

Slow motions in systems with inertially excited vibrations
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A nonlinear system with two degrees of freedom consisting of a rigid platform and mechanical vibroactuator is considered. The platform, connected to an immovable base by means of elastic and damping elements can move along a fixed direction. The mechanical vibroactuator is an unbalanced rotor, mounted on the platform and driven with an electric drive. Such a system is a model of many vibrational machines and technological units.

During the speed up of the actuator to the working frequency ω^* exceeding the free oscillation frequency p of the platform, a remarkable phenomenon can be observed: capture of the current frequency ω near resonance frequency p . Further increase of the supply power of the drive leads to a jump transition from $\omega \approx p$ to an above resonance frequency $\omega > p$. Such a phenomenon was first described by an eminent German physicist A.Sommerfeld. In 1953 one of the authors of this work gave physical explanation and mathematical description of this phenomenon and coined the term "Sommerfeld effect" [1].

Later a comprehensive study of Sommerfeld effect was carried on in numerous publications including a number of books [2 – 4]. In [5, 6, 7] (see also book [4]) it was discovered by means of classical methods of nonlinear mechanics and their modifications that "semi-slow" oscillations of rotor frequency may appear in the area of Sommerfeld effect. Such an effect can be interpreted as appearance of "internal pendulum" in the system. Natural frequency of internal pendulum is less than resonance frequency of the system. In the above cited papers representation of system solutions by expansions in powers of square root of a small parameter allowing to study multi-scale motions are used. Using internal pendulum and semi-slow oscillations of the rotor is important for a number of methods for control of vibration units with inertia excitation of vibrations allowing to significantly reduced the motor power required for passage through resonance zone [8, 9].

The main contribution of this paper is analysis of existence and dynamics of internal pendulum. The problem of passage through resonance zone is solved by an iterative method combined with direct method of separation of motions. Though such an approach looks more primitive than the previous ones, it allows to obtain two autonomous second order equations for slow motions (for rotation frequency) and for semi-slow motions (for oscillations of rotation frequency) which can be solved separately. Both equations are valid both in below resonance and in above resonance area. Expression for the frequency of semi-slow oscillations (internal pendulum) in below resonance area can be derived from the obtained equations and provides an important contribution of the paper. This frequency depends essentially on rotation frequency ω and decreases down to zero when ω approaches the resonance frequency p .

A remarkable overturning property of an internal pendulum is discovered: in the below resonance area its equilibrium near the lower position is stable, while in the above resonance area its lower equilibrium becomes unstable and its equilibrium near upper position becomes stable. A comparison of the obtained analytical results with numerical results obtained by simulation of initial system equations is given demonstrating a good concordance of the results.

The results of the paper can be used for improvement of control methods for vibration units in the start-up mode.

System description. The system under consideration consists of an unbalanced rotor 2 mounted on a rigid platform 1, see Fig.1. Rotor is driven by an electric motor. The platform moving along

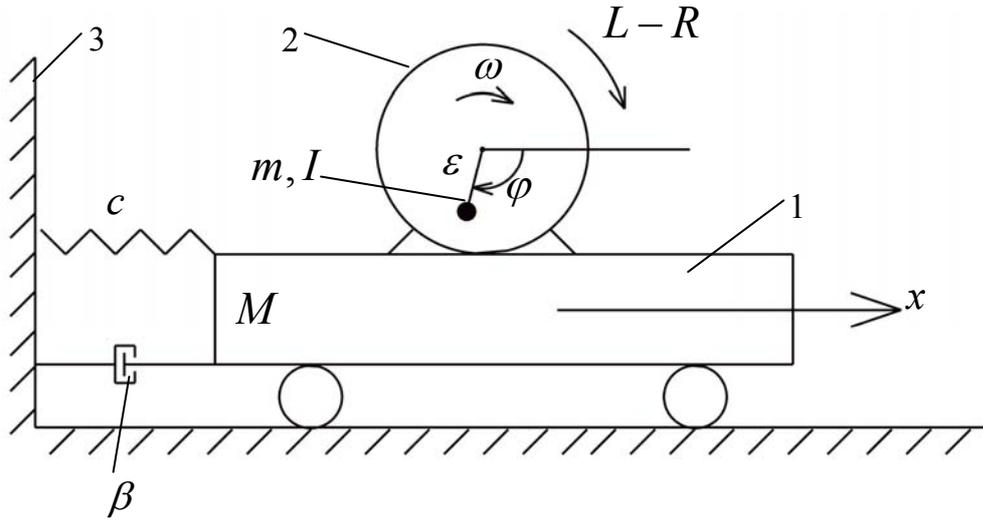


Fig.1. System schematics.

