Chaos synchronization with time-delayed couplings

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Networks of chaotic units with time delayed couplings and feedbacks are investigated analytically and numerically. Recent results and several conjectures on chaos synchronization are summarized.

I. INTRODUCTION

Chaos synchronization is a counter intuitive phenomen. On one hand, a chaotic system is unpredictable. Two chaotic systems, starting from almost identical initial states, end in completely different trajectories. On the other hand, two identical chaotic units which are coupled to each other can synchronize to a common trajectory. The system is still chaotic, but after a transient the two chaotic trajectories are locked to each other [1, 2]. This phenomenon is attracting a lot of research, since it has the potential to be applied for novel secure communication systems [3]. In addition, networks of chaotic units have been realized with electronic circuits and lasers and they are being discussed in the context of neural networks [4, 5].

In this contribution, we report some results of our recent work on chaos synchronization with time-delayed couplings. In particular, we emphasize some conjectures which are supported by our results but are waiting for a general proof.

II. MODEL AND METHODS

We consider complete synchronization, only. This means, a network of chaotic units synchroinzes to a common identical trajectory, without any time shift and identical in amplitude and phase. For simplicity, we discuss the phenomena with a network of coupled maps. This model allows analytical and extensive numerical investigations. The dynamic is defined as

$$x_t^j = (1-\epsilon) f(x_{t-1}^j) + \epsilon\kappa f(x_{t-\tau_d}^j) + \epsilon(1-\kappa)\sum_k G_{jk} f(x_{t-\tau_c}^k) \tag{1}$$

x_t^j are the dynamic variables, t is the discrete time and $i = 1, ..., N$ is the index of the N nodes. ϵ is the strength of all delay terms while κ denotes the relative strength of the self-feedback term. G_{jk} is the coupling matrix and $f(x)$ is a map of the unit interval to itself.

In the following, we will use the Bernoulli shift $f(x) = (\alpha x) \mathrm{mod} 1$ with $\alpha > 1$, which gives a chaotic trajectory.

We define the matrix G_{jk} such that the completely synchronized trajectory $x_t^1 = x_t^2 = ... = x_t^N = x_t$ is a solution of Eq.(1). The stability of this synchronization manifold is determined by linearizing Eq.(1). With $d_t^j = x_t^j - x_t$ one obtains

$$d_t^j = (1-\epsilon)\alpha d_{t-1}^j + \epsilon\kappa\alpha d_{t-\tau_d}^j + \epsilon(1-\kappa)\alpha \sum_{k\neq j} G_{jk} d_{t-\tau_c}^k \tag{2}$$

This equation can be analyzed in terms of the eigenvalues of the matrix G. In general, one obtains the master stability function which is analyzed numerically [6], but for the Bernoulli maps Eq.(2) gives a polynomial of degree τ_c (for $\tau_c \geq \tau_f$) which determines the spectrum of τ_c Lyapunov exponents for each eigenmode [7].

Usually, the eigenvalue zero describes perturbation tangential to the synchronization manifold whereas all other eigenvalues determine the transverse Lyapunov exponents.

III. COMPLETE SYNCHRONIZATION

Consider two units, $N = 2$ and $G_{12} = 1 = G_{21}$, with identical feedback and coupling times $\tau = \tau_c = \tau_d$. In the limit of infinite τ, one finds an analytic result of the phase diagram shown in Fig. 1 [7, 8].

In regions I and II the two units synchronize to a common chaotic trajectory $x_t^1 = x_t^2$. After a transient, synchronization occurs without any time shift although the transmitted signal is delayed by the time τ which can be arbitrarily long. Note that without self-feedback, $\kappa = 0$, the two units cannot synchronize. This is a general result which holds for any pair of chaotic maps of flows. But with a triangle of three units, $G_{12} = G_{21} = G_{23} = G_{32} = G_{13} = G_{31}$, one finds a region of synchronization even without feedback.

For a master-slave configuration, two units with uni-directional coupling $G_{12} = 1$, $G_{21} = 0$, the system synchronizes in region II and III.

Bi-directional coupling is different from uni-directional one, interaction is more than drive.

This phenomenon is discussed in the context of public channel cryptography [3, 9–12].

