On D-stability of Mechanical Systems

Larisa A.Burlakova

***Abstract*— The well-known concept of D-stability of matrices is applied to a special kind of matrices, which belong to the class of matrices for linear (linearized) mechanical systems. Systems with two and three degrees of freedom are considered.**

I. Introduction

The concept of D-stability of matrices appeared pretty long ago, initially in works on mathematical economics [1]. Later it has found applications in mathematical methods of ecology. As far as $n \times n-$matrices of general form are concerned, only some necessary and sufficient conditions of D-stability are known ([2], [3],[4], [5] etc.). Necessary and sufficient conditions of D-stability are known for 2nd and 3rd-order matrices. As far as $4 \times 4-$matrices are concerned, the paper [6] discusses one of the algorithms intended for verification of the property of D-stability, which has been implemented for a particular case, and the paper [7] discusses the analytical results obtained.

II. Principal Definitions

Let $M_n(R)$ be a set of quadratic $n \times n-$matrices over the domain R of real numbers; $\sigma(A)$ be the spectrum of matrix $A \in M_n(R)$; $D_n \subset M_n(R)$ be a class of diagonal matrices with positive elements on the main diagonal.

Definition 1: Matrix $A \in M_n(R)$ is called D-stable if $Re(\lambda) < 0$ for all $\lambda \in \sigma(DA)$ for any $D \in D_n$.

Definition 2: Matrix $Q \in M_n(R)$ belongs to the class P_0, when all the main minors of matrix Q are nonnegative, and for each $k \leq n$ there exists a strongly positive minor of matrix Q, which has the order k [2].

The requirement $A \in (-P_0)$ is the necessary condition of D-stability for matrices $A \in M_n(R)$ [2], it ensures positivity of coefficients of the characteristic polynomial of matrix DA for all $D \in D_n$.

Let matrix A be the matrix in the differential equation of a linear mechanical system:

$$\ddot{x} - B\dot{x} - Cx = 0, \quad x \in R^m, \ \dot{x} = \frac{dx}{dt}, \tag{1}$$

$$A = \begin{pmatrix} B & C \\ E & 0 \end{pmatrix}, \tag{2}$$

where B is an $m \times m-$matrix of velocity forces (both dissipative and gyroscopic ones), C is an $m \times m-$matrix of positional forces (conservative and nonconservative ones), E is a unit matrix.

The work has been supported by an INTAS-SB RAS grant No.06-1000013-9019

L. A. Burlakova , Institute for System Dynamics and Control Theory, Siberian Branch, Russian Academy of Sciences, 134, Lermontov Str., Irkutsk, 664033, Russia; irteg@icc.ru

If the matrix of a linear differential system is D-stable, then let us call the system D-stable. The pithy character of such a concept is confirmed by the fact that there exist asymptotically stable mechanical systems, which either possess or fail to possess the property of D-stability.

When the concept of D-stability is formally applied to (2), the matrix $D \in D_{2m}$ is arbitrary. In order to retain the structure of the matrix after multiplying it by D, it is necessary to put $d_{m+1} = ... = d_{2m} = 1$, what represents a particular case. If matrices C and B are diagonal ones, then the system (1) is D-stable, when both these matrices are definite negative. This is an obvious corollary of Thomson–Tait–Chetayev's theorem on the possibility of stabilization of a stable conservative system by any dissipative forces with complete dissipation.

III. A System with Two Degrees of Freedom

In the process of investigation of the property of D-stability, the problem is reduced to verification of positiveness everywhere in the positive orthant of Hurwitz determinants for the matrix DA, which are real polynomials of n variables. When matrix $A \in M_4(R)$, the Hurwitz polynomial of matrix DA represents a 6th-order polynomial of four d_i for each 3rd-order polynomial d_i. The necessary conditions of positiveness of such a polynomial in the positive orthant, which are complementary to the property $A \in (-P_0)$, write [7]:

$$\begin{gathered}\Delta_{44} > 0,\\ -\Big(\sqrt{-A_{3_3}A_{2_{2,4}}} + \sqrt{-A_{3_2}\Delta_{22}} + \sqrt{-A_{2_{2,3}}\Delta_{33}}\Big)^2 a_{1,1} \geq a_{1,1}^2\Delta_{44} \geq 0,\\ -\Big(\sqrt{-A_{3_3}A_{2_{1,4}}} + \sqrt{-A_{3_1}\Delta_{22}} + \sqrt{-A_{2_{1,3}}\Delta_{33}}\Big)^2 a_{2,2} \geq a_{2,2}^2\Delta_{44} \geq 0,\\ -(\sqrt{-A_{3_2}A_{2_{1,4}}} + \sqrt{-A_{3_1}A_{2_{2,4}}} + \sqrt{-A_{2_{1,2}}\Delta_{33}})^2 a_{3,3} \geq a_{3,3}^2\Delta_{44} \geq 0,\\ -(\sqrt{-A_{3_3}A_{2_{1,2}}} + \sqrt{-A_{3_2}A_{2_{1,3}}} + \sqrt{-A_{3_1}A_{2_{2,3}}})^2 a_{4,4} \geq a_{4,4}^2\Delta_{44} \geq 0;\end{gathered} \tag{3}$$

