SWITCHED LINEAR SYSTEMS. GEOMETRIC APPROACH

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Abstract
We consider equivalence relation between switched linear systems and compute the dimension of equivalence classes, after proving they can be obtained as the orbits of the action of a Lie group on the differentiable manifold of matrices defining the subsystems the system consists of and deduce miniversal deformations.

Key words
Switched linear system,Lie group action, miniversal deformation.

1 Introduction

Switched linear systems constitute a class of nonlinear systems having a behavior far from that of classical systems. They have been studied recently because of their interesting properties and the great number of applications from which they arise. We define natural equivalence relations in the space of matrices defining switched linear systems, which include basis changes in the state variables, inputs and outputs spaces, state feedback and output injection. Equivalence classes coincide with the orbits under suitable Lie group actions on the differentiable manifold of matrices defining the systems. From this identification, the dimension of equivalence classes may be deduced and miniversal deformations are obtained, following Arnold's techniques in [Arnold, 1971] and [Tannenbaum, 1981].

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers, $M_{n\times m}(\mathbb{R})$ the set of matrices having n rows and m columns and entries in $\mathbb{R}$ (in the case where $n = m$, we will simply write $M_n(\mathbb{R})$) and by $Gl_n(\mathbb{R})$ the group of non-singular matrices in $M_n(\mathbb{R})$.

2 Switched linear systems

Switched linear systems consist of different subsystems of linear equations and a rule orchestrating the changes between them. More concretely, we can define them as follows.

Definition 2.1. *A switched non-singular linear system is a system which consists of several non-singular linear subsystems and a rule that determines the switching between them. It can be written as*

$$\Sigma_\sigma \begin{cases} \dot{\mathbf{x}}(t) = A_\sigma \mathbf{x}(t) + B_\sigma u(t) \\ \mathbf{y}(t) = C_\sigma \mathbf{x}(t) \end{cases}$$

with σ a piecewise constant signal taking values from an index set $M = \{1, \dots, \ell\}$.

Example 2.1. *Let us consider a PWM (pulse-width modulator) boost-converter. We will (as usual) denote by L an inductance, C a capacitance, R a load resistance and $e_S(t)$ the source voltage. It allows to transform the source voltage $e_S(t)$ into a higher voltage $e_C(t)$ over the load R.*

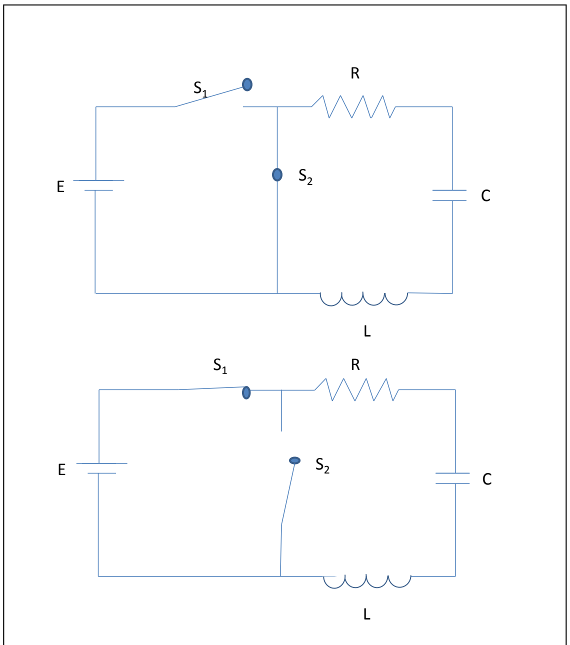

Figure 1. PWM boost-converter

Equations when the switch S_1 is closed are:

$$\begin{pmatrix} \dot{Q} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} E$$

and when the switch S_2 is closed are:

$$\begin{pmatrix} \dot{Q} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{L} \end{pmatrix} E$$

It is modeled by a switched linear system defined by matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ -\frac{1}{L} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

3 Equivalence relations

For simplicity's sake, we will restrict ourselves to the case where the system consists of two subsystems ($\ell = 2$). The cases where there are more than two subsystems can be handled in analogous way. Then a switched linear system is defined by a $6-$tuple of matrices $(A_1, A_2, B_1, B_2, C_1, C_2) \in \mathcal{X} = M_n(\mathbb{R}) \times M_n(\mathbb{R}) \times M_{n\times m}(\mathbb{R}) \times M_{n\times m}(\mathbb{R}) \times M_{p\times n}(\mathbb{R}) \times M_{p\times n}(\mathbb{R})$.

