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DYNAMICS OF LOWDIMENSIONAL ENSEMBLES OF PHASE SYSTEMS WITH UNIDIRECTIONAL COUPLING

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Abstract

In current work we study collective dynamics of lowdimensional chain of unidirectionally coupled phase-locking systems. The existence conditions of synchronous regimes are obtained. For such values of control parameters, when there is no fixed points in the phase space, the asynchronous autooscillatory regimes and transitions between them are researched. It was discovered that the chain with $n + 1$ elements inherits the structure of parameter space of chain with n elements.

Key words

Phase-locking, synchronisation, nonlinear dynamics, coupled oscillators.

1 Introduction

Mathematical models of coupled phase-locking systems ensemble are of great interest as a main theoretical tool for investigating the dynamics of various multielemental oscillatory systems for various applications, e.g. phased antenna arrays, frequency etalon cascades, communication networks, electrical grids [Kapranov et. al., 1984; Afraimovich et. al., 1995] etc. Recently the interest to this subject have noticeably increased due to appearance of new problems of coherent power summation [Mishagin and Shalfeev, 2006; Brignon, 2013], neuron networks researches [Matrosov et. al., 2013], Kuramoto oscillator ensembles [Abrams and Strogatz, 2006], social and economical dynamic models [Osipov, Kurths and Zhou, 2007]. On the one hand intense researchers interest in phase-locking system ensemble analysis relates to comparably simple dynamics of individual elements. On the other hand this interest is related to the possibility of coupling such elements into a networks with various configuration in order to investigate the collective dynamics of oscillatory systems (emergence of complex dynamics, formation and evolution of structures etc.)

Series of publications are devoted to the research of phase systems collective dynamics (phase-locking systems) [Kapranov et. al., 1984; Afraimovich et. al., 1995; Osipov, Kurths and Zhou, 2007]. Most of this investigations concerned the existence problems of the stable phase-locking regimes for the elements in chain or lattice oscillatory ensembles. It is worth mentioning, that due to significant difficulties in analytical treatment in this articles simplified models are turned to account, which assuming chain- or lattice-like topology of the ensemble and, generally, the uniformity of the elements.

Exploration could be much simpler if one consider the ensemble with low number of elements – a **small ensemble**. Such models are of interest by itself and as an extreme case of the large ensembles. Research of small ensembles by means of nonlinear dynamics in the combination with numerical analysis allows to obtain sufficiently complete description of the collective dynamics in such ensembles. Analysis of existing articles demonstrates, that despite of the attractiveness of small ensemble models, the dynamics of such systems is insufficiently studied in comparison with the dynamics of Kuramoto oscillator ensembles [Strogatz, 2000]. In this sense the most intension of this research is the study of the small ensemble of coupled phase-locking systems dynamics.

The main goal of this work – is the investigation of the synchronous and asynchronous autooscillatory regimes in the elements of lowdimensional chain of unidirectionally coupled phase-locking systems and localisation of the regions in the parameter space that response to a qualitatively different dynamical regimes.

2 Basic model

Consider the multichannel system (Fig. 1) that contains the chain of phase-locking loops **(PLL)**, coupled in the way that i-th generator **(G)** is driven by both the signal from i-th and $i-1$-th phase discriminators **(PD)** (see Fig. 1). For the simplified analysis there were

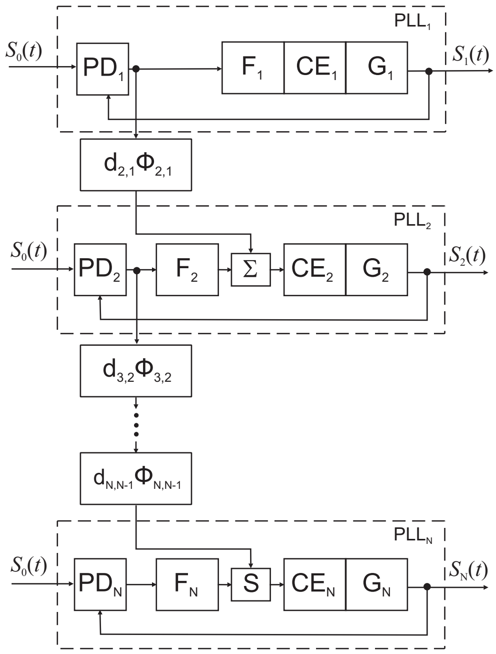

Figure 1. General model of unidirectionally coupled PLLs. Here $S_0(t)$ – reference signal, $S_n^n(t)$ – n-th PLL output signal, G_n – controllable generator, F_n – frequency filter, CE_n – control element, PD_n – phase discriminator, $\Phi_{n,n-1}$ – signal from $(n-1)$-th phase discriminator (normed), $d_{n,n-1}$ – coupling parameter, Σ – summator.