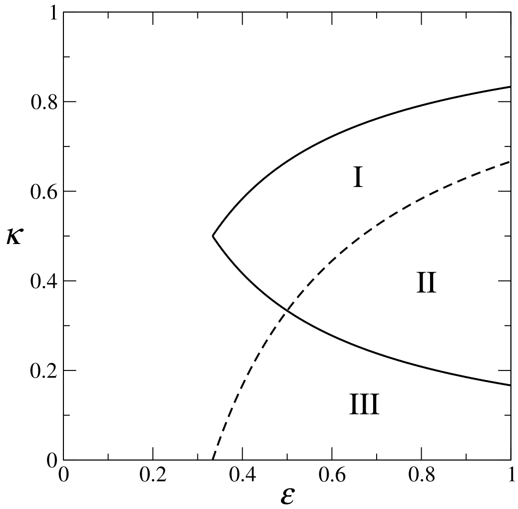

Figure 1: Phase diagram in the (ϵ, κ)-space (with ϵ and κ as coupling parameters) for the Bernoulli-shift parameter$\alpha = 3/2$ (analytical result). The areas I,II and III refer to different regimes of (sublattice-) synchronization.

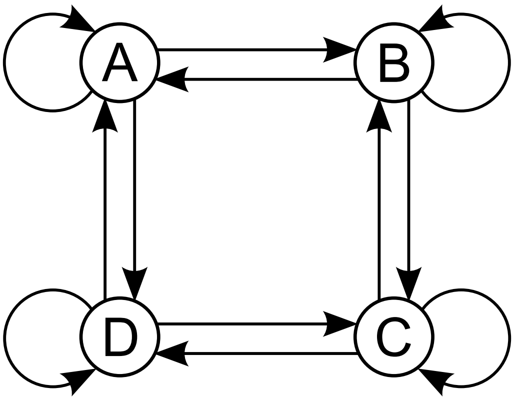

Figure 2: Ring of four bidirectionally coupled units, each unit with self-feedback.

IV. SUBLATTICE SYNCHRONIZATION

Complete synchronization is found in other networks, as well. For example, consider a ring of 4 units with $G_{ij} = G_{ji} = \frac{1}{2}$ for neighbours, as in Fig.2. The analytic stability analysis yields complete synchronization in region II of Fig.1. In addition, however, we find a new kind of synchronization. In region III the unit A and C are synchronized and B and D are synchronized to a different trajectory. One finds sublattice synchronization, the network relaxes to the pattern $\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}$. Note that other patterns are solutions of Eq.(1), as well. In this case, the patterns $\begin{smallmatrix} A & A \\ B & B \end{smallmatrix}$ and $\begin{smallmatrix} A & B \\ A & B \end{smallmatrix}$ are solutions, in agreement with the general classification of Golubitsky et. al. These configurations break the symmetry of the square, and we find that they are unstable [8]. We made similar observations in other networks, as well. Hence we conjecture:

For any networks, stable patterns of synchronized chaotic trajectories do not break the symmetry of the network.

This conjecture is based on some examples, but we do not have a proof. We encourage the reader to find a counter example.

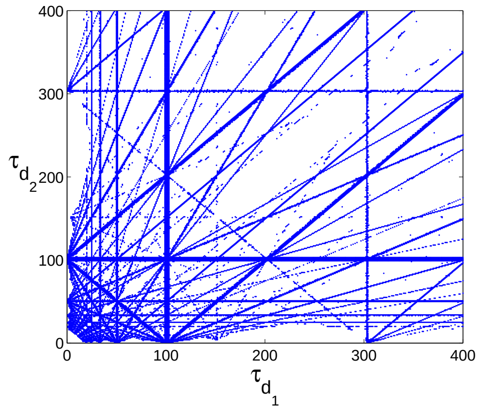

Figure 3: Simulations and semi-analytic results for the complete synchronization points for a system with double self-feedback in the (τ_{d_1}, τ_{d_2})-space for the delayed coupling $\tau_c = 101$, the Bernoulli-shift parameter $\alpha = 1.1$, and the coupling parameters $\epsilon = 0.9$, $\kappa = 0.8$.

V. MULTIPLE FEEDBACK DELAYS

For two interacting units, complete syncrhonization is only possible if the feedack delay τ_d is carefully adjusted to the coupling delay τ_c. This means a severe restriction on possible applications for secure communication with bi-directional coupling [3].