We will consider the following elementary transformations:

ET1 Change of basis of the state variables
$(A_1, A_2, B_1, B_2, C_1, C_2) \to$
$(TA_1T^{-1}, TA_2T^{-1}, TB_1, TB_2, C_1T^{-1}, C_2T^{-1})$

ET2 Change of basis of the input variables
$(A_1, A_2, B_1, B_2, C_1, C_2) \to$
$(A_1, A_2, B_1V, B_2V, C_1, C_2)$

ET3 Change of basis of the output variables
$(A_1, A_2, B_1, B_2, C_1, C_2) \to$
$(A_1, A_2, B_1, B_2, WC_1, WC_2)$

ET4 State feedback
$(A_1, A_2, B_1, B_2, C_1, C_2) \to$
$(A_1 + B_1K, A_2 + B_2K, B_1, B_2, C_1, C_2)$

ET5 Output injection
$(A_1, A_2, B_1, B_2, C_1, C_2) \to$
$(A_1 + UC_1, A_2 + UC_2, B_1, B_2, C_1, C_2)$

and the equivalence relation which includes the elementary transformations above.

Definition 3.1. *We define equivalent 6-tuples*

$$(A_1, A_2, B_1, B_2, C_1, C_2) \sim (A_1^{'}, A_2^{'}, B_1^{'}, B_2^{'}, C_1^{'}, C_2^{'})$$

as those such that there exist $T \in Gl_n(\mathbb{R}), V \in Gl_m(\mathbb{R}), W \in Gl_p(\mathbb{R}), K \in M_{m\times n}(\mathbb{R}), U \in M_{n\times p}(\mathbb{R})$ with
$(A_1^{'}, A_2^{'}, B_1^{'}, B_2^{'}, C_1^{'}, C_2^{'}) =$
$= (TA_1T^{-1}+TB_1K+UC_1T^{-1}, TA_2T^{-1}+TB_2K+UC_2T^{-1}, TB_1V, TB_2V, WC_1T^{-1}, WC_2T^{-1})$

We also consider the following elementary transformation:

ET6 Premultiplication of the state equations by invertible matrices
$(A_1, A_2, B_1, B_2, C_1, C_2) \to$
$(Q_1A_1, Q_2A_2, Q_1B_1, Q_2B_2, C_1, C_2)$

When including this new elementary transformation, we obtain a new equivalence relation, which we will denote by $\sim_g$ (generalized equivalence relation).

Definition 3.2. *We define equivalent 6-tuples with respect to generalized equivalence*

$$(A_1, A_2, B_1, B_2, C_1, C_2) \sim_g (A_1^{'}, A_2^{'}, B_1^{'}, B_2^{'}, C_1^{'}, C_2^{'})$$

as those such that there exist $Q_1 \in GL_n(\mathbb{R}), Q_2 \in GL_n(\mathbb{R}), T \in Gl_n(\mathbb{R}), V \in Gl_m(\mathbb{R}), W \in Gl_p(\mathbb{R}), K \in M_{m\times n}(\mathbb{R}), U \in M_{n\times p}(\mathbb{R})$ with
$(A_1^{'}, A_2^{'}, B_1^{'}, B_2^{'}, C_1^{'}, C_2^{'}) = (Q_1TA_1T^{-1}+ Q_1TB_1K + UC_1T^{-1}, Q_2TA_2T^{-1} + Q_2TB_2K +UC_2T^{-1}, Q_1TB_1V, Q_2TB_2V, WC_1T^{-1}, WC_2T^{-1})$

In order to compute the dimension of the equivalence classes, a key point is to view them as orbits under a suitable Lie group action. This is a particular case of the more general situation where a Lie group acts on a differentiable manifold. The equivalence classes under the equivalence relation defined above are, in fact, differentiable manifolds. More concretely, we can state this as follows.