made following assumptions: 1) all PLL-s are identical; 2) there is no filter in the control circuit; 3) characteristic function of phase discriminator is sinusoidal. Thus using assumptions above the mathematical model of the ensemble is like follows:

$$\begin{cases} \frac{d\varphi_1}{d\tau} = \gamma - \sin\varphi_1, \\ \frac{d\varphi_n}{d\tau} = \gamma - \sin\varphi_n - \delta\sin\varphi_{n-1},\ n = \overline{2,N}, \end{cases} \quad (1)$$

where $\forall n = \overline{1,N}$ phase variable $\varphi_n = \theta_0 - \theta_n$ – is the phase difference between n-th generator and the reference signal, γ - arbitrary parameter characterizing the initial frequency detuning, δ – arbitrary coupling parameter.

3 Dynamical regimes

Partial element of the chain could function in the following regimes [Osipov, Kurths and Zhou, 2007]: *Synchronous regime* of the element‘s generator and the reference signal, such that their frequencies are equal; *Quazysynchronous regime* such that reference signal and generator frequencies are equal on the average and on the output from the PLL one can observe the oscillations with the angular modulation of the mean frequency stabilized by the frequency of reference signal; *Asynchronous regime* takes place if average frequency difference is not equal to zero and absolute value of phase difference between generator and reference signal infinitely increases. In the phase space

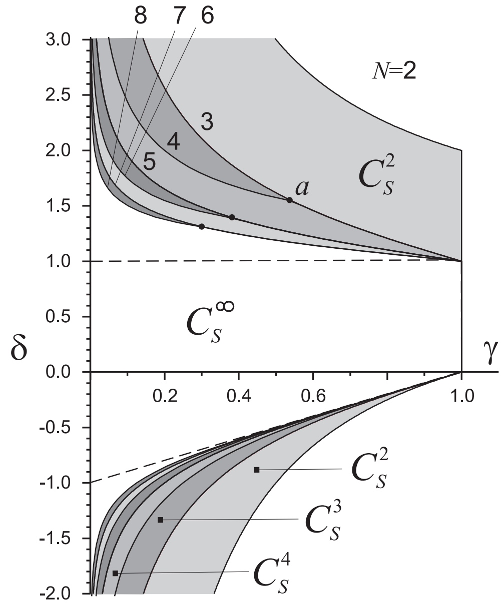

Figure 2. Synchronous regimes of the system (1). C_S^N – region in parameter space, where synchronization of N-elemental chain takes place. C_S^∞ – region of global synchronization.

of the partial system synchronous regime corresponds to a single point, quazisinchronous and asynchronous regimes correspond to finite and infinite trajectories accordingly.

Considering dynamics of the ensemble as a complex of elements with individual dynamics, one can determine [Afraimovich et. al., 1995] *globally synchronous* regime, such that all the partial systems are in synchronous regime; *partially synchronous* regime – only part of the oscillators are in synchronous regime. Partially synchronous regime could be divided into partially synchronized regimes with more delicate structure, for instance regimes taking to account number of synchronized elements and also their distribution over the space of the chain. If there is no synchronized elements, at the first place will be analysed number of quazisynchronous elements and their distribution over the space of the ensemble, besides there is an individual interest in partition of the oscillations to regular and chaotic and transformations of this oscillations from element to element as well.

4 Synchronization

Exploration of the global synchronization regime reduces to the analysis of the fixed points of the model (1) – to exploring the stable equilibrium points and defining regions of parameters where this points exist.