a fixed direction x is connected to an unmovable base 3 by means of elastic link with stiffness c and a damper with viscous friction coefficient β . Equations of motion are as follows (see [3], p.143)

$$I\ddot{\varphi} = L(\dot{\varphi}) - R(\dot{\varphi}) + m\varepsilon\ddot{x}\sin\varphi, \quad (1)$$

$$M\ddot{x} + \beta\dot{x} + cx = m\varepsilon(\dot{\varphi}^2 \cos\varphi + \ddot{\varphi}\sin\varphi). \quad (2)$$

Here φ is the rotor rotation angle; x is deflection of the platform; M is mass of the platform; m is mass of the rotor; I is inertia moment of the rotor; $M = M + m$ is total mass of the system; $L(\dot{\varphi})$ is driving torque of the motor (static response*); $R(\dot{\varphi})$ is the torque of resistance forces. Gravity and motor internal dynamics are neglected.

First approximation, Sommerfeld effect. The following initial approximation is taken:

$$\varphi_1 = \omega t, \quad x_1 = P\sin\omega t + Q\cos\omega t, \quad (3)$$

where $\omega = \omega(t)$, P , Q are slow functions of time and x_1 is fast function of time.

This approximation was studied in [3] by direct separation of motions and the following equation for rotation frequency was obtained:

$$I\dot{\omega} = L(\omega) - R(\omega) + V(\omega), \quad (4)$$

where

$$V(\omega) = m\varepsilon \langle \ddot{x}_1 \sin\omega t \rangle = -\frac{(m\varepsilon\omega)^2}{M} \frac{n\omega^3}{(\omega^2 - p^2)^2 + 4n^2\omega^2} = -n\omega MA^2 \quad (5)$$

is the so called vibration torque providing an additional load over the rotor caused by oscillations of the platform. It is vibration torque that can explain Sommerfeld effect. Angular brackets in (5) denote averaging over the period $T = 2\pi$ by "fast" time $\tau = \omega t$, other notations are as follows:

* Dependence of the static characteristic only of $\dot{\varphi}$ is typical, e.g. for induction motors or DC motors.

$$p^2 = c/M, \quad 2n = \beta/M, \quad A = \frac{m\varepsilon\omega^2}{M\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}}. \quad (6)$$

The value A is the amplitude of steady-state oscillations of the platform (3) described by the equation

$$M\ddot{x}_1 + \beta\dot{x}_1 + cx_1 = m\varepsilon\omega^2 \sin \omega t, \quad (7)$$

that is obtained from (2) for $\varphi = \varphi_0 = \omega t$. Amplitude A is linked to the values P and Q in (3) by the relations

$$A = \sqrt{P^2 + Q^2}, \quad (8)$$

$$P = \frac{m\varepsilon\omega^2}{M} \frac{2n\omega}{(p^2 - \omega^2)^2 + 4n^2\omega^2}, \quad Q = \frac{m\varepsilon\omega^2}{M} \frac{p^2 - \omega^2}{(p^2 - \omega^2)^2 + 4n^2\omega^2}. \quad (9)$$

The presented solution is valid when the relative change of the frequency ω , is sufficiently slow

$$\dot{\omega}/\omega \ll \omega, \quad (10)$$

or when the frequency has reached its steady-state value $\omega = \text{const}$.

