NONSMOOTH STRUCTURED CONTROL DESIGN

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Abstract: Feedback controllers with specific structure arise frequently in applications because they are easily apprehended by design engineers and facilitate on-board implementations and re-tuning. This work is dedicated to H_∞-synthesis with structured controllers. In this context, straightforward application of traditional synthesis techniques fails, which explains why only a few ad-hoc methods have been developed over the years. In response, we propose a systematic way to design H_∞-optimal controllers with fixed structure using local optimization techniques. We apply non-smooth optimization techniques to compute locally optimal solutions. See the paper full version (Apkarian *et al.*, 2007) for more details and applications discussed at length.

Keywords: Nonsmooth optimization, H_∞ synthesis, structured controllers, PID, NP-hard problems.

1. INTRODUCTION

Considerable efforts have been made over the past two decades to develop new and powerful control methodologies. H_∞ synthesis (Doyle *et al.*, n.d.) is certainly the most prominent outcome of this search. In spite of its theoretical success, it turns out that structured controllers such as PID, lead-lag, observed-based, and others, are still preferred in industrial control. The reason is that controllers designed with modern control techniques are usually of high order, difficult to implement and often impossible to re-tune in case of model changes. But those are precisely the properties which make structured controllers so popular for practitioners. Easy to implement and to understand, and easy to re-tune whenever performance or stability specifications change.

Structured control design is generally a difficult problem. Even the simple static output feedback stabilization problem is known to be NP-hard (V. Blondel, 1997). Due to their importance for practice, a number of innovative techniques and heuristics for structured control have been proposed in the literature. Some authors use branch-and-bound techniques to construct globally optimal solution to the design problem (Balakrishnan and Boyd, 1992). In the same vein, Wong and Bigras (Wong and Bigras, 2003) propose evolutionary optimization to reduce the computational overhead, while still aiming at globally optimal solutions. These approaches are certainly of interest for small problems, but quickly succumb when problems get sizable.

A fairly disparate set of heuristic techniques for structured control design was developed in the realm of linear matrix inequalities (LMIs) (Boyd *et al.*, 1994; Grigoriadis and Skelton, 1996; Ghaoui and Balakrishnan, 1994; Iwasaki, 1997; Han and Skelton, 2003; Hassibi *et al.*, 1999). Iterative solving of SDPs based on successive linearizations is yet another idea, but often leads to prohibitive running times. In (Bao *et al.*, 1999), 2 hours cputime were necessary to compute a decentralized PID controller for a 2×2 process on a Pentium II 333 MHz computer. A relatively rich literature addresses specific controller structures such as decentralized or PIDs. In (Miyamoto and Vinnicombe, 1997), Miyamoto and Vinnicombe discuss a coordinate scheme for H_∞ loop-shaping with decentralized constraints. In (Tan *et al.*, 2002), again in the loop shaping context, the authors adopt a truncation procedure to reduce a full-order controller to a PID controller. Those are heuristic procedures, because closed-loop performance is not necessarily inherited by the final controller. In (Saeki, 2006), sufficient conditions are given under which PID synthesis reduces to solving LMIs.

In (Rotkowitz and Lall, 2006), Rotkowitz and Lall fully characterize a class of problems for which structured controller design can be solved using convex programming. See also (Xin *et al.*, 2004) and the analysis in (Scherer, 2002). Note that these concepts and tools only apply to particular problem classes and do not easily lend themselves to generalization for finer controller structures.

In our opinion local optimization is the approach best suited for these difficult design problems. We mention that early approaches to structured design based on tailored optimization techniques can be traced back to the work of Mäkilä and Toivonen (Mäkilä and Toivonen, 1987) for parametric LQ problems, or Polak and Wardi (Polak and Wardi, 1982). More recently, we have used nonsmooth analysis to fully caracterize the subdifferential properties of closed-loop mappings of the form $\|.\|_\infty \circ T_{w\to z}$ acting on the controller space, where $T_{w\to z}(K)$ denotes the closed-loop transfer function from w to z at a given controller K. These results are used to develop nonsmooth descent algorithms for various design problems (Apkarian and Noll, 2006*b*; Apkarian and Noll, 2006*d*; Apkarian and Noll, 2006*c*). Here we extend our results to structured controller design and elaborate the case of MIMO PID controllers.