Proposition 3.1. *Equivalence relation $\sim$ is the one induced by the action of the Lie group:*
$\mathcal{G} = \{(T, V, W, K, U) \in Gl_n(\mathbb{R}) \times Gl_m(\mathbb{R}) \times Gl_p(\mathbb{R}) \times M_{m\times n}(\mathbb{R}) \times M_{n\times p}(\mathbb{R})\}$ *on $\mathcal{X}$.*

And generalized equivalence relation $\sim_g$ is the one induced by the action of the Lie group:
$\mathcal{G}_g = \{(Q_1, Q_2, T, V, W, K, U) \in Gl_n(\mathbb{R}) \times Gl_n(\mathbb{R}) \times Gl_n(\mathbb{R}) \times Gl_m(\mathbb{R}) \times Gl_p(\mathbb{R}) \times M_{m\times n}(\mathbb{R}) \times M_{n\times p}(\mathbb{R})\}$ *on $\mathcal{X}$.*
In particular, the elementary transformations ET1,ET2, ET3, ET4, ET5 above are equivalence relations induced by the action of the Lie subgroups of $\mathcal{G}$:

$$\begin{aligned} \mathcal{G}_1 &= \{(T, I_m, I_p, 0, 0) \in \mathcal{G}\} \\ \mathcal{G}_2 &= \{(I_n, V, I_p, 0, 0) \in \mathcal{G}\} \\ \mathcal{G}_3 &= \{(I_n, I_m, W, 0, 0) \in \mathcal{G}\} \\ \mathcal{G}_4 &= \{(I_n, I_m, I_p, K, 0) \in \mathcal{G}\} \\ \mathcal{G}_5 &= \{(I_n, I_m, I_p, 0, U) \in \mathcal{G}\} \end{aligned}$$

and ET6 by the Lie subgroup of $\mathcal{G}_g$:

$$\mathcal{G}_6 = \{(Q_1, Q_2, I_n, I_m, I_p, 0, 0) \in \mathcal{G}_g\}$$

on the differentiable manifold $\mathcal{X}$.

Proof. It is straightforward to check that the equivalence classes acording to $\sim$ (respectively, $\sim_g$) coincide with the orbits under $\mathcal{G}$ (respectively, $\mathcal{G}_g$), being the actions on X as follows.

$$\alpha : \mathcal{G} \times \mathcal{X} \longrightarrow \mathcal{X}$$

where if $G = (T, V, W, K, U) \in \mathcal{G}$ and $X = (A_1, A_2, B_1, B_2, C_1, C_2) \in \mathcal{X}$, then:

$$\begin{aligned}\alpha(G, X) = (&TA_1T^{-1} + TB_1K + UC_1T^{-1},\\ &TA_2T^{-1} + TB_2K + UC_2T^{-1},\\ &TB_1V, TB_2V,\\ &WC_1T^{-1}, WC_2T^{-1})\end{aligned}$$

and $\alpha_X : \mathcal{G} \longrightarrow \mathcal{X}$, $\alpha_X(G) = \alpha(G, X)$.

Analogously, we define the action:

$$\alpha_g : \mathcal{G}_g\mathcal{X} \longrightarrow \mathcal{X}$$

where if $G_g = (Q_1, Q_2, T, V, W, K, U) \in \mathcal{G}_g$ and $X = (A_1, A_2, B_1, B_2, C_1, C_2) \in \mathcal{X}$, then:

$$\begin{aligned}\alpha_g(G_g, X) = (&Q_1TA_1T^{-1} + Q_1TB_1K + UC_1T^{-1},\\ &Q_2TA_2T^{-1} + Q_2TB_2K + UC_2T^{-1},\\ &Q_1TB_1V, Q_2TB_2V,\\ &WC_1T^{-1}, WC_2T^{-1})\end{aligned}$$

and $\alpha_{gX} : \mathcal{G}_g \longrightarrow \mathcal{X}$, $\alpha_{gX}(G_g) = \alpha_g(G_g, X)$.

Given any set of matrices $(A_1, A_2, B_1, B_2, C_1, C_2) \in \mathcal{X}$, we will denote by $\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ and $\mathcal{O}_g(A_1, A_2, B_1, B_2, C_1, C_2)$ its orbits under equivalence and generalized equivalence, respectively.

Proposition 3.2. *Equivalence classes under both equivalence relations are locally closed differentiable submanifolds of $\mathcal{X}$ and their boundaries are a union of equivalence classes or orbits of strictly lower dimension. In particular, equivalence classes or orbits of minimal dimension are closed.*

The Proposition below describes the tangent spaces and provides a way to obtain a basis of the normal spaces with regard to the following Euclidean scalar product in $\mathcal{X}$:
$\langle (A_1, A_2, B_1, B_2, C_1, C_2), (Y_1, Y_2, Z_1, Z_2, T_1, T_2)\rangle =$
$\mathrm{tr}(A_1Y_1^t) + \mathrm{tr}(A_2Y_2^t) + \mathrm{tr}(B_1Z_1^t) + \mathrm{tr}(B_2Z_2^t) + \mathrm{tr}(C_1T_1^t) + \mathrm{tr}(C_2Z_2^t)$