$$\begin{gathered}\big(\sqrt{-A_{2_{1,2}}a_{2,2}} + \sqrt{-A_{2_{1,3}}a_{3,3}} + \sqrt{-A_{2_{1,4}}a_{4,4}}\big)^2 \geq (-A_{3_1}) \geq 0,\\ \big(\sqrt{-A_{2_{1,2}}a_{1,1}} + \sqrt{-A_{2_{2,3}}a_{3,3}} + \sqrt{-A_{2_{2,4}}a_{4,4}}\big)^2 \geq (-A_{3_2}) \geq 0,\\ \big(\sqrt{A_{2_{1,3}}(-a_{1,1})} + \sqrt{A_{2_{2,3}}(-a_{2,2})} + \sqrt{\Delta_{22}(-a_{4,4})}\big)^2 \geq (-A_{3_3}) \geq 0,\\ \big(\sqrt{A_{2_{1,4}}(-a_{1,1})} + \sqrt{A_{2_{2,4}}(-a_{2,2})} + \sqrt{\Delta_{22}(-a_{3,3})}\big)^2 \geq (-\Delta_{33}) \geq 0.\end{gathered} \tag{4}$$

Here $A_{j_{i_1,\ldots,i_{n-j}}}$ are the main minors of order j; the subindices indicate the numbers of deleted rows and columns of matrix A in the order of increase of the numbers; Δ_{kk} are the main diagonal minors of order k. Note, simultaneous equality on the right with respect to the groups of conditions (3) and (4) is not admitted.

The system of necessary inequalities $A \in (-P_0)$, (3) and (4) for the 2nd-order matrix A (2) has the form:

$$\begin{aligned} b_{1,1} < 0,\ b_{2,2} \le 0, -c_{1,2}c_{2,1} + c_{1,1}c_{2,2} > 0, \\ -b_{2,2}c_{1,1} + b_{1,2}c_{2,1} \le 0, \\ -b_{1,2}b_{2,1} + b_{1,1}b_{2,2} \ge 0, \\ b_{2,1}c_{1,2} - b_{1,1}c_{2,2} < 0,\ c_{2,2} < 0,\ c_{1,1} \le 0, \\ b_{1,1}c_{1,2}(-b_{2,1}c_{1,1} + b_{1,1}c_{2,1}) \ge 0, \\ b_{2,2}c_{2,1}(b_{2,2}c_{1,2} - b_{1,2}c_{2,2}) \ge 0, \\ b_{1,2}c_{2,1}(b_{2,2}c_{1,1} - b_{1,2}c_{2,1}) \ge 0, \\ b_{2,1}c_{1,2}(-b_{2,1}c_{1,2} + b_{1,1}c_{2,2}) \ge 0. \end{aligned} \tag{5}$$

Hence, in order to obtain D-stability of matrix $A \in M_4(R)$ (2), it is necessary that $c_{1,1} < 0$, $c_{2,2} < 0$ and $b_{1,1} < 0$, $b_{2,2} \le 0$, and in the case, when there are no zero terms in matrices B and C, $c_{1,2}c_{2,1} > 0$, $b_{1,2}c_{1,2} > 0$, $b_{1,2}b_{2,1} > 0$.

Matrix (2) is D-stable when, in addition to (5), for any positive d_i there takes place the condition

$$\begin{aligned} & d_1^2 d_3 d_4 b_{1,1}c_{1,2}(-b_{2,1}c_{1,1} + b_{1,1}c_{2,1}) + \\ & d_1^2 d_3^2 b_{1,1}c_{1,1}(b_{2,2}c_{1,1} - b_{1,2}c_{2,1}) - \\ & d_1 d_2^2 d_3 b_{2,2}(b_{1,2}b_{2,1} - b_{1,1}b_{2,2})(-b_{2,2}c_{1,1} + b_{1,2}c_{2,1}) + \\ & d_1^2 d_2 d_3 b_{1,1}(-b_{1,2}b_{2,1} + b_{1,1}b_{2,2})(-b_{2,2}c_{1,1} + b_{1,2}c_{2,1}) - \\ & d_1 d_2 d_3^2 b_{1,2}c_{2,1}(-b_{2,2}c_{1,1} + b_{1,2}c_{2,1}) - \\ & d_1 d_2^2 d_4 b_{2,2}(b_{1,2}b_{2,1} - b_{1,1}b_{2,2})(b_{2,1}c_{1,2} - b_{1,1}c_{2,2}) - \\ & d_1 d_2 d_4^2 b_{2,1}c_{1,2}(b_{2,1}c_{1,2} - b_{1,1}c_{2,2}) - \\ & d_2^2 d_4^2 b_{2,2}c_{2,2}(b_{2,1}c_{1,2} - b_{1,1}c_{2,2}) - \\ & d_1^2 d_2 d_4 b_{1,1}(-b_{1,2}b_{2,1} + b_{1,1}b_{2,2})(-b_{2,1}c_{1,2} + b_{1,1}c_{2,2}) + \\ & d_2^2 d_3 d_4 b_{2,2}c_{2,1}(b_{2,2}c_{1,2} - b_{1,2}c_{2,2}) + \\ & d_1 d_2 d_3 d_4(-2(b_{1,2}b_{2,1} - b_{1,1}b_{2,2})c_{1,2}c_{2,1} - \\ & b_{1,1}(b_{2,2}c_{1,1} - b_{1,2}c_{2,1})c_{2,2} + \\ & \qquad b_{2,2}c_{1,1}(b_{2,1}c_{1,2} - b_{1,1}c_{2,2})) > 0. \end{aligned} \tag{6}$$