We use concepts from nonsmooth analysis covered by (Clarke, 1983). For a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, $\partial f(x)$ denotes its Clarke subdifferential at x.

2. NONSMOOTH H_∞ DESIGN TECHNIQUE

Consider a plant P in state-space form

$$P(s): \quad \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector of P, $u \in \mathbb{R}^{m_2}$ the vector of control inputs, $w \in \mathbb{R}^{m_1}$ the vector of exogenous inputs, $y \in \mathbb{R}^{p_2}$ the vector of measurements and $z \in \mathbb{R}^{p_1}$ the controlled or performance vector. Without loss, it is assumed throughout that $D_{22} = 0$.

The focus is on H_∞ synthesis with structured controllers, which consists in designing a dynamic output feedback controller $K(s)$ with feedback law $u = K(s)y$ for the plant in (1) having the following properties:

- **Controller structure:** $K(s)$ has a prescribed structure.
- **Internal stability:** $K(s)$ stabilizes the original plant $P(s)$ in closed-loop.
- **Performance:** Among all stabilizing controllers with that structure, $K(s)$ minimizes the H_∞ norm $\|T_{w\to z}(K)\|_\infty$. Here $T_{w\to z}(K)$ denotes the closed-loop transfer function from w to z.

2.1 Subdifferential of the H_∞ map

For the time being we leave apart structural constraints and assume that $K(s)$ has the frequency domain representation:

$$K(s) = C_K(sI - A_K)^{-1}B_K + D_K,\ A_K \in \mathbb{R}^{k\times k}, (2)$$

where k is the order of the controller, and where the case $k = 0$ of a static controller $K(s) = D_K$ is included. A further simplification is obtained if we assume that preliminary dynamic augmentation of the plant $P(s)$ has been performed:

$$A \to \begin{bmatrix} A & 0 \\ 0 & 0_k \end{bmatrix}, \quad B_1 \to \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \text{etc.}$$

so that manipulations will involve a static matrix

$$\mathcal{K} := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}. \quad (3)$$

With this proviso, the following closed-loop notations will be useful:

$$\begin{bmatrix} \mathcal{A}(\mathcal{K}) & \mathcal{B}(\mathcal{K}) \\ \mathcal{C}(\mathcal{K}) & \mathcal{D}(\mathcal{K}) \end{bmatrix} := \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \mathcal{K} \left[C_2 \quad D_{21} \right] (4)$$

Owing to its special composite structure, the function $f = \|.\|_\infty \circ T_{w\to z}$, which maps the set $\mathcal{D} \subset$

$\mathbb{R}^{(m_2+k)\times(p_2+k)}$ of stabilizing controllers into $\mathbb{R}^+$, is Clarke subdifferentiable (Noll and Apkarian, 2005; Apkarian and Noll, 2006*b*; Apkarian and Noll, 2006*a*). Its Clarke subdifferential can be described as follows. Introduce the set $\Omega(\mathcal{K})$ of active frequencies at a given $\mathcal{K}$

$$\{\omega \in [0, +\infty] : \overline{\sigma}\left(T_{w\to z}(\mathcal{K}, j\omega)\right) = f(\mathcal{K})\}. \quad (5)$$

We assume throughout that $\Omega(\mathcal{K})$ is a finite set and we refer the reader to (Bompart *et al.*, 2006) for a justification of this hypothesis. We shall also need the notation:

$$\begin{bmatrix} T_{w\to z}(\mathcal{K}, s) & G_{12}(\mathcal{K}, s) \\ G_{21}(\mathcal{K}, s) & \star \end{bmatrix} := \begin{bmatrix} \mathcal{C}(\mathcal{K}) \\ C_2 \end{bmatrix} (sI - \mathcal{A}(\mathcal{K}))^{-1} \left[\mathcal{B}(\mathcal{K}) \quad B_2\right] + \begin{bmatrix} \mathcal{D}(\mathcal{K}) & D_{12} \\ D_{21} & \star \end{bmatrix}. \quad (6)$$