The equation (4) has either three such steady-state solutions $\omega_1, \omega_2, \omega_3$, either one ω_3 , see Fig.2, where the curves L correspond to static characteristics of electric motors. The solution $\omega_1 < p$ is below-resonance, $\omega_2 > p$ is above-resonance, while ω_3 is “far-above-resonance” solution. It can be shown that the solutions ω_1 and ω_3 are stable, while ω_2 is unstable. The solution ω_3 corresponds to a steady-state working mode of “above-resonance” vibration machines. The curve L_1 is responsible for the capture of the system near resonance at the frequency ω_1 (Sommerfeld effect), the curve L_3 , corresponding to a more powerful motor demonstrates achievement of a nominal steady-state mode. The curve L_2 corresponds to a jump transition from a resonance mode ω_1 to a far-above-resonance mode ω_3 .

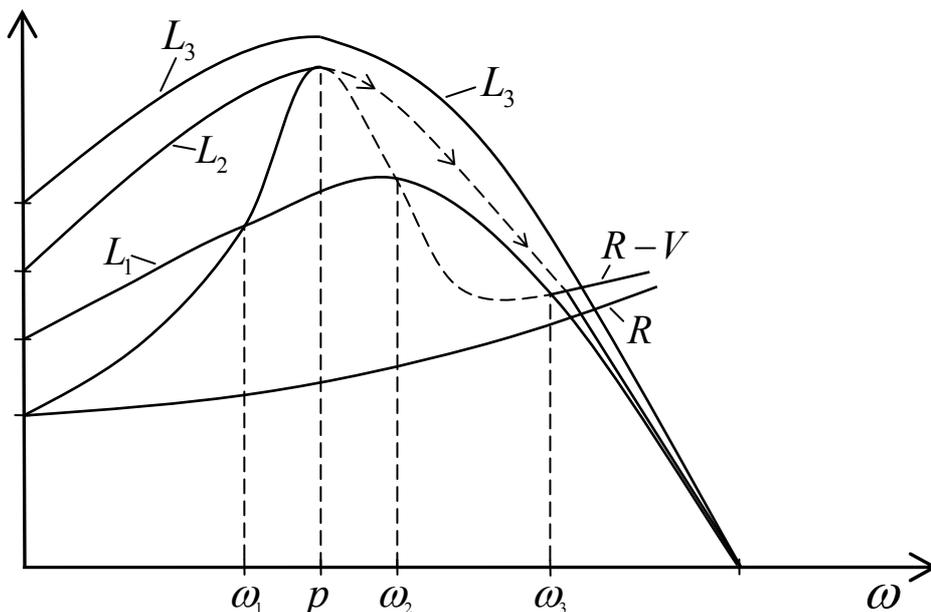


Fig.2. Steady-state values of the rotor frequency (explanation of Sommerfeld effect).

Therefore already the first approximation allows one to explain and analyze Sommerfeld effect (see more detail in the book [3]). The second approximation of this study is aimed at analysis of the deviations of the rotor frequency $\dot{\varphi}$. As was mentioned before, existence of such oscillations in the near resonance zone $\omega_1 < p$ allows one to significantly reduce the torque of the motor L_* , required for passage through resonance by means of control.

Second approximation. Semi-slow oscillations of the rotor. To obtain second approximation make the following assumptions for initial equations (1), (2)

$$\varphi = \varphi_2 = \omega t + \psi, \quad x = x_2 = x_1 + y, \quad (11)$$

and assume that ω and x_1 satisfy (4), (7). Then the following equations for ψ and y hold:

$$I\ddot{\psi} + k\dot{\psi} = m\varepsilon \left[\ddot{x}_1 \sin(\omega t + \psi) - \langle \ddot{x}_1 \sin \omega t \rangle \right] + m\varepsilon \ddot{y} \sin(\omega t + \psi), \quad (12)$$

$$M\ddot{y} + \beta\dot{y} + cy = m\varepsilon \left[(\dot{\omega} + \dot{\psi}) \sin(\omega t + \psi) + (\omega + \dot{\psi})^2 \cos \omega t - \omega^2 \cos \omega t \right]. \quad (13)$$