Only the last two coefficients at $d_1d_2d_3d_4$ in the expression (6) are nonpositive due to the necessary conditions (5), all the rest of the coefficients at the products d_i are nonnegative.

When all the elements of matrices B and C are nonzero, the system of inequalities (5) has the solutions:

$$\begin{aligned} & c_{1,1} < 0,\ c_{2,2} < 0,\ b_{1,1} < 0,\ b_{2,2} < 0,\ c_{1,2} > 0, \\ & 0 < c_{2,1} < \frac{c_{1,1}c_{2,2}}{c_{1,2}}, \\ & 0 < b_{2,1} \le \frac{b_{1,1}c_{2,1}}{c_{1,1}},\ 0 < b_{1,2} \le \frac{b_{2,2}c_{1,2}}{c_{2,2}} \end{aligned}$$

or

$$\begin{aligned} & c_{1,1} < 0,\ c_{2,2} < 0,\ b_{1,1} < 0,\ b_{2,2} < 0, \\ & c_{1,2} < 0,\ \tfrac{c_{1,1}c_{2,2}}{c_{1,2}} < c_{2,1} < 0, \\ & \tfrac{b_{1,1}c_{2,1}}{c_{1,1}} \le b_{2,1} < 0,\ \tfrac{b_{2,2}c_{1,2}}{c_{2,2}} \le b_{1,2} < 0. \end{aligned} \tag{7}$$

Unfortunately, we fail to demonstrate the satisfaction of condition (6) due to (7). But numerical experiments, when only two elements are unknown in matrix A (2), give evidence that (6) is satisfied under the conditions (7).

Example 1.

Let matrix A be such that $c_{1,1} = -1$, $c_{2,2} = -4$, $b_{1,1} = -1$, $b_{2,2} = -5$, $b_{1,2} = -1$, $c_{1,2} = -4/5$. Conditions (7) give the following relations for the rest of the system's elements: $-5 < c_{2,1} < 0$, $c_{2,1} \le b_{2,1} < 0$ (compared are numerical values). Hence inequality (6) is satisfied for any $d_i > 0$, and the system with such parameters is D-stable. The trajectory of the point for $c_{2,1} = b_{2,1} = -4$ is shown in Fig. 1.

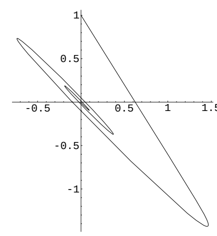

Fig. 1. Example 1

Example 2. Let parameters in matrix A have the following values: $c_{1,1} = -10, c_{2,2} = -4, b_{1,1} = -1, b_{2,2} = -5, b_{1,2} = -1$, $c_{1,2} = -40/5, c_{2,1} = -4, b_{2,1} = 30$. Such a system is asymptotically stable, but the Hurwitz determinant of matrix DA (6) for $d_3 = d_4 = 1$

$$1446d_1^2 - 34260d_1d_2 + 5075d_1^2d_2 + 2800d_2^2 + 25375d_1d_2^2$$

may be negative for $d_1 \le 2.72859$, for example, for $d_1 = 1.1$ and $0.0588353 \le d_2 \le 0.968279$. Consequently, such a system does not possess the property of D-stability. The trajectory of the point is shown in Fig. 2

Consider the variants for (5) and (6), when matrices C and B have zero elements.

