COMMENTS ON THE METHOD OF HARMONIC BALANCE FOR NONLINEAR CONSERVATIVE SINGLE-DEGREE-OF-FREEDOM SYSTEMS

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Abstract

Although the method of harmonic balance is well-known to analyze approximately nonlinear vibrations, its application includes always a certain risk if it results in a good approximation or if it fails. In this contribution some facts on this method are summarized, particularly a lower bound for the oscillation period is given by the method of harmonic balance. Together with the experience of many academic and practical examples some comments are presented resulting in recommendations when the method of harmonic balance may be or may be not applied.

Key words

Nonlinear vibration, method of harmonic balance, single-degree-of-freedom system.

1 Introduction

For a periodic solution of the single-degree-of-freedom system

$$\ddot{x} + f(x) = 0 \tag{1}$$

about the equilibrium position $x = x_0$ the oscillation period T can be calculated by

$$T = 2[T_1(a_1) + T_2(a_2)] \tag{2}$$

with

$$T_1(a_1) = \frac{1}{\sqrt{2}} \int_{a_1}^{x_0} \frac{dy}{\sqrt{E - U(y)}},$$
$$T_2(a_2) = \frac{1}{\sqrt{2}} \int_{x_0}^{a_2} \frac{dy}{\sqrt{E - U(y)}}.$$

Here, the restoring force $f(x)$ is assumed to be a uniquely defined integrable function with $f(x)(x - x_0) \geq 0$ in $[a_1, a_2]$, where $a_1, a_2 (a_1 < x_0 < a_2)$ are the extreme amplitudes of the oscillation. Further, $U(x) = \int_{x_0}^{x} f(y)dy$ is the potential energy and $E = U(a_1) = U(a_2)$ represents the energy constant of the system.

In general the integrals (2) cannot be solved in closed form; only in special cases they can be determined by elementary functions or by elliptic integrals. Therefore, in many problems approximation methods have to be applied to obtain approximate solutions of (1) and to determine approximately the oscillation period (2). One of the most well-known and most applied approximation methods is the method of harmonic balance [3,4]. For nonlinear control problems the method of harmonic balance has been adapted in the frequency domain and is called the describing function method [1,2].

Without loss of generality in the following the restoring force is simplified to an odd function

$$f(x) = -f(-x) \quad , \quad x_0 = 0 \quad , \quad xf(x) \geq 0. \tag{3}$$

Then for an approximate analysis of (1) the method of harmonic balance is used replacing (1) by an approximate system

$$\ddot{x}_h + \omega_h^2(a)x_h = 0 \tag{4}$$

with

$$\omega_h^2(a) = \frac{1}{\pi a}\int_0^{2\pi} f(a\sin u)\sin u du$$
$$= \frac{4}{\pi a^2}\int_0^a f(y)\frac{y}{\sqrt{a^2-y^2}}dy \qquad (5)$$

$(a = a_2 = -a_1 > 0)$ where the characteristic $\omega_h^2(a)x$ is the Tschebyscheff polynomial of first degree with respect to $f(x)$.

The analysis of system (1) is approximately realized by the analysis of linear system (2) where the eigenfrequency $\omega_h(a)$ depends on the oscillation amplitude a.

The method of harmonic balance is applied to more or less all types of nonlinearities. The functions $f(x)$ may be smooth or discontinuous, it may include jumps (e. g. the sign function) or even it may represent relay functions with hysteresis (violating the assumption of a unique function), cf. [1,4]. In many of these applications the approximation is reasonable or even good (confirmed by a difficult theoretical discussion of (1) or (2) or by comparisons with simulations or experiments), but there are also examples where the method fails completely. Unfortunately, there is no theoretical statement on the error of the approximation in general. Only for small nonlinearities error estimates are possible by the perturbation method, but for most interesting, strong nonlinearities there does not exist a theoretical justification. In the following some comments and recommendations are given.

2 Lower Bound for $T(a)$

The oscillation period

$$T(a) = 4T_2(a)$$

is bounded from below by the period of the approximation system (4):

$$T(a) \geq T_h(a) = \frac{2\pi}{\omega_h(a)} \quad . \qquad (6)$$

This fact has been firstly observed by many examples, cf. [4], but then it was explicitly proved [5]. The proof is based on the inequality $\frac{1}{\sqrt{1-A}} \geq 1 + \frac{A}{2}$ for $A < 1$ leading to

$$T(a) = 2\sqrt{2}\int_0^a \frac{dy}{\sqrt{E-U(y)}}$$
$$= 2\sqrt{2}\int_0^a \frac{dy}{\sqrt{\frac{1}{2}\omega_h^2(a^2-y^2)}\sqrt{1-A(y)}}$$
$$\geq 2\sqrt{2}\int_0^a \frac{1+\frac{A(y)}{2}}{\sqrt{\frac{1}{2}\omega_h^2(a^2-y^2)}}dy$$
$$= T_h(a) + \sqrt{2}\int_0^a \frac{A(y)dy}{\sqrt{\frac{1}{2}\omega_h^2(a^2-y^2)}}$$

where

$$A(x) = 1 - \frac{E-U(x)}{\frac{1}{2}\omega_h^2(a^2-x^2)} \quad .$$

Having regard to (5) it is shown by partial integration that the second term in the last part of the inequality vanishes [5]. This results in the lower bound (6) of the oscillation period and, as a by-product, to an additional relation for ω_h:

$$\omega_h^2(a) = \frac{4}{\pi}\int_0^a \frac{E-U(y)}{(a^2-y^2)^{\frac{3}{2}}}dy \quad . \qquad (7)$$

The inequality (6) holds for all types of restoring forces (which are unique and integrable in $[-a, a]$), i.e. for overlinear (hardening), underlinear (softening) or mixed-type spring forces. For other approximation methods the bound (6) does not hold generally, cf. [5].