Assume that the expressions for L and R can be linearized near $\dot{\varphi} = \omega$:

$$L(\omega + \dot{\psi}) = L(\omega) - k_L \dot{\psi}, \quad R(\omega + \dot{\psi}) = R(\omega) + k_R \dot{\psi}, \quad (14)$$

and introduce total damping coefficient $k = k_L + k_R$ ($k_L > 0, k_R > 0$). Then apply direct separation of motions for the system (12), (13) with

$$\psi = \Psi + \gamma, \quad y = Y + \delta, \quad (15)$$

where Ψ and Y are slow terms while γ and δ are fast 2π -periodic in $\tau = \omega t$ terms having zero mean when averaging in fast time $\tau = \omega t$:

$$\langle \gamma \rangle = 0, \quad \langle \delta \rangle = 0. \quad (16)$$

Substitution of (15) in (12), (13) yields equations of fast and slow motions (it is known that the equations for fast motion can be solved approximately without making a serious error in the equations for slow motion). For our purpose is it possible to make a further simplification and derive slow motion equations for Ψ under assumption that the fast variable γ is small with respect to Ψ , and y is small with respect to x_1 . Then substituting the first expression of (15) into (12), we arrive at the following equation for Ψ :

$$I\ddot{\Psi} + k\dot{\Psi} = m\varepsilon \left\langle \ddot{x}_1 (\sin \omega t \cos \Psi + \cos \omega t \sin \Psi) - \langle \ddot{x}_1 \sin \omega t \rangle \right\rangle.$$

After performing averaging and taking into account (3), (9) this equation takes the form

$$\ddot{\Psi} + 2n_1 \dot{\Psi} + b \sin \Psi - \rho^2 \sin^2 \frac{\Psi}{2} = 0, \quad (17)$$

where

$$\begin{aligned} 2n_1 &= k/I; \\ b &= \frac{(m\varepsilon\omega^2)^2}{2MI} \frac{p^2 - \omega^2}{(p^2 - \omega^2)^2 + 4n^2\omega^2} = \frac{1}{2} \frac{M}{I} (p^2 - \omega^2) A^2; \\ \rho^2 &= \frac{(m\varepsilon\omega^2)^2}{MI} \frac{2n\omega}{(p^2 - \omega^2)^2 + 4n^2\omega^2} = \frac{M}{I} 2n\omega A^2. \end{aligned} \quad (18)$$

and A is initial approximation for the amplitude of the platform oscillations determined by (6).

Under condition (10) the frequency of the rotor ω is changing slowly. In an under resonance region $\omega < p$ the value

$$q = \sqrt{b} = \sqrt{M(p^2 - \omega^2)/2IA} \quad (19)$$

is nothing but the frequency of small free oscillations of “internal pendulum” (with neglected damping force). It is seen that this frequency equals to zero when $p = \omega$.

For the equation (17) to be valid it is necessary that the frequency q would be significantly less than ω , i.e. the fast variable γ would be small with respect to Ψ , and y would be small with respect to x_1 . For practice the following relation is sufficient:

$$q/\omega < \frac{1}{3}. \quad (20)$$

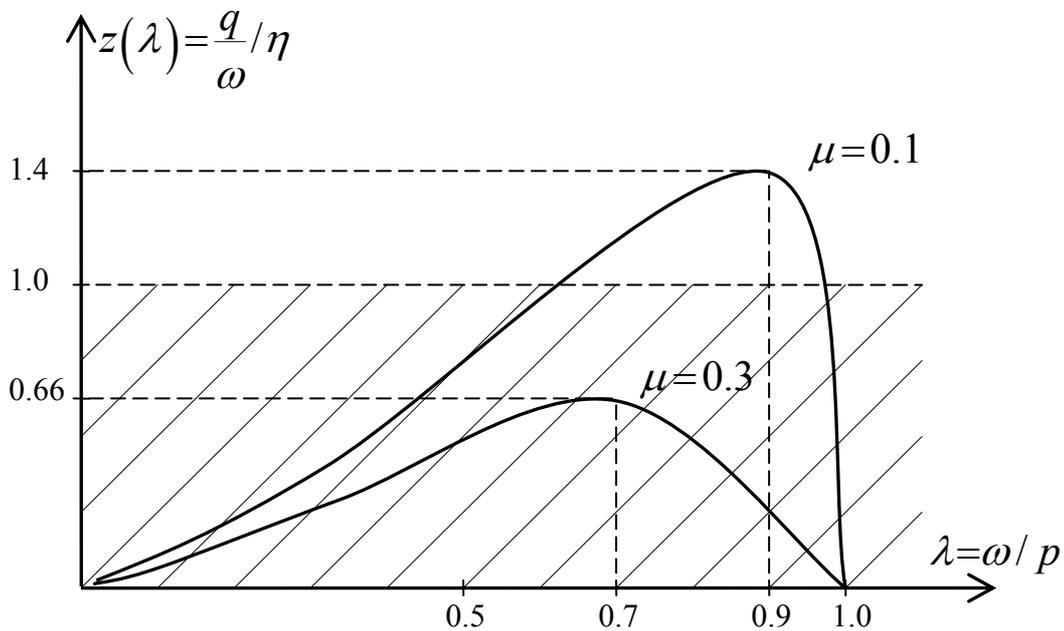


Fig.3. Dependence of the relative frequency of slow oscillations on its relative frequency. The plot of the function

$$z(\lambda) = \frac{q}{\omega} : \eta = \lambda \sqrt{\frac{1 - \lambda^2}{(1 - \lambda^2)^2 + 4\nu^2 \lambda^2}}, \quad (21)$$

where

$$\lambda = \frac{\omega}{p}, \quad \nu = \frac{n}{p}, \quad \eta = \frac{m\varepsilon}{\sqrt{2MI}}, \quad (22)$$

is presented in Fig.3. The region satisfying inequality (20) for $\eta = 1/3$ is shaded. The function has the maximum value $0,5/\sqrt{\nu(1 + \nu)}$ at $\lambda = 1/\sqrt{1 + 2\nu}$. Therefore, the condition (20) holds for all $\lambda = \omega/p$ if the following inequality is valid

$$\frac{4}{9}\nu(1 + \nu) > \eta^2. \quad (23)$$

Assuming that $\nu \ll 1$ in (23) and using (6), (22) to come back to dimension variables we

obtain the relation

$$\frac{m^2 \varepsilon^2}{MI} < \frac{4 \beta}{9 Mp}. \quad (24)$$

In other words, (24) means that the squared relative static moment of the rotor $m\varepsilon/\sqrt{MI}$ should be less than the relative damping $4\beta/9Mp$. The region, corresponding to (23), is shaded in Fig.4.

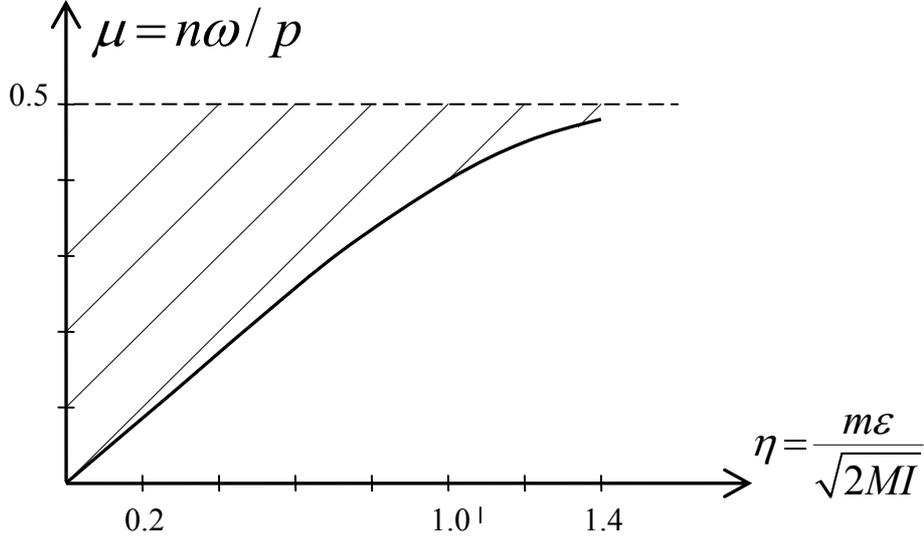


Fig.4. Second approximation validity region.