If matrix C is diagonal, and B is symmetric diagonal or having extra-diagonal elements, then the system is D-stable, when both the matrices are definite negative, and this condition is the necessary one:

$$\begin{aligned} c_{1,1} < 0,\ c_{2,2} < 0,\ b_{1,1} < 0,\ b_{2,2} < 0, \\ -\sqrt{b_{1,1}b_{2,2}} < b_{1,2} < \sqrt{b_{1,1}b_{2,2}}. \end{aligned}$$

This property may also be obtained as an obvious corollary of Thomson–Tait's theorem on stabilization of a linear conservative system with the secular stability at the expense of dissipative forces with complete dissipation and adding gyroscopic forces. If the matric B of dissipative forces is

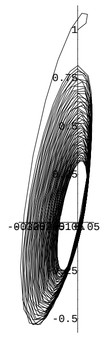

Fig. 2. Example 2

diagonal, and if there are extra-diagonal elements in matrix $C = C^T$, then the system is D-stable when

$$b_{1,1} < 0,\ b_{2,2} \leq 0,\ c_{1,1} < 0,\ c_{2,2} < 0,$$
$$-\sqrt{c_{1,1}c_{2,2}} < c_{1,2} < \sqrt{c_{1,1}c_{2,2}}\ ,\ (c_{1,2} \neq 0).$$

It is obvious from the latter property that the following statement is valid.

Theorem 1: A stable conservative linear system with matric C, for which $c_{1,2} \neq 0$, becomes D-stable under the effect of some linear dissipative force depending on one of the velocities.

Let the positional forces be such that $c_{1,1} < 0,\ \ c_{2,2} < 0,$ $c_{2,1} = 0,\ \ c_{1,2} \neq 0$ (or $c_{1,2} = 0,\ \ c_{2,1} \neq 0$), i.e. there are nonconservative forces acting in the system. Such a system may be stabilized up to D-stability at the expense of only such linear dissipative and gyroscopic forces for which $b_{1,1} < 0,\ b_{2,1} = 0,\ b_{2,2} < 0,\ b_{1,2}$ is arbitrary (or $b_{1,1} < 0,$ $b_{1,2} = 0,\ \ b_{2,2} < 0,\ b_{2,1}$ is arbitrary).

If matrix C is diagonal, and $b_{2,2} = 0$ in matrix B, then only such a linear system may be D-stable, for which $b_{1,1} < 0,\ \ c_{1,1} < 0,\ \ c_{2,2} < 0,\ \ b_{1,2}\ b_{2,1} < 0$, i.e. only when there are gyroscopic forces, which ensure that $b_{1,2}\ b_{2,1} < 0$.

If all the elements in matrix C are nonzero, $b_{2,2} = 0$ and $b_{1,2} = 0$, then such a system is D-stable if and only if

$$b_{1,1} < 0,\ c_{1,1} < 0,\ c_{2,2} < 0,\ c_{2,1} > 0,$$
$$0 < c_{1,2} < \frac{c_{1,1}c_{2,2}}{c_{2,1}}, 0 \leq b_{2,1} \leq \frac{b_{1,1}c_{2,1}}{c_{1,1}},$$

or

$$b_{1,1} < 0,\ c_{1,1} < 0, c_{2,2} < 0,\ c_{2,1} < 0,$$
$$\frac{c_{1,1}c_{2,2}}{c_{2,1}} < c_{1,2} < 0, \frac{b_{1,1}c_{2,1}}{c_{1,1}} \leq b_{2,1} \leq 0,$$
$$(c_{1,2}\ c_{2,1} > 0,\ b_{2,1}\ c_{1,2} \geq 0).$$

Hence the following statement is valid.

Theorem 2: The stable system (2) with positional forces such that $c_{1,2}c_{2,1} > 0$ may be stabilized up to D-stability by the effect of a dissipative force, which depends on the velocity with respect to one coordinate.

Example 3. Let numerical valuers of the matrix elements be such that $\{c_{1,1} = -10, c_{2,2} = -4, b_{1,1} = 0,\ b_{2,2} = 0, c_{1,2} = -1,\ c_{2,1} = -1, b_{2,1} = b_{1,2} = 0\}$. Such a conservative system is stable (the trajectory of the point is shown in Fig.3).

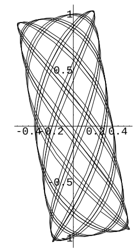

Fig. 3. Example 3, a conservative system

In case of adding a dissipative force, whose matrix has one element $b_{1,1} = -1$, the necessary conditions of D-stability are satisfied, and the inequality (6) holds always. Such a system is D-stable (the point's trajectory is shown in Fig.4) .

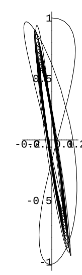

Fig. 4. Example 3, a D-stable system

IV. A System Having 3 Degrees of Freedom

Matrix (2) for a system having 3 degrees of freedom has the dimension of 6×6. Introduce the following denotations:

$$Q = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & c_{1,1} & c_{1,2} & c_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} & c_{2,1} & c_{2,2} & c_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} & c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

$$P_{2_{2,3,4,5}} = \det \begin{pmatrix} b_{1,1} & c_{1,3} \\ b_{3,1} & c_{3,3} \end{pmatrix},$$

$$P_{2_{1,3,4,5}} = \det \begin{pmatrix} b_{2,2} & c_{2,3} \\ b_{3,2} & c_{3,3} \end{pmatrix},$$