On Fig. 2 regions of existence of equilibrium points of the model (1) C_S^N are presented for number of elements $N = 2,3,4,5,6,7,8$. With values of parameters from this regions corresponding chain is globally stable and all the elements are synchronized with the reference signal. Area of C_S^N decreases over increasing N and with $N \to \infty$ region C_S^N degenerate into $C_S^\infty = \{\gamma - 1 < \delta < 1, \gamma < 1\}$. When $\gamma > 1$ synchronous regimes are absent. The boundaries of C_S^N are smooth for odd $N > 2$ and for even N from region $\delta < 0$. Boundaries C_S^N in region $\delta > 0$ for even N have fractures in the points $a(\delta^*, \gamma^*{=}1-\delta^*)$, where δ^* is the solution of $\delta^{N-1} - 2\left[\sum_{i=0}^{N-2}(-\delta)^i\right] = 0, N = 4, 6, 8, \ldots$. Boundaries C_S^{N-1} and C_S^N to the right of points a are identical.

5 Asynchronous regimes

Lets consider the uniform chain containing two elements. The dynamical processes of this chain are governed by the system of two first equations from (1):

$$\begin{cases} \frac{d\varphi_1}{d\tau} = \gamma - \sin\varphi_1, \\ \frac{d\varphi_2}{d\tau} = \gamma - \sin\varphi_2 - \delta\sin\varphi_1. \end{cases} \quad (2)$$

Right hand sides of this system are 2π-periodical functions with both phase coordinates φ_1 φ_2, so the phase space of this system is twodimensional torus surface: $T_2 = \{\varphi_1(\mathrm{mod}2\pi), \varphi_2(\mathrm{mod}2\pi)\}$. Possible limit sets of the system (2) are equilibrium points, homoclinic orbits and limit cycles. Due to the topology of phase space limit cycles could be of two types: 1-st type limit cycle that envelope equilibrium points and 2-nd type limit cycles that envelope phase torus in the direction of the parallel and (or) meridian. Features of trajectory behavior are determined by the rotation number [Pliss, 1966]:

$$\nu_1 = \lim_{\varphi_1 \to +\infty} \frac{\Phi(\varphi_1, \varphi_2^0)}{\varphi_1},$$

where $\Phi(\varphi_1, \varphi_2^0)$ is the solution of the equation (2). Integral trajectories are closed or unclosed depending on the rationality or irrationality of the rotation number.

First equation from (2) solvable analytically and for different values of parameter γ solution behaves in a different way: with $\gamma \le 1$ solution converges to the stationary state $\varphi_1^* = \arcsin\gamma$; with $\gamma > 1$ solution

$$\varphi_1(\tau) = 2\arctan\left[\frac{\gamma - 1}{\sqrt{\gamma^2 - 1}} \times \tan\frac{\sqrt{\gamma^2 - 1}}{2}(\tau + \tau_0)\right]$$

infinitely grows on the equivalence class $[-\pi, \pi)$ [Katok, 1995] for all τ_0 and with continuous increasing of τ. All solutions of the first equation from (2) can not be both oscillatory and bounded so there is no 1-st type limit cycles in the (2) phase space.

Starting with $\gamma \le 1$ and considering stationary regimes it can be shown, that the second equation reduces to a topological analog of the first equation:

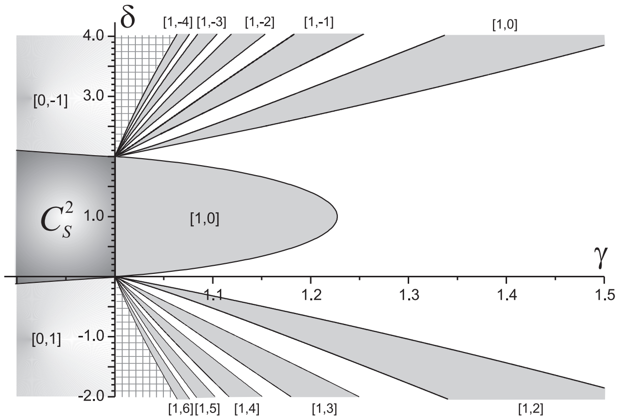

Figure 3. The parameter space (γ, δ) of model (2). In the region $\gamma > 1$ gray subregions correspond to periodic solutions with rotation number noted in square brackets, white subregions correspond to a quaziperiodic solutions. Region with square hatching contains numerable number of subregions corresponding to periodic solutions separated with the numerable number of subregions corresponding to quaziperiodic solutions. Note that when $\gamma \to 1{+}0$ second rotation number decreases infinitely.