Proposition 3.3. *Let us denote by $T\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ the tangent space and by $N\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ the normal space to the orbit of $(A_1, A_2, B_1, B_2, C_1, C_2)$ under equivalence and $T_g\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ the tangent space and by $N_g\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ the normal space to the orbit of $(A_1, A_2, B_1, B_2, C_1, C_2)$ under generalized equivalence at the point $(A_1, A_2, B_1, B_2, C_1, C_2)$. Then:*

(a) $T\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ *is the set* $\{([T, A_1] + B_1K + UC_1, [T, A_2] + B_2K + UC_2, TB_1 + B_1V, TB_2 + B_2V, WC_1 - C_1T, WC_2 - C_2T), (T, V, W, K, U) \in T_I\mathcal{G}\}$
where
$T_I\mathcal{G}$ *is* $\{(T, V, W, K, U) \in M_n(\mathbb{R}) \times M_m(\mathbb{R}) \times M_p(\mathbb{R}) \times M_{m\times n}(\mathbb{R}) \times M_{n\times p}(\mathbb{R})\}$.

(b) $N\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ *is the vector subspace consisting of 6-tuples*
$(Y_1, Y_2, Z_1, Z_2, T_1, T_2) \in \mathcal{X}$ *such that*

$$\begin{aligned}[A_1, Y_1^t] + B_1Z_1^t - T_1^tC_1 + [A_2, Y_2^t] + B_2Z_2^t - T_2^tC_2 &= 0\\ Z_1^tB_1 + Z_2^tB_2 &= 0\\ C_1Y_1^t + C_2Y_2^t &= 0\\ Y_1^tB_1 + Y_2^tB_2 &= 0\\ C_1T_1^t + C_2T_2^t &= 0\end{aligned}$$

(c) $T_g\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ *is the set* $\{([T, A_1] + Q_1A_1 + B_1K + UC_1, [T, A_2] + Q_2A_2 + B_2K + UC_2, TB_1 + Q_1B_1 + B_1V, TB_2 + Q_2B_2 + B_2V, WC_1 - C_1T, WC_2 - C_2T), (Q_1, Q_2, T, V, W, K, U) \in T_I\mathcal{G}_g\}$
where
$T_I\mathcal{G}_g$ *is* $\{(Q_1, Q_2, T, V, W, K, U) \in M_n(\mathbb{R}) \times M_n(\mathbb{R}) \times M_n(\mathbb{R}) \times M_m(\mathbb{R}) \times M_p(\mathbb{R}) \times M_{m\times n}(\mathbb{R}) \times M_{n\times p}(\mathbb{R})\}$.

(d) $N_g\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ *is the vector subspace consisting of 6-tuples* $(Y_1, Y_2, Z_1, Z_2, T_1, T_2) \in \mathcal{X}$ *such that*

$$\begin{aligned}A_1Y_1^t + B_1Z_1^t &= 0\\ A_2Y_2^t + B_2Z_2^t &= 0\\ [A_1, Y_1^t] + B_1Z_1^t - T_1^tC_1 + [A_2, Y_2^t] + B_2Z_2^t - T_2^tC_2 &= 0\\ Z_1^tB_1 + Z_2^tB_2 &= 0\\ C_1Y_1^t + C_2Y_2^t &= 0\\ Y_1^tB_1 + Y_2^tB_2 &= 0\\ C_1T_1^t + C_2T_2^t &= 0\end{aligned}$$