Surprisingly, a whole interval of τ_c values is possible when multiple feedback delays are added to each unit [13]. In Eq.(1) the self-feedback delay is replaced by M terms

$$\frac{\epsilon\kappa}{M} \sum_{l=1}^{M} f(x^j_{t-\tau_{dl}}) \tag{3}$$

For the Bernoulli system with two units, $N = 2$, and two feedback times, $M = 2$, we find lines of synchronization as shown in Fig. 3.

It turns out that these lines are described by the following equation

$$\sum_{l=1}^{M} n_l \tau_{dl} = m\tau_c \tag{4}$$

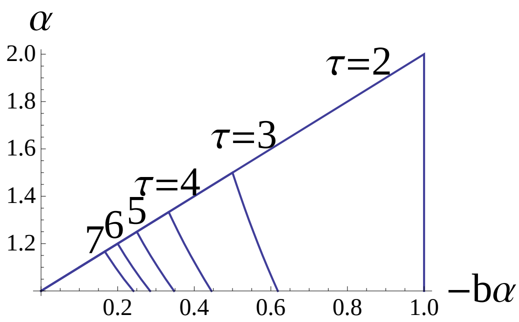

Figure 4: Regimes of stable complete synchronization. Inside the triangle the master stability function for a chaotic network is negative. α is the parameter of the Bernoulli shift with $\lambda_0 = \ln \alpha$ and b is the rescaled coupling constant. With increasing delay time τ the region of synchronization shrinks.

where $n_l \in \{0, \pm 1, ..., \pm n_{max}\}$ and $m \in \{\pm 1, \pm 3\}$. For a given set of feedback delays τ_{dl}, this equation opens several intervals of τ_c-values, where n_{max} depends on the parameters of the system. For example, with $n_{max} = 2$ and $\tau_{d1} = 1$, $\tau_{d2} = 7$, $\tau_{d3} = 49$ one obtains all integer values of τ_c up to $\tau_c = 171$.

VI. ABSENCE OF SYNCHRONIZATION

As mentioned in the introduction, chaos synchronization is a counter intuitive phenomenon. A chaotic unit has a irregular unpredictable motion, and it is surprising that a set of these units can be tamed to a common trajectory by coupling them. However, a detailed analysis shows that this is only true when the transmission delay is not too large. We conjecture [14]:

A network of chaotic units cannot be synchronized if the coupling delay times are much larger than the characteristic time scales of the individual units.

This conjecture is based on the stability equation (2). The corresponding master stability equation for any Bernoulli network is

$$\zeta_t = \alpha \zeta_{t-1} + b\alpha \zeta_{t-\tau} \tag{5}$$

with $\tau_d = \tau_c = \tau$, and b contains the eigenvalues of the coupling matrix, including the feedback term.

Fig.4 shows the analytic result of the regions of synchronzation. The isolated unit without feedback is chaotic for $\alpha > 1$ with Lyapunov exponent $\lambda_0 = \ln \alpha$. The network can only be synchronized if τ is smaller than

$$\tau_{max} = \frac{\mathrm{e}^{\lambda_0}}{\mathrm{e}^{\lambda_0} - 1} \sim \frac{1}{\lambda_0} \tag{6}$$

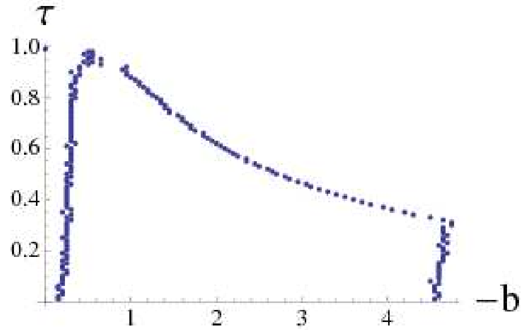

Figure 5: Regime of stable synchronization for a network of chaotic Roessler units. b is the rescaled coupling constant and τ is the delay time of the transmitted signals.

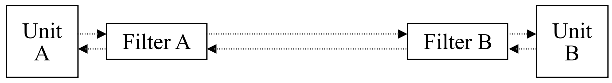

Figure 6: Two bidirectional coupled units, each with a private filter.

For large λ_0, the maximal transmission delay is given by the Lyapunov time $1/\lambda_0$. Fig. 5 shows the corresponding stability regime for any network of Roessler units [14].

Again, synchronization of chaotic units is only possible for small values of coupling delay τ. In this example, τ is even smaller than the Lyapunov time $1/\lambda_0 \simeq 10$.