This leads to the following result

Theorem 2.1. Assume the controller $K(s)$ stabilizes $P(s)$ in (1), that is, $\mathcal{K} \in \mathcal{D}$. With the notations (5) and (6), let Q_ω be a matrix whose columns form an orthonormal basis of the eigenspace of $T_{w\to z}(\mathcal{K}, j\omega)T_{w\to z}(\mathcal{K}, j\omega)^H$ associated with the largest eigenvalue $\lambda_1\left(T_{w\to z}T_{w\to z}^H\right)$. Then, the Clarke subdifferential of the mapping f at $\mathcal{K} \in \mathcal{D}$ is the compact and convex set $\partial f(\mathcal{K}) = \{\Phi_Y : Y \in \mathcal{S}(\mathcal{K})\}$, where $f(\mathcal{K})\Phi_Y =$

$$\sum_{\omega\in\Omega(\mathcal{K})} \Re\left\{G_{21}(j\omega)\, T_{w\to z}(j\omega)^H Q_\omega Y_\omega (Q_\omega)^H G_{12}(j\omega)\right\}^T, \quad (7)$$

and $S(\mathcal{K})$ is the spectraplex set

$$\mathcal{S}(\mathcal{K}) = \{Y = (Y_\omega)_{\omega\in\Omega(\mathcal{K})} : Y_\omega \succeq 0, \sum_{\omega\in\Omega(\mathcal{K})} \operatorname{Tr} Y_\omega = 1\}. \quad (8)$$

Proof See (Clarke, 1983) and (Polak and Salcudean, 1989; Apkarian and Noll, 2006*b*; Apkarian and Noll, 2006*a*) for a proof and further details.

In geometric terms, the subdifferential of f is a linear image of the spectraplex set $S(\mathcal{K})$.

2.2 Structured controllers

Note that we have assumed so far that controllers have no specific structure. We now extend the results in section 2.1 to structured controllers using chain rules.

Assume $\mathcal{K}$ defined in (3) depends smoothly on a free parameter $\kappa \in \mathbb{R}^q$, that is, $\mathcal{K} = \mathcal{K}(\kappa)$, where $\mathcal{K}(\cdot)$ is smooth. Then the subgradients with respect to κ of the mapping $g = \|.\|_\infty \circ T_{w\to z}(.) \circ \mathcal{K}(.)$ at κ are obtained as $\mathcal{K}'(\kappa)^* \partial f(\mathcal{K})$, where $\partial f(\mathcal{K})$ is given in Theorem 2.1, $\mathcal{K}'(\kappa)$ is the derivative of $\mathcal{K}()$ at κ, and where $\mathcal{K}'(\kappa)^*$ is its adjoint. This is a direct application of the chain rule in (Clarke, 1983). Note that the adjoint $\mathcal{K}'(\kappa)^*$ acts on elements $F \in \mathbb{R}^{(m_2+k)\times(p_2+k)}$ via

$$\mathcal{K}'(\kappa)^* F = \left[\operatorname{Tr}\left(\frac{\partial\mathcal{K}(\kappa)}{\partial\kappa_1}^T F\right), \ldots, \operatorname{Tr}\left(\frac{\partial\mathcal{K}(\kappa)}{\partial\kappa_q}^T F\right)\right]^T .$$

We infer the following

Corollary 2.2. Assume the controller $\mathcal{K}(\kappa)$ stabilizes $P(s)$ in (1), that is, $\mathcal{K}(\kappa) \in \mathcal{D}$. With the notations of Theorem 2.1, the Clarke subdifferential of the mapping $g = \|.\|_\infty \circ T_{w\to z}(.) \circ \mathcal{K}(.)$ at $\kappa \in \mathbb{R}^q$ is the compact and convex set $\partial g(\kappa) =$

$$\left\{\left[\operatorname{Tr}\left(\frac{\partial\mathcal{K}}{\partial\kappa_1}^T \Phi_Y\right), \ldots, \operatorname{Tr}\left(\frac{\partial\mathcal{K}}{\partial\kappa_q}^T \Phi_Y\right)\right]^T : \Phi_Y \in \partial f(\mathcal{K})\right\}. \quad (9)$$