Return to examination of equation (17). Its equilibrium solutions $\Psi = \text{const}$ satisfy equation

$$b \sin \Psi - \rho^2 \sin^2 \Psi / 2 = 0. \quad (25)$$

One of such solutions is

$$\Psi = \Psi_1 = 0, \quad (26)$$

while the other one satisfies the relation

$$2b \cos \Psi_2 / 2 - \rho^2 \sin \Psi_2 / 2 = 0. \quad (27)$$

An equilibrium solution is stable if the following inequality holds:

$$R = b \cos \Psi - \frac{1}{2} \rho^2 \sin^2 \Psi > 0$$

(it can be easily derived from analysis of the linearized equation (17). For Ψ_1 it leads to the inequality $b > 0$, while for Ψ_2 the opposite inequality $b < 0$ holds (the latter follows easily from (27) multiplied by $\cos \Psi_2 / 2$). Therefore the solution $\Psi_1 = 0$ is stable in the below-resonance region ($\omega < p$) and unstable in the above-resonance region ($\omega > p$), while for the solution Ψ_2 the situation is opposite.

Note that the stable steady-state solutions of the equation (17) without damping ($\rho = 0$) are $\Psi = \Psi_1 = 0$ below resonance and $\Psi = \Psi_1 = \pi$ above resonance, respectively. These values coincide with the phase shift between external force and displacement of the platform for forced vibrations in linear systems when the driving force $F = m\varepsilon\omega^2 \sin \omega t$ is given (system with the energy source of infinite power). Since (17) describes the oscillations with respect to the above equilibria, one can say that passage through resonance leads to overturning of the “internal pendulum”.

Discussion. According to the proposed results the steady-state or slowly varying value of the rotor angular velocity ω is determined by the equation (4), while “semi-slow” deviations of this angular velocity are described by (18). Both equations hold under condition of sufficiently slow change of ω and Ψ with respect to the deviations of ω . If condition (10) holds, then the changes of the frequency q in (18) in the course of t are slow too, and the frequency can be considered as approximately constant. The relative location of the characteristic frequencies along the ω -axis is shown schematically in Fig.5.

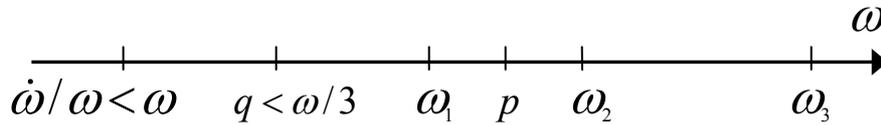


Fig.5. Relative location of the system critical frequencies.

Note that the equation (17) can be reduced to the equation

$$\ddot{\chi} + 2n_1\dot{\chi} + W \sin \chi = \rho^2/2, \quad (28)$$

where $\chi = \Psi + \Psi_0$, and Ψ_0 , W are determined from the relations

$$b = W \cos \Psi_0, \quad \rho^2/2 = W \sin \Psi_0.$$

The equation (28) is nothing but the well known equation of the “biased” pendulum which is important for theory of synchronous electrical machines. Its global behavior on the phase plane $\dot{\chi}, \chi$ is studied in numerous papers, see survey in [17]. However the known results apply only for the case $\omega \equiv \text{const}$. In a more general case of slowly varying ω , satisfying (4), one needs to consider a “set of phase portraits” for different values of W and ρ , and for jump change of the angle Ψ_0 when passage of ω through resonance value p (since the sign of the coefficient b changes at resonance, according to (18)).