$$P_{2_{1,3,4,6}} = \det \begin{pmatrix} b_{1,1} & c_{1,2} \\ b_{2,1} & c_{2,2} \end{pmatrix},$$

$$P_{2_{1,3,4,5}} = \det \begin{pmatrix} b_{2,2} & c_{2,3} \\ b_{3,2} & c_{3,3} \end{pmatrix},$$

$$P_{2_{1,3,5,6}} = \det \begin{pmatrix} b_{1,2} & c_{1,1} \\ b_{2,2} & c_{2,1} \end{pmatrix},$$

$$P_{2_{1,2,5,6}} = \det \begin{pmatrix} b_{1,3} & c_{1,1} \\ b_{3,3} & c_{3,1} \end{pmatrix},$$

$$P_{2_{1,2,4,6}} = \det \begin{pmatrix} b_{2,3} & c_{2,2} \\ b_{3,3} & c_{3,2} \end{pmatrix}.$$

In the characteristic polynomial of matrix DA

$$\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 \quad (8)$$

the main diagonal minors of the Hurwitz matrix

$$H = \begin{pmatrix} a_1 & a_3 & a_5 & 0 & 0 \\ 1 & a_2 & a_4 & a_6 & 0 \\ 0 & a_1 & a_3 & a_5 & 0 \\ 0 & 1 & a_2 & a_4 & a_6 \\ 0 & 0 & a_1 & a_3 & a_5 \end{pmatrix}$$

must be positive. In (8), B_{2_i}, C_{2_i} are the main minors of 2nd-order matrices B and C (the subindex indicates the number of the row and the column deleted); $A_{3_{i,j,k}}$ are determinants of the 3rd-order matrices obtained by deletion of the columns, which have the numbers indicated in the subindices, from Q:

$$\begin{aligned} &a_1 = -d_1b_{1,1} - d_2b_{2,2} - d_3b_{3,3}, \quad a_2 = B_{2_3}d_1d_2 + \\ &B_{2_2}d_1d_3 + B_{2_1}d_2d_3 - d_1d_4c_{1,1} - d_2d_5c_{2,2} - \\ &d_3d_6c_{3,3}, \quad a_3 = -B_3d_1d_2d_3 - d_2d_3d_5P_{2_{1,2,4,6}} \\ &-d_1d_3d_4P_{2_{1,2,5,6}} + d_2d_3d_6P_{2_{1,3,4,5}} \\ &+d_1d_2d_5P_{2_{1,3,4,6}} - d_1d_2d_4P_{2_{1,3,5,6}} + d_1d_3d_6P_{2_{2,3,4,5}}, \\ &a_4 = -A_{3_{1,5,6}}d_1d_2d_3d_4 + A_{3_{2,4,6}}d_1d_2d_3d_5 \\ &-A_{3_{3,4,5}}d_1d_2d_3d_6 + d_2d_3d_5d_6c_{2_1} + d_1d_3d_4d_6C_{2_2} \\ &+d_1d_2d_4d_5C_{2_3}, \ a_5 = d_1d_2d_3\big(-A_{3_{1,2,6}}d_4d_5 \\ &+A_{3_{1,3,5}}d_4d_6 - A_{3_{2,3,4}}d_5d_6\big), \\ &a_6 = -d_1d_2d_3d_4d_5d_6C_3, \end{aligned}$$

Conditions $A \in (-P_0)$ for the positivity of the coefficients of the characteristic polynomial (8) write:

$$\begin{aligned} &b_{1,1} \le 0, b_{2,2} \le 0, \ b_{3,3} \le 0, b_{1,1} + b_{2,2} + b_{3,3} \ne 0, \\ &c_{1,1} \le 0, c_{2,2} \le 0, c_{3,3} \le 0, \ B_{2_3} \ge 0, B_{2_2} \ge 0, \\ &B_{2_1} \ge 0, \ B_{2_3} + B_{2_2} + B_{2_1} - c_{1,1} - c_{2,2} - c_{3,3} \ne 0, \\ &-B_3 \ge 0, \ -P_{2_{1,2,4,6}} \ge 0, \ -P_{2_{1,2,5,6}} \ge 0, \ P_{2_{1,3,4,5}} \ge 0, \\ &P_{2_{1,3,4,6}} \ge 0, -P_{2_{1,3,5,6}} \ge 0, P_{2_{2,3,4,5}} \ge 0, \\ &-B_3 - P_{2_{1,2,4,6}} - P_{2_{1,2,5,6}} + P_{2_{1,3,4,5}} + P_{2_{1,3,4,6}} - \\ &P_{2_{1,3,5,6}} + P_{2_{2,3,4,5}} \ne 0, \ -A_{3_{1,5,6}} \ge 0, \ A_{3_{2,4,6}} \ge 0, \\ &-A_{3_{3,4,5}} \ge 0, \ C_{2_1} \ge 0, \ C_{2_2} \ge 0, \ C_{2_3} \ge 0, \\ &-A_{3_{1,5,6}} + A_{3_{2,4,6}} - A_{3_{3,4,5}} + C_{2_1} + C_{2_2} + \\ &C_{2_3} \ne 0, \ C_3 < 0, \ -A_{3_{1,2,6}} \ge 0, \ A_{3_{1,3,5}} \ge 0, \\ &-A_{3_{2,3,4}} \ge 0, -A_{3_{1,2,6}} + A_{3_{1,3,5}} - A_{3_{2,3,4}} \ne 0. \end{aligned} \quad (9)$$