Proof. (a) If we consider $I = (I_n, I_m, I_p, 0, 0)$:
$d\alpha_X(I + \varepsilon G) = ((I_n + \varepsilon T)A_1(I_n + \varepsilon T)^{-1} + (I_n + \varepsilon T)B_1\varepsilon K + \varepsilon UC_1(I_n + \varepsilon T)^{-1}$,
$(I_n + \varepsilon T)A_2(I_n + \varepsilon T)^{-1} + (I_n + \varepsilon T)B_2\varepsilon K + \varepsilon UC_2(I_n + \varepsilon T)^{-1}, (I_n + \varepsilon T)B_1(I_m + \varepsilon V)$,
$(I_n + \varepsilon T)B_2(I_m + \varepsilon V), (I_p + \varepsilon W)C_1(I_n + \varepsilon T)^{-1}, (I_p + \varepsilon W)C_2(I_n + \varepsilon T)^{-1}) =$
$= ((I_n + \varepsilon T)A_1(I_n - \varepsilon T + \ldots) + (I_n + \varepsilon T)B_1\varepsilon K + \varepsilon UC_1(I_n - \varepsilon T + \ldots)$,
$(I_n + \varepsilon T)A_2(I_n - \varepsilon T + \ldots) + (I_n + \varepsilon T)B_2\varepsilon K + \varepsilon UC_2(I_n - \varepsilon T + \ldots), (I_n + \varepsilon T)B_1(I_m + \varepsilon V), (I_n + \varepsilon T)B_2(I_m + \varepsilon V), (I_p + \varepsilon W)C_1(I_n - \varepsilon T + \ldots), (I_p + \varepsilon W)C_2(I_n - \varepsilon T + \ldots)^{-1}) = (A_1, A_2, B_1, B_2, C_1, C_2) + \varepsilon(TA_1 - A_1T + B_1K + UC_1, TA_2 - A_2T + B_2K + UC_2, TB_1 + B_1V, TB_2 + B_2V, WC_1 - C_1T, WC_2 - C_2T) + \varepsilon^2(\ldots$
and the statement follows.

(b) For any $(Y_1, Y_2, Z_1, Z_2, T_1, T_2) \in \mathcal{X}$, this 6-tuple is in $N\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ if, and only if, $\langle (TA_1 - A_1T + B_1K + UC_1, TA_2 - A_2T + B_2K + UC_2, TB_1 + B_1V, TB_2 + B_2V, WC_1 - C_1T, WC_2 - C_2T), (Y_1, Y_2, Z_1, Z_2, T_1, T_2)\rangle =$

$= \text{tr}((TA_1 - A_1T + B_1K + UC_1)Y_1^t) + \text{tr}((TA_2 - A_2T + B_2K + UC_2)Y_2^t) + \text{tr}((TB_1 + B_1V)Z_1^t) + \text{tr}((TB_2 + B_2V)Z_2^t) + \text{tr}((WC_1 - C_1T)T_1^t) + \text{tr}((WC_2 - C_2T)T_2^t) =$

$= \text{tr}(TA_1Y_1^t - A_1TY_1^t + B_1KY_1^t + UC_1Y_1^t) + \text{tr}(TA_2Y_2^t - A_2TY_2^t + B_2KY_2^t + UC_2Y_2^t) + \text{tr}(TB_1Z_1^t + B_1VZ_1^t) + \text{tr}(TB_2Z_2^t + B_2VZ_2^t) + \text{tr}(WC_1T_1^t - C_1TT_1^t) + \text{tr}(WC_2T_2^t - C_2TT_2^t) =$

$= \text{tr}(A_1Y_1^tT) - \text{tr}(Y_1^tA_1T) + \text{tr}(Y_1^tB_1K) + \text{tr}(C_1Y_1^tU) + \text{tr}(A_2Y_2^tT) - \text{tr}(Y_2^tA_2T) + \text{tr}(Y_2^tB_2K) + \text{tr}(C_2Y_2^tU) + \text{tr}(B_1Z_1^tT) + \text{tr}(Z_2^tB_1V) + \text{tr}(B_2Z_2^tT) + \text{tr}(Y_2^tZ_2V) + \text{tr}(C_1T_1^tW) - \text{tr}(T_2^tC_1T) + \text{tr}(C_2T_2^tW) - \text{tr}(T_2^tC_2T) =$

$= \text{tr}((A_1Y_1^t - Y_1^tA_1 + A_2Y_2^t - Y_2^tA_2 + B_1Z_1^t + B_2Z_2^t - T_1^tC_1 - T_2^tC_2)T) + \text{tr}((Z_1^tB_1 + Z_2^tB_2)V) + \text{tr}((C_1T_1^t + C_2T_2^t)W) + \text{tr}((Y_1^tB_1 + Y_2^tB_2)K) + \text{tr}((C_1Y_1^t + C_2Y_2^t)U) =$
$= \langle (A_1Y_1^t - Y_1^tA_1 + A_2Y_2^t - Y_2^tA_2 + B_1Z_1^t + B_2Z_2^t - T_1^tC_1 - T_2^tC_2, Z_1^tB_1 + Z_2^tB_2, C_1T_1^t + C_2T_2^t, Y_1^tB_1 + Y_2^tB_2, C_1Y_1^t + C_2Y_2^t, (T^t, V^t, W^t, K^t, U^t)\rangle = 0, \quad \forall (T^t, V^t, W^t, K^t, U^t)$.
And we obtain the equations:

$$[A_1, Y_1^t] + B_1Z_1^t - T_1^tC_1 + [A_2, Y_2^t] + B_2Z_2^t - T_2^tC_2 = 0$$
$$Z_1^tB_1 + Z_2^tB_2 = 0$$
$$C_1Y_1^t + C_2Y_2^t = 0$$
$$Y_1^tB_1 + Y_2^tB_2 = 0$$
$$C_1T_1^t + C_2T_2^t = 0$$