$\dot{\varphi_2} = \gamma_2 - \sin\varphi_2$, where $\gamma_2 = \gamma - \delta\gamma = const$ and thus all the dynamics depends on the values of γ_2 and described above.

If $\gamma > 1$ then $\dot{\varphi_2}$ is modulated by φ_1 and dynamics becomes much complex. Bifurcational analysis of the system (2) is presented on the Fig. 3. On this figure with the parameter from regions marked with $[i, j]$ system (2) have the only attractor – limit cycle that envelopes the phase torus i times over the meridian and j times over the parallel. Crossing of the boundary of the gray region always leads to a saddle-node bifurcation of the limit cycle except the case of crossing the boundary $(\gamma = 1, 0 \le \delta \le 2)$ – in this case crossing goes with the conjunction of the limit cycle and homoclinic trajectory.

Details of the region marked with square hatching displayed on the Fig. 4. Points on the figure are obtained as a trajectory points of the descrete map on the secant $\varphi_2 = -\pi$, generated by the trajectories of the twodimensional system. Due to complexity of the limit cycle form there exist set of points where trajectory intersects the secant and for any of this point there is always one point where trajectory intersects the secant in opposite direction – that is why on the diagram appear curves l^+ and l^-.

With increasing number of elements in the chain of PLL structure of model (1) parameter space becomes more complex, but the model of dimension N inherits the structure of parameter space from the model of dimension $N - 1$. Lets consider the transformations of the fragments from parameter space, determined by the increase of N on the example of the transition between model (2) to (1) with $N = 3$.

Model (1) in the case of $N = 3$ takes the following

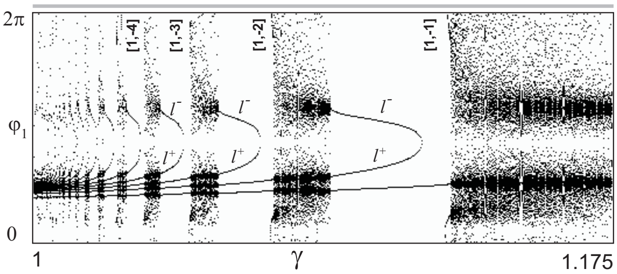

Figure 4. One-parameter bifurcation diagram for $\delta = 4.5$ on the secant $\phi_2 = -\pi$. Second rotation number grows infinitely when $\gamma \to 1 + 0$. Curves l^- and l^+ appear due to a complex structure of the corresponding limit cycle.

form:

$$\begin{cases} \frac{d\varphi_1}{d\tau} = \gamma - \sin\varphi_1, \\ \frac{d\varphi_2}{d\tau} = \gamma - \sin\varphi_2 - \delta\sin\varphi_1, \\ \frac{d\varphi_3}{d\tau} = \gamma - \sin\varphi_3 - \delta\sin\varphi_2. \end{cases} \quad (3)$$

This model is determined on the three-dimensional phase torus $T_3 = \{\varphi_1(\text{mod}2\pi), \varphi_2(\text{mod}2\pi), \varphi_3(\text{mod}2\pi)\}$. Similarly with the case T_2 we will designate trajectories in T_3 with three numbers $[i, j, k]$. Due to unidirectional type of coupling i and j are inherited from the trajectories of the system (2) and remain unchanged. The rotation numbers ν_1 and ν_2, where

$$\nu_2 = \lim_{\varphi_2 \to +\infty} \frac{\Phi(\varphi_2, \varphi_3^0)}{\varphi_2},$$

will be used to characterize the solutions on the T_3. Trajectories will be closed if ν_1 and ν_2 are rational, otherwise trajectories will be unclosed. Consequently follows the conclusion that periodic orbits for model (3) could exist only when parameters are chosen from the regions of existence of periodic orbits and synchronous regimes of model (2).

The transition between parameter space of the ensemble with $N = 2$ to parameter space of the ensemble with $N = 3$ demonstrated on Fig. 5. Gray regions from the parameter space for $N = 2$ now divided in the same manner on subregions of periodic solutions separated by the subregions corresponding to a quaziperiodic solutions. White quaziperiodic regions remain unchanged. Square hatching marks the same regions as it was for the case of $N = 2$ with according modifications.