Using vectorization, the subgradients in (9) can be rewritten as

$$\left[\operatorname{vec}\frac{\partial\mathcal{K}(\kappa)}{\partial\kappa_1}, \ldots, \operatorname{vec}\frac{\partial\mathcal{K}(\kappa)}{\partial\kappa_q}\right]^T \operatorname{vec}\Phi_Y . \quad (10)$$

An important special case in practice is when the maximum singular values $\overline{\sigma}(T_{w\to z}(\mathcal{K}(\kappa), j\omega))$ have multiplicity one for every $\omega \in \Omega(\mathcal{K}(\kappa))$. Then the subgradients Φ_Y reduce in vector form to $\operatorname{vec}\Phi_Y = \Psi\,\xi$ where $\sum_{\omega\in\Omega(\mathcal{K}(\kappa))} \xi_\omega = 1$, $\xi_\omega \geq 0, \forall\omega \in \Omega(\mathcal{K}(\kappa))$ and matrix Ψ is constructed columnwise as $\Psi :=$

$$\left(\operatorname{vec}\Re\left\{G_{21}(\omega)\, T_{w\to z}(j\omega)^H Q_\omega (Q_\omega)^H G_{12}(j\omega)\right\}^T\right)_{\omega\in\Omega} .$$

Combining this expression with (10), the subdifferential $\partial g(\kappa)$ at κ admits a simpler representation in the form of a linear image of a simplex, $\partial g(\kappa) =$

$$\left\{\left[\operatorname{vec}\frac{\partial\mathcal{K}(\kappa)}{\partial\kappa_1}, \ldots, \operatorname{vec}\frac{\partial\mathcal{K}(\kappa)}{\partial\kappa_q}\right]^T \Psi\xi,\ \xi \in \text{simplex}\right\} .$$

2.3 PID controllers

If this section we specialize the above results to PID controllers. A common representation of MIMO PID controllers is

$$K(s) = K_p + \frac{K_i}{s} + \frac{K_d s}{1 + \epsilon s}, \quad (11)$$

where K_p, K_i and K_d are the proportional, the integral and the derivative gains, respectively. The PID gains K_p, K_i and K_d all belong to $\mathbb{R}^{m_2\times m_2}$ for a square plant with m_2 inputs and outputs. ϵ is a small scalar which determines how close the last term in (11) comes to a pure derivative action. Using partial fraction expansion, an alternative representation can be obtained in the form

$$K(s) = D_K + \frac{R_i}{s} + \frac{R_d}{s + \tau}, \quad (12)$$

with the correspondence

$$D_K := K_p + \frac{K_d}{\epsilon},\ R_i := K_i,\ R_d := -\frac{K_d}{\epsilon^2},\ \tau := \frac{1}{\epsilon} .$$

Note that these two representations are in one-to-one correspondence via

$$K_d = -\epsilon^2 R_d,\, K_p = D_K + \epsilon R_d,\, K_i = R_i,\, \epsilon = \frac{1}{\tau}\,.$$

From (12) we obtain a linearly parameterized state-space representation of a MIMO PID controller

$$\mathcal{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & R_i \\ 0 & -\tau I & R_d \\ \hline I & I & D_K \end{array}\right]. \qquad (13)$$

Since the state-space representation of the PID controller is affine in the parameters τ, R_i, R_d and D_k, the same is true for its vectorized form and we can write

$$\operatorname{vec}\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = \operatorname{vec}\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline I & I & 0 \end{array}\right] + T\overbrace{\begin{bmatrix} \tau \\ \operatorname{vec} R_i \\ \operatorname{vec} R_d \\ \operatorname{vec} D_K \end{bmatrix}}^{\kappa},$$

for a suitable matrix $T \in \mathbb{R}^{(k+m_2)(k+p_2)\times(3m_2^2+1)}$. The subdifferential of the mapping $g = \|.\|_\infty \circ T_{w\to z}(.) \circ \mathcal{K}(.)$ at κ, where $\mathcal{K}(\kappa)$ describes a MIMO PID controller (11) or (12) above, is the compact and convex set of subgradients

$$\partial g(\kappa) = \left\{T^T \operatorname{vec}\Phi_Y \,:\, \Phi_Y \in \partial f(\mathcal{K}(\kappa))\right\}. \qquad (14)$$

Note that the outlined procedure to describe subdifferentials is easily extended to decentralized MIMO PID controllers as well as to any controller structure of practical interest.