VII. PUBLIC CHANNEL CRYPTOGRAPHY

As mentioned before in sec. III, bi-directional coupling may open the possibility to apply chaos synchronization to public channel cryptography [3]. This means, that two partners who want to send a secret message are not allowed to exchange secret keys before the transmission. any attacker who is recording any exchanged signal has complete knowledge about the details of the algorithms and equipments. But we assume that he cannot influence the two partners.

For chaos synchronization, this poses the following question: Two chaotic systems A and B synchronize by exchanging signals. An attacker E can record the exchanged signals and has the same equipment as A and B. Can E synchronize, as well? Recently, we have suggested a configuration, as in Fig.6, which - to our present understanding - can realize such a public channel synchronization [11, 12]. The method is based on several principles:

1. Each partner selects a private secret filter through which all exchanged signals are transmitted.

2. The communication is periodically switched on and off, and the filters are changed randomly during the off period.

3. Integer values and nonlinearities are used for the transmitted signals.

The secret filter may be a convolution with a random kernel, for example for the iterated maps the transmitted signal is defined by

$$T_t = \sum_{s=0}^{N} K_s f(x_{t-s}) \tag{7}$$

The first ingredient ensures that an attacking unit which is driven by the two exchanged signals cannot synchronize. But an attacker E may be able to calculate the private secret filters of A and B. Therefore, the number of equations which E can use is limited by ingredient 2. Ingredient 3 relates this problem to the solution of equations with integers, which is proven to be in the complexity class of NP problems. Hence, we finally conjecture:

> It is possible that two chaotic units synchronize whereas a third unit, being driven by the transmitted signals, cannot synchronize.

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[1] L. M. Pecora and T. L. Carroll. Synchronization in chaotic systems. *Phys. Rev. Lett.*, 64(8):821–824, 1990.

[2] A. Pikovsky, M. Rosenblum, and J. Kurths. *Synchronization, a universal concept in nonlinear sciences.* Cambridge University Press, Cambridge, 2001.

[3] W. Kinzel and I. Kanter. Secure communication with chaos synchronization. In E. Schöll and H.G. Schuster, editors, *Handbook of Chaos Control.* Wiley-VCH, Weinheim, second edition, 2008.

[4] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou. The synchronization of chaotic systems. *Phys. Rep.*, 366:1–2, August 2002.

[5] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang. Complex networks: Structure and dynamics. *Phys. Rep.*, 424:175–308, February 2006.

[6] Louis M. Pecora and Thomas L. Carroll. Master stability functions for synchronized coupled systems. *Phys. Rev. Lett.*, 80(10):2109–2112, Mar 1998.

[7] J. Kestler, W. Kinzel, and I. Kanter. Sublattice synchronization of chaotic networks with delayed couplings. *Phys. Rev. E*, 76:035202, 2007.

[8] J. Kestler, E. Kopelowitz, I. Kanter, and W. Kinzel. Patterns of chaos synchronization. *Phys. Rev. E*, 77(4):046209, 2008.

[9] E. Klein, N. Gross, M. Rosenbluh, W. Kinzel, L. Khaykovich, and I. Kanter. Stable isochronal synchronization of mutually coupled chaotic lasers. *Phys. Rev. E*, 73(6):066214, 2006.

[10] E. Klein, , R. Mislovaty, I. Kanter, and W. Kinzel. Public-channel cryptography using chaos synchronization. *Phys. Rev. E*, 72:016214, 2005.

[11] E. Klein, N. Gross, E. Kopelowitz, M. Rosenbluh, L. Khaykovich, W. Kinzel, and I. Kanter. Public-channel cryptography based on mutual chaos pass filters. *Phys. Rev. E*, 74(4):046201, 2006.

[12] I. Kanter, E. Kopelowitz, and W. Kinzel. Public channel cryptography: Chaos synchronization and hilbert's tenth problem. *Physical Review Letters*, 101(8):084102, 2008.

[13] M Zigzag, M Butkovski, A. Englert, W. Kinzel, and I. Kanter. Emergence of zero-lag synchronization in generic mutually coupled chaotic systems. to be published, see http://arxiv.org/abs/0811.4066.

[14] W. Kinzel, A. Englert, and I. Kanter. Synchronization of networks of chaotic units with time-delayed couplings. to be published.