It is worth to note also that in this paper we did not consider the “fast” oscillations of the rotor velocity $\dot{\gamma}$. It is easy to see that the frequency of such oscillations is 2ω . Indeed, the velocity of the platform crosses zero level twice per period $2\pi/\omega$. Hence, its maximum kinetic energy $\frac{1}{2}M(A\omega)^2$ is twice added to the rotor kinetic energy $\frac{1}{2}I\omega^2$ the amplitude value of the frequency oscillations satisfies the following expression: $\Delta\omega \approx MA^2\omega/2I$. These oscillations are clearly seen at Fig.6 obtained by computer simulation, see below.

The final comment is that the obtained results admit simple interpretation in terms of vibrational mechanics [3]. For example, equation (17) can be presented in the form

$$I\ddot{\Psi} + k\dot{\Psi} = V(\Psi),$$

where

$$V(\Psi) = -d\Pi_V/d\Psi = -b \sin \Psi + \rho^2 \sin^2 \Psi,$$

is vibration torque,

$$\Pi_V = b \cos \Psi - \rho(\Psi - \sin \Psi)/2$$

is potential energy of the so called “vibrational forces”. Minimal points of this energy correspond to the stable steady-state motions. Therefore the system in question belongs to a class of so called “potential in the average” dynamical systems (with respect to the variable Ψ). The vibration torque is potential in spite of essential nonconservativeness of initial system (1), (2).

Comparison with computer simulation results. Appearance of slow oscillations of the rotor angular velocity is clearly seen in Fig.6 obtained by computer simulations of the system (1), (2).

In the picture the plots of the angular velocity $\dot{\varphi}(t)$ for constant values of external torque $L(\omega) - R(\omega) = 0.51 \text{ kg}\cdot\text{m}/\text{s}^2$ (passage through resonance) and 0.50 (capture). The parameter values in (1), (2), (14) correspond to parameters of the experimental stand SV-2 [18]: $m = 1.5 \text{ kg}$, $M = 12 \text{ kg}$, $I = 0.014 \text{ kg}\cdot\text{m}^2$, $\varepsilon = 0.04 \text{ m}$, $c = 5300 \text{ kg}\cdot\text{m}^2/\text{s}^2$, $\beta = 0.005$, $k = 5$.

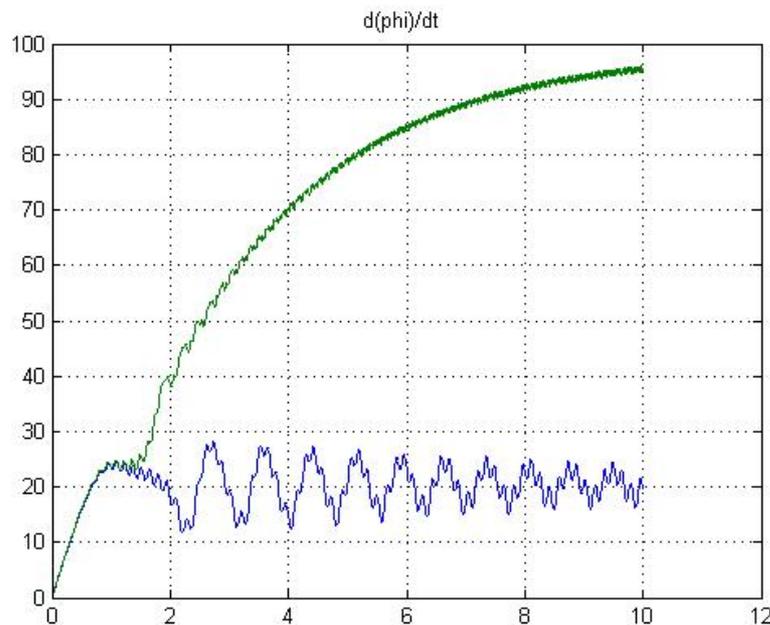


Fig.6. Change of the rotor angular velocity in the start-up mode (computer simulation results): upper curve - passage through resonance, lower curve - capture (Sommerfeld effect).

For the above parameter values the expression (19) yields $q = 5.7 \text{ c}^{-1}$, which corresponds to the value obtained from the simulation (Fig.6) with good accuracy. In addition, in Fig.6 the fast oscillations of the angular velocity of the rotor with frequency 2ω mentioned above can be clearly seen.

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