(c) If we consider $I = (I_n, I_n, I_n, I_m, I_p, 0, 0)$:
$d\alpha_{g_X}(I + \varepsilon G_g) = ((I_n + \varepsilon Q_1)(I_n + \varepsilon T)A_1(I_n + \varepsilon T)^{-1} + (I_n + \varepsilon Q_1)(I_n + \varepsilon T)B_1\varepsilon K + \varepsilon UC_1(I_n + \varepsilon T)^{-1}, (I_n + \varepsilon Q_2)(I_n + \varepsilon T)A_2(I_n + \varepsilon T)^{-1} + (I_n + \varepsilon Q_2)(I_n + \varepsilon T)B_2\varepsilon K + \varepsilon UC_2(I_n + \varepsilon T)^{-1}, (I_n + \varepsilon Q_1)(I_n + \varepsilon T)B_1(I_m + \varepsilon V), (I_n + \varepsilon Q_2)(I_n + \varepsilon T)B_2(I_m + \varepsilon V), (I_p + \varepsilon W)C_1(I_n + \varepsilon T)^{-1}, (I_p + \varepsilon W)C_2(I_n + \varepsilon T)^{-1}) =$
$= ((I_n + \varepsilon Q_1)(I_n + \varepsilon T)A_1(I_n - \varepsilon T + \ldots) + (I_n + \varepsilon Q_1)(I_n + \varepsilon T)B_1\varepsilon K + \varepsilon UC_1(I_n - \varepsilon T + \ldots), (I_n + \varepsilon Q_2)(I_n + \varepsilon T)A_2(I_n - \varepsilon T + \ldots) + (I_n + \varepsilon Q_2)(I_n + \varepsilon T)B_2\varepsilon K + \varepsilon UC_2(I_n - \varepsilon T + \ldots), (I_n + \varepsilon Q_1)(I_n + \varepsilon T)B_1(I_m + \varepsilon V), (I_n + \varepsilon Q_2)(I_n + \varepsilon T)B_2(I_m + \varepsilon V), (I_p + \varepsilon W)C_1(I_n - \varepsilon T + \ldots), (I_p + \varepsilon W)C_2(I_n - \varepsilon T + \ldots)^{-1}) =$
$= (A_1, A_2, B_1, B_2, C_1, C_2) + \varepsilon(TA_1 - A_1T + Q_1A_1 + B_1K + UC_1, TA_2 - A_2T + Q_2A_2 + B_2K + UC_2, TB_1 + Q_1B_1 + B_1V, TB_2 + Q_2B_2 + B_2V, WC_1 - C_1T, WC_2 - C_2T) + \varepsilon^2(\ldots)$
and the statement follows.

(d) For any $(Y_1, Y_2, Z_1, Z_2, T_1, T_2) \in$, this 6-tuple is in X $N\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ if and only if, $\langle (TA_1 - A_1T + Q_1A_1 + B_1K + UC_1, TA_2 - A_2T + Q_2A_2 + B_2K + UC_2, TB_1 + Q_1B_1 + B_1V, TB_2 + Q_2B_2 + B_2V, WC_1 - C_1T, WC_2 - C_2T), (Y_1, Y_2, Z_1, Z_2, T_1, T_2)\rangle =$

$= \text{tr}((TA_1 - A_1T + Q_1A_1 + B_1K + UC_1)Y_1^t) + \text{tr}((TA_2 - A_2T + Q_2A_2 + B_2K + UC_2)Y_2^t) + \text{tr}((TB_1 + Q_1B_1 + B_1V)Z_1^t) + \text{tr}((TB_2 + Q_2B_2 + B_2V)Z_2^t) + \text{tr}((WC_1 - C_1T)T_1^t) + \text{tr}((WC_2 - C_2T)T_2^t) =$