2.4 Setpoint filter design

When PID feedback alone is not sufficient to achieve suitable performance, prefilters or setpoint filters must be introduced. In figure 1, a typical model following strategy is shown. The setpoint filter $F(s)$ is used in such a way that the responses of the feedback controlled plant $G(s)$ match as closely as possible those of a reference model $G_{ref}(s)$. Finding such a filter could also be cast as an H_∞ synthesis problem, where the transfer function from the reference signal r to the tracking error e is minimized:

$$\underset{F(s)}{\text{minimize}}\, \|T_{r\to e}(F)\|_\infty\,. \qquad (15)$$

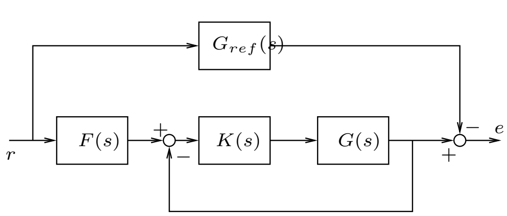

Fig. 1. setpoint filter design

To solve the setpoint filter design problem, we suggest once again the use of nonsmooth optimization methods. In order to illustrate the construction, consider the case of a two-input two-outputs system. To achieve decoupling and good quality responses, the setpoint filter is sought in the form (Tan *et al.*, 2002),

$$F(s) = \begin{bmatrix} \dfrac{1}{\tau_1 s+1} & \dfrac{a_1 s}{b_1 s+1} \\ \dfrac{a_2 s}{b_2 s+1} & \dfrac{1}{\tau_2 s+1} \end{bmatrix}. \qquad (16)$$

Setting

$$\kappa_1 = \frac{1}{\tau_1}, \kappa_2 = \frac{1}{b_1}, \kappa_3 = \frac{a_1}{b_1},$$
$$\kappa_4 = \frac{1}{\tau_2}, \kappa_5 = \frac{1}{b_2}, \kappa_6 = \frac{a_2}{b_2},$$

a state-space representation of the filter is obtained as

$$\mathcal{F}(\kappa) := \left[\begin{array}{c|c} A_F & B_F \\ \hline C_F & D_F \end{array}\right] = \left[\begin{array}{cccc|cc} -\kappa_1 & 0 & 0 & 0 & \kappa_1 & 0 \\ 0 & -\kappa_2 & 0 & 0 & 0 & -\kappa_3 \\ 0 & 0 & -\kappa_4 & 0 & 0 & \kappa_4 \\ 0 & 0 & 0 & -\kappa_5 & -\kappa_6 & 0 \\ \hline 1 & \kappa_2 & 0 & 0 & 0 & \kappa_3 \\ 0 & 0 & 1 & \kappa_5 & \kappa_6 & 0 \end{array}\right].$$

This means there exists a matrix U such that

$$\operatorname{vec}\mathcal{F}(\kappa) = \operatorname{vec}\mathcal{F}(0) + U\kappa, \qquad \kappa \in \mathbb{R}^6\,.$$

We immediately deduce the relevant subgradient formulas for program (15). With $v := \|.\|_\infty \circ T_{r\to e}(.) \circ \mathcal{F}(.)$, the subdifferential of v at κ, where $\mathcal{F}(\kappa)$ is a setpoint filter, is the compact and convex set of subgradients $\partial v(\kappa) =$

$$\left\{U^T \operatorname{vec}\Phi_Y \,:\, \Phi_Y \in \partial\left(\|.\|_\infty \circ T_{r\to e}\right)(\mathcal{F}(\kappa))\right\}. (17)$$

The remaining expression for the subdifferential is directly obtained from Theorem 2.1.