The Hurwitz determinants for the characteristic polynomial (8) are homogeneous forms of six variables d_i. According to the Lienard-Chipart criterion, it is sufficient to verify positivity of the main 3rd– and 5th–order minors of matrix H. The 3rd-order Hurwitz determinant has the general degree of 6 with respect to d_i, not more than 3 with respect to each d_i, the total number of the polynomial's terms is 70. The 5th-order Hurwitz determinant is reduced to the polynomial of the general degree of 12, not more than 4 with respect to each d_i, the total number of the polynomial's terms is 485. The complete investigation of such polynomials in symbolic form is rather complicated. So, some necessary conditions may be obtained from the 2nd-order determinant:

$$\begin{aligned} f_2 = d_1d_2d_3\Big(&n_{123} + 2\sqrt{-B_{2_3}b_{1,1}}\sqrt{-B_{2_1}b_{3,3}} + \\ &2\sqrt{-B_{2_2}b_{1,1}}\sqrt{-B_{2_1}b_{2,2}} + \\ &2\sqrt{-B_{2_3}b_{2,2}}\sqrt{-B_{2_2}b_{3,3}}\Big) + \\ &d_2\Big(d_1\sqrt{-B_{2_3}b_{1,1}} - d_3\sqrt{-B_{2_1}b_{3,3}}\Big)^2 + \\ &\Big(\sqrt{-B_{2_2}b_{1,1}}d_1 - \sqrt{-B_{2_1}b_{2,2}}d_2\Big)^2 d_3 + \\ &d_1\Big(d_2\sqrt{-B_{2_3}b_{2,2}} - d_3\sqrt{-B_{2_2}b_{3,3}}\Big)^2 + \\ &d_1^2d_4b_{1,1}k_{1,1} + d_2^2d_5b_{2,2}k_{2,2} + d_3^2d_6b_{3,3}k_{3,3} + \\ &d_1d_2d_4n_{124} + d_1d_2d_5n_{125} + d_1d_3d_4n_{134} + \\ &d_1d_3d_6n_{136} + d_2d_3d_5n_{235} + d_2d_3d_6n_{236}, \end{aligned}$$

where

$$\begin{aligned} &n_{123} = B_3 - B_{2_1}b_{1,1} - B_{2_2}b_{2,2} - B_{2_3}b_{3,3}, \\ &n_{124} = P_{2_{1,3,5,6}} + b_{2,2}k_{1,1}, \\ &n_{134} = P_{2_{1,2,5,6}} + b_{3,3}k_{1,1}, \\ &n_{125} = -P_{2_{1,3,4,6}} + b_{1,1}k_{2,2}, \\ &n_{235} = P_{2_{1,2,4,6}} + b_{3,3}k_{2,2}, \\ &n_{136} = -P_{2_{2,3,4,5}} + b_{1,1}k_{3,3}, \\ &n_{236} = -P_{2_{1,3,4,5}} + b_{2,2}k_{3,3} \end{aligned}$$

The condition

$$\begin{aligned} \Big(&n_{123} + 2\sqrt{-B_{2_3}b_{1,1}}\sqrt{-B_{2_1}b_{3,3}} + \\ &2\sqrt{-B_{2_2}b_{1,1}}\sqrt{-B_{2_1}b_{2,2}} + \\ &2\sqrt{-B_{2_3}b_{2,2}}\sqrt{-B_{2_2}b_{3,3}}\Big) \ge 0 \end{aligned}$$

provides for positiveness of the coefficient for the maximum degree of the polynomial's variables, and it is the necessary one for the positiveness of the polynomial f_2 in the positive orthant. This condition may be transformed to the form:

$$B_3 + \Big(\sqrt{-B_{2_1}b_{1,1}} + \sqrt{-B_{2_2}b_{2,2}} + \sqrt{-B_{2_3}b_{3,3}}\Big)^2 \ge 0. \quad (10)$$

If one considers f_2 as a polynomial with respect to all d_i, then it is necessary that the following inequalities be satisfied: $n_{136} \ge 0$, $n_{236} \ge 0, n_{125} \ge 0, n_{235} \ge 0, n_{124} \ge 0, n_{134} \ge 0$, what is unnecessary for a mechanical system. This group of conditions identifies a class of systems, for which

$$\begin{aligned} &b_{3,1}c_{1,3} \ge 0, \ b_{3,2}c_{2,3} \ge 0, \ b_{2,1}c_{1,2} \ge 0, \\ &b_{2,3}c_{3,2} \ge 0, \ b_{1,2}c_{2,1} \ge 0, \ b_{1,3}c_{3,1} \ge 0. \end{aligned} \quad (11)$$

When conditions (9), (10) and (11) are satisfied, the 2nd-order Hurwitz determinant is positive for any $D \in D_6$.