$= \text{tr}(TA_1Y_1^t - A_1TY_1^t + Q_1A_1Y_1^t + B_1KY_1^t + UC_1Y_1^t) + \text{tr}(TA_2Y_2^t - A_2TY_2^t + Q_2A_2Y_2^t + B_2KY_2^t + UC_2Y_2^t) + \text{tr}(TB_1Z_1^t + Q_1B_1Z_1^t + B_1VZ_1^t) + \text{tr}(TB_2Z_2^t + Q_2B_2Z_2^t + B_2VZ_2^t) + \text{tr}(WC_1T_1^t - C_1TT_1^t) + \text{tr}(WC_2T_2^t - C_2TT_2^t) =$

$= \text{tr}(A_1Y_1^tT) - \text{tr}(Y_1^tA_1T) + \text{tr}(A_1Y_1^tQ_1) + \text{tr}(Y_1^tB_1K) + \text{tr}(C_1Y_1^tU) + \text{tr}(A_2Y_2^tT) - \text{tr}(Y_2^tA_2T) + \text{tr}(A_2Y_2^tQ_2) + \text{tr}(Y_2^tB_2K) + \text{tr}(C_2Y_2^tU) + \text{tr}(B_1Z_1^tT) + \text{tr}(B_1Z_1^tQ_1) + \text{tr}(Z_2^tB_1V) + \text{tr}(B_2Z_2^tT) + \text{tr}(B_2Z_2^tQ_2) + \text{tr}(Y_2^tZ_2V) + \text{tr}(C_1T_1^tW) - \text{tr}(T_1^tC_1T) + \text{tr}(C_2T_2^tW) - \text{tr}(T_2^tC_2T) =$

$= \text{tr}((A_1Y_1^t - Y_1^tA_1 + A_2Y_2^t - Y_2^tA_2 + B_1Z_1^t + B_2Z_2^t - T_1^tC_1 - T_2^tC_2)T) + \text{tr}((Z_1^tB_1 + Z_2^tB_2)V) + \text{tr}((C_1T_1^t + C_2T_2^t)W) + \text{tr}((Y_1^tB_1 + Y_2^tB_2)K) + \text{tr}((C_1Y_1^t + C_2Y_2^t)U) + \text{tr}((A_1Y_1^t + B_1Z_1^t)Q_1) + \text{tr}((A_2Y_2^t + B_2Z_2^t)Q_2) =$

$= \langle (A_1Y_1^t + B_1Z_1^t, A_2Y_2^t + B_2Z_2^t, A_1Y_1^t - Y_1^tA_1 + A_2Y_2^t - Y_2^tA_2 + B_1Z_1^t + B_2Z_2^t - T_1^tC_1 - T_2^tC_2, Z_1^tB_1 + Z_2^tB_2, C_1T_1^t + C_2T_2^t, Y_1^tB_1 + Y_2^tB_2, C_1Y_1^t + C_2Y_2^t), (Q_1^t, Q_2^t, T^t, V^t, W^t, K^t, U^t)\rangle = 0, \quad \forall (Q_1^t, Q_2^T, T^t, V^t, W^t, K^t, U^t)$.
And we obtain the equations:

$$A_1Y_1^t + B_1Z_1^t = 0$$
$$A_2Y_2^t + B_2Z_2^t = 0$$
$$[A_1, Y_1^t] + B_1Z_1^t - T_1^tC_1 + [A_2, Y_2^t] + B_2Z_2^t - T_2^tC_2 = 0$$
$$Z_1^tB_1 + Z_2^tB_2 = 0$$
$$C_1Y_1^t + C_2Y_2^t = 0$$
$$Y_1^tB_1 + Y_2^tB_2 = 0$$
$$C_1T_1^t + C_2T_2^t = 0$$

4 Dimension of equivalence classes

It is a classical technique for computing the dimension of orbits or equivalence classes to calculate the codimension of the normal spaces (of any set of matrices in the same equivalence class), since they coincide. In particular, we can state the following

Proposition 4.1. *The dimension of orbits $\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ coincides with the rank of matrix $M(A_1, A_2, B_1, B_2, C_1, C_2)$ the matrix associated to the linear system yielding the normal space to the orbit of the given 6-tuple, and the dimension of orbits $\mathcal{O}_g(A_1, A_2, B_1, B_2, C_1, C_2)$ coincides with the rank of matrix $M_g(A_1, A_2, B_1, B_2, C_1, C_2)$ associated to the linear system yielding the normal space to the orbit of the given 6-tuple.*