2.5 Nonsmooth descent method

For a more detailed discussion we refer the reader to (Apkarian and Noll, 2006*b*; Apkarian and Noll, 2006*d*). We start by representing the composite functions $f = \|\cdot\|_\infty \circ T_{w\to z}$ or more generally $g = \|\cdot\|_\infty \circ T_{w\to z} \circ \mathcal{K}(\cdot)$ under the form

$$g(\kappa) = \max_{\omega\in[0,+\infty]} g(\kappa,\omega),$$

where each $g(\kappa,\omega)$ is a composite maximum singular value function

$$g(\kappa,\omega) = \overline{\sigma}\left(\mathcal{G}(\kappa, j\omega)\right).$$

Here $\mathcal{G}(\kappa, j\omega) = T_{w\to z}(\mathcal{K}(\kappa), j\omega)$. At a given parameter κ, we can compute the set $\Omega(\kappa) := \Omega(\mathcal{K}(\kappa))$ of active frequencies, which is either finite, or coincides with $[0,+\infty]$ in those rare cases where the closed-loop system is all-pass. Excluding this case, we assume $\Omega(\kappa)$ finite and construct a finite extension $\Omega_e(\kappa)$ by adding frequencies according to the strategy presented in (Apkarian and Noll, 2006*b*; Apkarian and Noll, 2006*d*).

Following the general trend of Polak (Polak, 1997), we now define the optimality function $\theta_e(\kappa) := \min_{h\in\mathbb{R}^q} \max_{\omega\in\Omega_e(\kappa)} \max_{Y_\omega\succeq 0, \mathrm{Tr}(Y_\omega)=1}$

$$-g(\kappa) + g(\kappa,\omega) + h^T\phi_{Y_\omega} + \tfrac{1}{2}h^TQh, \qquad (18)$$

where for every fixed ω, ϕ_{Y_ω} is a subgradient of $g(\kappa,\omega)$ at κ obtained as $\phi_{Y_\omega} :=$

$$\left[\mathrm{Tr}\,(\tfrac{\partial\mathcal{K}(\kappa)}{\partial\kappa_1}^T\Phi_{Y_\omega}),\ldots,\mathrm{Tr}\,(\tfrac{\partial\mathcal{K}(\kappa)}{\partial\kappa_q}^T\Phi_{Y_\omega})\right]^T,$$

where $g(\kappa,\omega)\Phi_{Y_\omega} = \Re G_{21}(j\omega)\,T_{w\to z}(j\omega)^H\times$

$$Q_\omega Y_\omega (Q_\omega)^H G_{12}(j\omega)\,, Y_\omega \succeq 0, \mathrm{Tr}\,Y_\omega = 1.$$

The model of the objective function represented by θ_e is in principle of first order, but the quadratic term h^TQh may in some cases be used to include second order information. In (Apkarian and Noll, 2006*b*; Apkarian and Noll, 2006*d*) we had worked with the basic choice $Q = \delta I \succ 0$, but we shall propose a more sophisticated choice here using BFGS updates.

Notice that independently of the choices of $Q \succ 0$ and the finite extension $\Omega_e(\kappa)$ of $\Omega(\kappa)$ used, the optimality function has the following property: $\theta_e(\kappa) \le 0$, and $\theta_e(\kappa) = 0$ if and only if $0 \in \partial g(\kappa)$, that is, κ is a critical point of g. In order to use θ_e to compute descent steps, it is convenient to obtain a dual representation of θ_e. To do this we use Fenchel duality to swap the max and min operators in (18). This means that we first replace the first inner supremum by a supremum over a convex hull which does not alter the value of θ_e. Then, after swapping max and min, the now inner infimum over $h \in \mathbb{R}^q$ becomes unconstrained and can be computed explicitely. Namely, for fixed Y_ω and τ_ω in the outer program, we obtain the solution of the form

$$h(Y,\tau) = -Q^{-1}\left(\sum_{\omega\in\Omega_e(\kappa)} \tau_\omega\phi_{Y_\omega}\right). \qquad (19)$$

Substituting this back we obtain the dual expression

$$\theta_e(\kappa) = \max_{\tau_\omega\ge 0,\ \sum_{\omega\in\Omega_e(\kappa)}\tau_\omega=1}\ \max_{Y_\omega\succeq 0,\ \mathrm{Tr}(Y_\omega)=1} \sum_{\omega\in\Omega_e(\kappa)} \tau_\omega \times$$
$$(g(\kappa,\omega) - g(\kappa))$$
$$-\tfrac{1}{2}\left(\sum_{\omega\in\Omega_e(\kappa)}\tau_\omega\phi_{Y_\omega}\right)^T Q^{-1}\left(\sum_{\omega\in\Omega_e(\kappa)}\tau_\omega\phi_{Y_\omega}\right). \qquad (20)$$

