $A_4 = (\mathbf{f}_1, 0, 0, 0)$, $A_5 = (\mathbf{f}_2, 0, 0, 0)$, and the no-trivial brackets

$$[A_1, A_2] = A_3,\ [A_1, A_3] = -A_2,\ [A_2, A_3] = A_1,$$

taking into account this notation, the system for the plate-ball problem is written as

$$\dot{q} = u_1 Y_1(q) + u_2 Y_2(q),$$

where $Y_1 = A_4 - A_2$ and $Y_2 = A_5 + A_1$, that is, a control affine system defined by an invariant distribution $\Delta = \{Y_1, Y_2\}$

The nilpotent approximation for a system with two control parameter in 5-dimensional manifold as the one written before, has been obtained in [Gauthier and Zakalyukin, 2007]. In that reference, privileged coordinates (x, y, z, w, v) around a tubular neighborhood of a non-admissible trajectory are considered, to write the nilpotent approximation $\{X_1, X_2\}$ of Δ as follows:

$$X_1 = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z} + \frac{y^2}{2}\frac{\partial}{\partial w} + \frac{xy}{2}\varphi(v)\frac{\partial}{\partial v},$$
$$X_2 = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z} - \frac{xy}{2}\frac{\partial}{\partial w} - \frac{x^2}{2}\varphi(v)\frac{\partial}{\partial v}.$$

It can be easily shown that the non-trivial brackets in this approximation are the following:

$$X_{12} = [X_1, X_2] = \frac{y}{2}\frac{\partial}{\partial w} - \frac{x}{2}\varphi(v)\frac{\partial}{\partial v},$$
$$X_{112} = [X_1, X_{12}] = -\frac{\varphi(v)}{2}\frac{\partial}{\partial v},$$
$$X_{212} = [X_2, X_{12}] = \frac{1}{2}\frac{\partial}{\partial w}.$$

It turns out that the nilpotent approximation yields the same 3-step nilpotent Lie algebra as the one of Cartan discussed before.

5 Conclusions

We have studied general properties for optimal trajectories of a problem defined by a driftless non-holonomic control system and a quadratic cost. We have considered the case when the Lie algebra generated by the distribution of vector fields is 3-step and 2-solvable. We have derived the geometric properties of the trajectories using the Pontryagin Maximum Principle and the associated Hamiltonian formalism. We have discussed a five dimensional case that models and interesting non-holonomic mechanical system.

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