Example 4.1. *We consider the circuit in Example 2.1. A basis of the normal space $N\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ to the orbit of the 6-tuple defining the system is:*

$$V_1 = \left\{ 0, \begin{pmatrix} -L & 0 \\ 0 & 0 \end{pmatrix}, 0, 0, 0, \begin{pmatrix} 0 & 1 \end{pmatrix} \right\},$$

$$V_2 = \left\{ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0, 0, 0 \right\},$$

$$V_3 = \left\{ 0, 0, 0, 0, \begin{pmatrix} 0 & 1 \end{pmatrix}, 0 \right\},$$

$$V_4 = \left\{ 0, 0, 0, 0, \begin{pmatrix} 1 & 0 \end{pmatrix}, 0 \right\},$$

$$V_5 = \left\{ 0, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, 0, 0, 0, \begin{pmatrix} 1 & 0 \end{pmatrix} \right\},$$

$$V_6 = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0, 0, 0 \right\},$$

$$V_7 = \left\{ 0, \begin{pmatrix} -L & 0 \\ 0 & 1 \end{pmatrix}, 0, 0, 0, 0 \right\}$$

On the other hand, a basis of the normal space $N_g\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ *is*

$$V_1 = \left\{ 0, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, 0, 0, 0, \begin{pmatrix} 1 & 0 \end{pmatrix} \right\},$$

$$V_2 = \left\{ 0, 0, 0, 0, \begin{pmatrix} 0 & 1 \end{pmatrix}, 0 \right\},$$

$$V_3 = \left\{ 0, 0, 0, 0, \begin{pmatrix} 1 & 0 \end{pmatrix}, 0 \right\},$$

$$V_4 = \left\{ 0, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, 0, 0, 0, \begin{pmatrix} 0 & 1 \end{pmatrix} \right\}$$

In this case, the dimension of the orbit under equivalence of this 6-tuple is 9, and the dimension under generalized equivalence is 12.

5 Miniversal deformations

When tackling the problem of how small perturbations of the system may lead to different structures a classical approach is to consider miniversal deformations, which provide all possible structures which can arise from small perturbations. Moreover, they can be applied to the study of singularities and bifurcations. We recall that the number of parameters in any miniversal deformation is always equal to the codimension of orbits.

Definition 5.1. *A deformation of the 6-tuple* $(A_1, A_2, B_1, B_2, C_1, C_2) \in \mathcal{X}$ *is a differentiable map* $\varphi : U \longrightarrow \mathcal{X}$*, with* U *an open neighborhood of the origin in* $\mathbb{R}^d$*, such that* $\varphi(0) = (A_1, A_2, B_1, B_2, C_1, C_2)$.

Let $\mathcal{G}$ be a Lie group acting on $\mathcal{X}$ through the action α.
A deformation $\varphi : U \longrightarrow \mathcal{X}$ of $(A_1, A_2, B_1, B_2, C_1, C_2)$ is called *versal* at 0 if for any other deformation of $(A_1, A_2, B_1, B_2, C_1, C_2)$, $\psi : V \longrightarrow \mathcal{X}$, there exists a neighborhood $V' \subseteq V$ with $0 \in V'$, a differentiable map $\gamma : V' \longrightarrow U$ with $\gamma(0) = 0$ and a deformation of the identity $I \in \mathcal{G}$, $\theta : V' \longrightarrow \mathcal{G}$, such that $\psi(\mu) = \alpha(\theta(\mu), \varphi(\gamma(\mu)))$ for all $\mu \in V'$.

A versal deformation with minimal number of parameters d is called *miniversal deformation*.

A miniversal deformation may be deduced from a basis of the normal space to the orbit of a given triple (see [Arnold, 1971], [Tannenbaum, 1981], where the main definitions and results about deformations and versatility can be found). This miniversal deformation is usually called the *orthogonal* miniversal deformation.

Theorem 5.1. *The mappings*

$$\begin{aligned} \mathbb{R}^d &\longrightarrow \mathcal{X} \\ (\eta_1, \ldots, \eta_d) &\mapsto (A_1, A_2, B_1, B_2, C_1, C_2) + \sum_{i=1}^{d} \eta_i V_i \end{aligned}$$

where $\{V_1, \ldots, V_d\}$ *is any basis of the vectorial subspace* $N\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ *or* $N_g\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ *is a miniversal deformation of* $(A_1, A_2, B_1, B_2, C_1, C_2)$ *with respect to equivalence or generalized equivalence of 6-tuples in* $\mathcal{X}$.

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