Notice that in its dual form, computing $\theta_e(\kappa)$ leads to a semidefinite program. Indeed, substituting $Z_\omega = \tau_\omega Y_\omega$, program (20) becomes

$$\theta_e(\kappa) = \max_{Z_\omega\succeq 0,\ \sum_{\omega\in\Omega_e(\kappa)}\mathrm{Tr}(Z_\omega)=1} \sum_{\omega\in\Omega_e(\kappa)} \mathrm{Tr}(Z_\omega)\times$$
$$(g(\kappa,\omega) - g(\kappa))$$
$$-\tfrac{1}{2}\left(\sum_{\omega\in\Omega_e(\kappa)}\phi_{Z_\omega}\right)^T Q^{-1}\left(\sum_{\omega\in\Omega_e(\kappa)}\phi_{Z_\omega}\right). \qquad (21)$$

The latter program is converted to an LMI problem using a Schur complement argument. As a byproduct we see that $\theta_e(\kappa) \le 0$ and that $\theta_e(\kappa) = 0$ implies κ is critical that is, $0 \in \partial g(\kappa)$.

What is important is that the direction $h(Y,\tau) = h(Z)$ in (19) is a descent direction of g at κ in the sense that the directional derivative satisfies the decrease condition

$$g'(\kappa; h(Z)) \le \theta_e(\kappa) - \tfrac{1}{2}\left(\sum_{\omega\in\Omega_e(\kappa)}\phi_{Z_\omega}\right) Q^{-1}\left(\sum_{\omega\in\Omega_e(\kappa)}\phi_{Z_\omega}\right),$$

where Z is the dual optimal solution. See (Apkarian and Noll, 2006*d*, Lemma 4.3) for a proof. In conclusion, we obtain the following algorithmic scheme:

Set Parameters $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \delta \ll 1$.
1. **Initilization**. Find a structured closed-loop stabilizing controller $\mathcal{K}(\kappa)$.
2. **Active frequencies**. Compute $g(\kappa)$ using the algorithm of (Boyd and Balakrishnan, 1990) in its refined version (Bompart *et al.*, 2006) and obtain set of active frequencies $\Omega(\kappa)$.
3. **Add frequencies**. Build finite extension $\Omega_e(\kappa)$ of $\Omega(\kappa)$ as proposed in (Apkarian and Noll, 2006*b*; Apkarian and Noll, 2006*d*), and choose $Q \succeq \delta I$.
4. **Step computation**. Compute $\theta_e(\kappa)$ by the dual SDP (20) and thereby obtain direction $h(Z)$ in (19). If $\theta_e(\kappa) = 0$ stop. Otherwise:
5. **Line search**. Find largest $t = \beta^k$ such that $g(\kappa + th(Z)) < g(\kappa) - \alpha t\theta_e(\kappa)$ and such that $\mathcal{K}(\kappa + th(Z))$ remains stabilizing.
6. **Step**. Replace κ by $\kappa + th(Z)$ and go back to step 2.

Notice that the line search in step 5 is successful because $t^{-1}\left(g(\kappa + th(Z)) - g(\kappa)\right) \to g'(\kappa; h(Z))$ as $t \to 0^+$, and because $\theta_e(\kappa) < 0$ and $0 < \alpha < 1$. Choosing t under the form $t = \beta^k$ with the largest possible k comes down to doing a backtracking line search, which safeguards against taking too small steps.

The reader is referred to the full version (Apkarian *et al.*, 2007) to see how matrix Q is computed to incorporate second-order information. Finally, we emphasize the important fact that when singular values $\overline{\sigma}\left(\mathcal{G}(\kappa, j\omega)\right)$ are simple on $\Omega_e(\kappa)$, which is the rule in practice, we have $Z_\omega = \operatorname{Tr} Z_\omega$ so that SDP (21) simplifies to a much faster convex QP.

3. CONCLUSION

We have presented and discussed a nonsmooth optimization technique for the synthesis of finely structured controllers with an H_∞ objective. Our approach is general and encompasses most controller structures and is endowed by a tractable convergence certificate.

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