6 Arbitrary length chain features

Consider the chain with arbitrary dimension that functions in partial synchronization regime. In such chain all possible dynamical regimes of partial elements are presented. First n^* elements function in synchronization regime, PLL_{n^*+1} functions in asynchronous

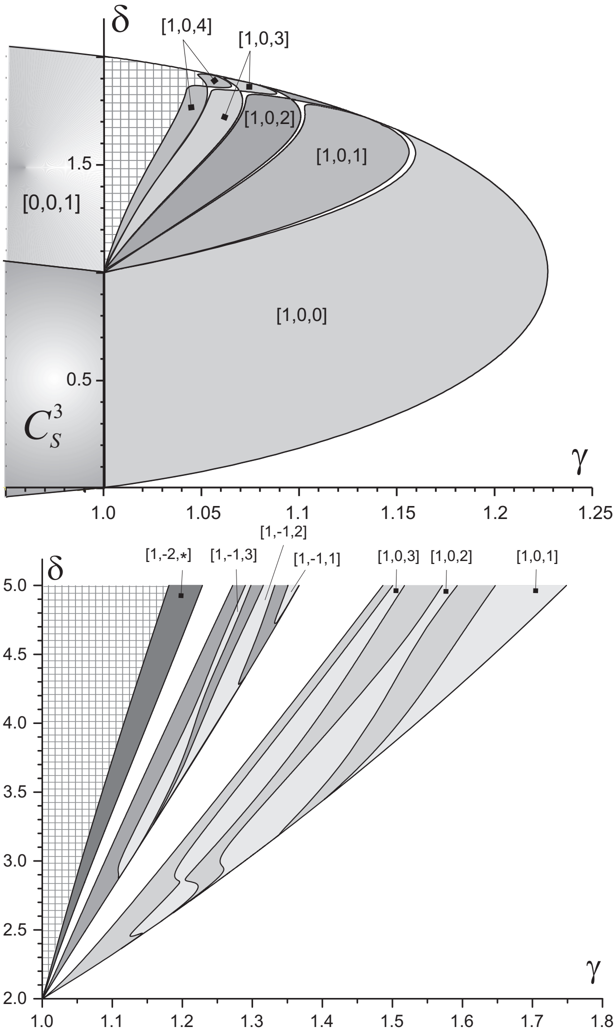

Figure 5. Evolution of the parameter space of the system (1) in the transition from $N = 2$ to $N = 3$. The "*"-symbol designates the regime of quazisynchronous dynamics of φ_3.

regime where φ_{n^*+1} evolution could be found analytically. Oscillations of consequent elements could be quaziperiodic or periodic, simple or complex. Form of oscillations would depend on the element number n and model parameters γ and δ.

Oscillatoins in PLL_n are governed by the equation:

$$\frac{d\varphi_n}{d\tau} = \gamma - \sin\varphi_n - \delta\sin\varphi_{n-1}, \quad (4)$$

and could be only regular. Chaotic dynamics could appear when φ_{n-1} changes chaotically, but the chain structure prevents this. For $n < n^*$ the variable φ_{n-1} is constant. For $n = n^* + 1$ equation (4), using analytical solution, transforms to a nonautonomous first order equation with regular dynamics. In this case dynamics of (4) is the same to the dynamics of the first element of the chain with $\gamma > 1$. For $n = n^* + 2$ the equation (4) is the first order dynamical system under external regular impact. In this case the dynamics of model (4) is the same as the dynamics of (2) and is regular. Consequent analysis of elements to the end of the chain demonstrates that oscillations of every n-th element $\forall n > 1$ is regular (due to regularity of φ_{n-1}), thus chaotic oscillations are not possible in the model (1).

7 Summary

In current work the results of the research of phase synchronization systems chain model for all real values of coupling and initial frequency detuning parameters are presented. Boundaries of the synchronous regime in the parameter space for every particular element were found and thus boundary of global synchronization regime was obtained. It is worth noticing that there is a limitation for the boundary of synchronous regime of even elements related to the form of the boundary for preceding element.

The boundaries of autooscillatory regimes with various rotation number were specified and features of mutual transitions between them were studied. Note that all the autooscillatory regimes regions are separated with the quazisynchronous regime regions and transitions between them always realizes due to saddle-node bifurcation of the corresponding limit cycle in the phase space (exept special case, discussed above). Consequently number increase of trajectory rotations before closure is not a result of period doubling cascade, but is the result of the birth of the trajectory with large number of the rotations due to the phase trajectory shrink on the phase torus.

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