The 3rd-order Hurwitz determinant f_3 may be written in the form

$$f_3 = d_1^3s_{31} + d_2^3s_{32} + d_3^3s_{33} + d_4^2s_{34} + d_5^2s_{35} + d_6^2s_{36} + k_{30},$$

where $s_{31} = s_{31}(d_4^2)$, $s_{32} = s_{32}(d_5^2)$, $s_{33} = s_{33}(d_6^2)$, or

$$d_1^3 k_{31} + d_2^3 k_{32} + d_3^3 k_{33} + d_4^2 d_1^2 k_{34} + d_5^2 d_2^2 k_{35} + d_6^2 d_3^2 k_{36} + k_{30}.$$

The polynomials s_{3i}, k_{3i} do not contain d_i; k_{30} is the polynomial of all d_j having the general degree of 6; not more than 2 – with respect to each d_1, d_2, d_3; the degree of 1 – with respect to d_4, d_5, d_6. The necessary conditions of positiveness of polynomial f_3 for any $d_i > 0$ are $k_{34} \geq 0$, $k_{35} \geq 0$, $k_{36} \geq 0$, $s_{31} \geq 0$, $s_{32} \geq 0$, $s_{33} \geq 0$. The first three inequalities hold when conditions (11) are satisfied. Having investigated the polynomials $s_{31} \geq 0$, $s_{32} \geq 0$, $s_{33} \geq 0$, we conclude on the necessity that the following inequalities be satisfied:

$$\begin{aligned}
&\Big(\Big(\sqrt{-B_{2_3} P_{2_{1,2,5,6}}} + \sqrt{-B_{2_2} P_{2_{1,3,5,6}}} + \sqrt{B_3 c_{1,1}}\Big)^2 - \\
&A_{3_{1,5,6}} b_{1,1}\Big) \geq 0, \\
&\Big(\Big(\sqrt{-B_{2_3} P_{2_{1,2,4,6}}} + \sqrt{B_{2_1} P_{2_{1,3,4,6}}} + \sqrt{B_3 c_{2,2}}\Big)^2 + \\
&A_{3_{2,4,6}} b_{2,2}\Big) \geq 0, \\
&\Big(\Big(\sqrt{B_{2_2} P_{2_{1,3,4,5}}} + \sqrt{B_{2_1} P_{2_{2,3,4,5}}} + \sqrt{B_3 c_{3,3}}\Big)^2 - \\
&A_{3_{3,4,5}} b_{3,3}\Big) \geq 0.
\end{aligned} \qquad (12)$$

If the following conditions hold in addition to (12)

$$\begin{aligned}
&-b_{1,1}(-b_{1,3}b_{2,1} + b_{1,1}b_{2,3})(-b_{3,1}c_{1,2} + b_{1,1}c_{3,2}) \geq 0, \\
&b_{1,1}c_{1,2}(-b_{2,1}c_{1,1} + b_{1,1}c_{2,1}) \geq 0, \\
&-b_{1,1}(-b_{1,2}b_{3,1} + b_{1,1}b_{3,2})(-b_{2,1}c_{1,3} + b_{1,1}c_{2,3}) \geq 0, \\
&-b_{1,1}(c_{2_2}b_{1,1} + b_{3,1}c_{1,1}c_{1,3} - b_{1,1}c_{1,1}c_{3,3}) \geq 0, \\
&-b_{2,2}(b_{1,3}b_{2,2} - b_{1,2}b_{2,3})(-b_{3,2}c_{2,1} + b_{2,2}c_{3,1}) \geq 0, \\
&b_{2,2}c_{2,1}(b_{2,2}c_{1,2} - b_{1,2}c_{2,2}) \geq 0, \\
&-b_{2,2}(b_{2,2}b_{3,1} - b_{2,1}b_{3,2})(b_{2,2}c_{1,3} - b_{1,2}c_{2,3}) \geq 0, \\
&b_{2,2}c_{2,3}(-b_{3,2}c_{2,2} + b_{2,2}c_{3,2}) \geq 0, \\
&-b_{3,3}(-b_{1,3}b_{3,2} + b_{1,2}b_{3,3})(b_{3,3}c_{2,1} - b_{2,3}c_{3,1}) \geq 0, \\
&-b_{3,3}(-b_{2,3}b_{3,1} + b_{2,1}b_{3,3})(b_{3,3}c_{1,2} - b_{1,3}c_{3,2}) \geq 0, \\
&b_{3,3}c_{3,1}(b_{3,3}c_{1,3} - b_{1,3}c_{3,3}) \geq 0, \\
&b_{3,3}c_{3,2}(b_{3,3}c_{2,3} - b_{2,3}c_{3,3}) \geq 0,
\end{aligned} \qquad (13)$$