3 Superposition

If the nonlinear restoring force $f(x)$ consists of a superposition of several individual characteristics,

$$f(x) = \sum_{i=1}^{n} f_i(x) \qquad (8)$$

where again relations (3) are assumed for each $f_i(x)$, then

$$\omega_h^2(a) = \omega_{h_1}^2(a) + \cdots + \omega_{h_n}^2(a) \qquad (9)$$

holds. Here, $\omega_{h_i}^2(a)$ are defined by (5) using the characteristic $f_i(x), i = 1, \ldots, n$. This is an old result which has been discussed again more recently [6]. It can be proved by direct calculation. The result (9) simplifies the calculation of the eigenfrequency of system (4). It represents a certain comfort. It should be mentioned that other averaging methods (approximation methods) do not have this superposition property.

4 Examples

A few examples and counter-examples shall show how good the approximation of the method of harmonic balance can be but how it also may fail. The comparison is based on the oscillation period. The explicit calculation of $T(a)$ and $T_h(a)$ is not shown. The results are taken over from the literature, especially from [4] and [6].

a) Sign function $f(x) = h \text{ sgn } x$:

$$\left.\begin{array}{l} T(a) = 4\sqrt{2}\sqrt{\frac{a}{h}} = 5,6569\sqrt{\frac{a}{h}} \\ T_h(a) = \pi\sqrt{\pi}\sqrt{\frac{a}{h}} = 5,5683\sqrt{\frac{a}{h}} \end{array}\right\} T > T_h$$

$$\frac{T - T_h}{T} \hat{=} 1,565\% \quad \text{independent on } a.$$

b) Quadratic function $f(x) = kx|x|$:

$$\left.\begin{array}{l} T(a) = 6,8699\frac{1}{\sqrt{ka}} \\ T_h(a) = 6,8198\frac{1}{\sqrt{ka}} \end{array}\right\} T > T_h$$

$$\frac{T - T_h}{T} \hat{=} 0,729\% \text{ independent on } a.$$

c) Cubic function $f(x) = kx^3$:

$$\left.\begin{array}{l} T(a) = 7,4164\frac{1}{a\sqrt{k}} \\ T_h(a) = \frac{4\pi}{a\sqrt{3k}} = 7,2552\frac{1}{a\sqrt{k}} \end{array}\right\} T > T_h$$

$$\frac{T - T_h}{T} \hat{=} 2,174\% \text{ independent on } a.$$

d) Sine function $f(x) = k \sin x$:

$$\left.\begin{array}{l} T(a) = 4\sqrt{k}K(a) \\ T_h(a) = 2\pi\sqrt{k}\sqrt{\frac{a}{2J_1(a)}} \end{array}\right\} T > T_h$$

$\frac{T-T_h}{T}$ depends on a: According to Fig.19 in [4] the error increases monotonically with a from 0 % for $a = 0$ (trivial) over 1 % for $a = \frac{\pi}{4}$ to about 2% for $a = \frac{\pi}{2}$, about 100 % for $a = \frac{2\pi}{3}$ and finally tends to infinity (%) for $a \rightarrow \pi$. Therefore, the error is small for small amplitudes but becomes larger and larger for large amplitudes increasing until infinity. Here, $K(a)$ is the complete elliptic integral of first kind, and $J_1(a)$ represents the Bessel function of first kind and first order.

e) Underlinear characteristic with piecewise linear behaviour:

$$f(x) = f_0(1 - \frac{x}{x_0}), x > 0; f(-x) = -f(x); a \leq x_0 :$$

$$\left.\begin{array}{l} T(a) = 4\sqrt{\frac{x_0}{f_0}} \ln \frac{1-\frac{a}{x_0}}{1-\sqrt{1-(1-\frac{a}{x_0})^2}} \\ T_h(a) = 2\pi\sqrt{\frac{x_0}{f_0}} \frac{1}{\sqrt{\frac{4x_0}{\pi a}-1}} \end{array}\right\} T > T_h$$

$\frac{T-T_h}{T}$ depends on $\frac{a}{x_0}$: According to Fig. 3 in [6] the error increases monotonically with $\frac{a}{x_0}$ from 0 % for $\frac{a}{x_0} = 0$ (trivial), over about 10 % for $\frac{a}{x_0} = 0,5$ to about 30 % for $\frac{a}{x_0} = 0,9$ and finally tends to infinity (%) for $\frac{a}{x_0} \rightarrow 1$. Again, for this underlinear restoring force the error is small for small amplitudes but increases dramatically to inifinty if the amplitude a tends to x_o.

While the examples a), b), and c) show small constant errors independent on the amplitudes, the underlinear restoring forces d) and e) show quite different behaviour. The method of harmonic balance yields good approximation as long as the vibration amplitudes are small. But with increasing amplitudes the approximation becomes worse and its error tends to infinity when the amplitude approximates the additional (unstable) equilibrium point. Here, the method of harmonic balance fails completely.

5 Conclusion

According to the bound (6) for the oscillation period and to the superposition result (9) it can be concluded that the method of harmonic balance can be well applied to overlinear vibration systems with hardening spring forces. With increasing vibration amplitudes the oscillation period of an overlinear system becomes smaller and smaller but it is still bounded from below by the period determined by the method of harmonic balance. Therefore, the results (6) and (9) are good reasons to recommend the approximation method of harmonic balance for overlinear systems. But for underlinear systems the method may be applied very carefully only. If additional equilibrium points of the restoring force exist (like in the examples d) and e)) then the approximation fails for amplitudes sufficiently large. Therefore, in those cases the method of harmonic balance should be not applied.

An open problem is whether a lower bound (6) exists also in case of self-excited forced nonlinear vibration systems.

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