then the polynomials s_{31}, s_{32}, s_{33} are nonnegative. Another group conditions may be obtained from the 4th-order Hurwitz determinant f_4. It represents a polynomial of 6 variables, which has a general degree of 10 (the number of its elements being 241), and has the following structure:

$$\begin{aligned}
&f_4 = \\
&s_{40} + d_1^4 s_{41} + d_2^4 s_{42} + d_3^4 s_{43} + d_4^3 s_{44} + d_5^3 s_{45} + d_6^3 s_{46},
\end{aligned}$$

where the polynomials s_{4i} (i=1,2,3) must be positive for any $d_j > 0$. Proceeding from this requirement, we obtain the necessary conditions

$$\begin{aligned}
&\Big(\sqrt{-B_3 C_{2_3}} + \sqrt{-A_{3_{1,5,6}} P_{2_{1,3,4,6}}} + \sqrt{-A_{3_{2,4,6}} P_{2_{1,3,5,6}}}\Big)^2 \\
&+A_{3_{1,2,6}} B_{2_3} \geq 0, \\
&\Big(\sqrt{-B_3 C_{2_3}} + \sqrt{-A_{3_{1,5,6}} P_{2_{1,3,4,6}}} + \sqrt{-A_{3_{2,4,6}} P_{2_{1,3,5,6}}}\Big)^2 \\
&+A_{3_{1,2,6}} B_{2_3} \geq 0, \\
&\Big(\sqrt{A_{3_{3,4,5}} P_{2_{1,2,5,6}}} + \sqrt{-B_3 C_{2_2}} + \sqrt{-A_{3_{1,5,6}} P_{2_{2,3,4,5}}}\Big)^2 \\
&-A_{3_{1,3,5}} B_{2_2} \geq 0
\end{aligned} \qquad (14)$$

The 5th-order Hurwitz determinant is reduced to the 12th-order polynomial with respect to six d_i, the degree with respect to each variable does not exceed 4:

$$\begin{aligned}
&f_5 = d_1 d_2 d_3 \big(s_6 + \\
&d_1^4 s_{51} + d_2^4 s_{52} + d_3^4 s_{53} + d_4^4 s_{54} + d_5^4 s_{55} + d_6^4 s_{56}\big),
\end{aligned}$$

where the polynomials s_{51}, s_{52}, s_{53} contain d_i^4 (i=4,5,6). The polynomials s_{51}, s_{52}, s_{53} are nonnegative when the following conditions holds:

$$\big(\sqrt{A_{3_{1,5,6}} A_{3_{2,3,4}}} + \sqrt{A_{3_{1,3,5}} A_{3_{2,4,6}}} + \sqrt{A_{3_{1,2,6}} A_{3_{3,4,5}}}\big)^2 \geq B_3 C_3.$$

Unfortunately, in virtue of the bulky character of the polynomials under scrutiny, we have failed to obtain the sufficient conditions of their positiveness. We have also failed to demonstrate sufficiency of the system of necessary inequalities obtained.

V. Conclusion

It has been shown that there exist asymptotically stable mechanical systems, which either possess the property of D-stability or do not obtain this property. Necessary conditions and sufficient conditions of D-stability have been obtained for the systems having 2 degrees of freedom, in particular cases, necessary and sufficient conditions have been obtained. A system of necessary conditions has been obtained for the system having 3 degrees of freedom.

References

[1] Arrow K.J., McManus M. A note of dynamic stability.// Econometrica. 1958, V.26, p.448-454.

[2] Johnson C.R., Sufficient Conditions for D-Stability.// Journal of Economic Theory, 9,(1974), p.53-62.

[3] Cross G.W., Three Types of Matrix Stability.// Linear Algebra and its Applications, 20, (1978), 253-263.

[4] Cain B.E., Real, 3×3 D-stable matrices // J. Research Nat.Bureau Standards U.S.A. V.B80, no 1,(1976), p.75-77.

[5] Kanovei G. V. and Logofet D. O., Relations, properties and invariant transformations of $D-$ and $\alpha D-$stable matrices. // Vestnik Moskovskogo universiteta, seriya 1, Matematika, Mekhanika, 2001, no 6, pp. 40-43.

[6] Kanovei G. V. and Logofet D. O., D-Stability of 4-by-4 Matrices // J.Computational Mathematics and Mathematical Physics, 38, no 9, 1998, pp. 1429-1435.

[7] Burlakova L.A.,Application of the Computer Algebra System in Investigation of D-Stability of Matrices.//12th International Conference on Applications of Computer Algebra (ACA 2006), June 26-29, 2006, Varna, Bulgaria, Institute of Mathematics and Informatics Bulgarian Academy of Sciences, 2006, p. 26