CLOSED-LOOP IMPULSE CONTROL OF OSCILLATING SYSTEMS [1]

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Abstract: This paper deals with the problem of damping the oscillations of a finite cascade of springs through impulsive controls in finite time. The control problem is treated as one of specifying a closed-loop (feedback) strategy and is solved by applying Hamiltonian technique in its Dynamic Programming version. A numerical algorithm is indicated and the limit case with control time tending to infinity is described.

Keywords: impulse control, control synthesis, dynamic programming, oscillating systems

1. INTRODUCTION

This paper deals with the problem of feedback control for systems with impulse controls. It is concerned with damping the oscillations of a cascade (suspended chain) of an arbitrary finite number of loaded oscillating springs through an impulsive force applied to a particular link. The system has to be fully stopped at given finite time. The solution is sought for in the form of a closed-loop *synthesized control strategy.* The required form of solution is reached by applying a respective version of the Dynamic Programming equation described through variational inequalities of the Hamilton-Jacobi type. The feedback control here results in a finite number of strokes (impulses) which stop the system at given time. A numerical scheme for calculating the solution is then described. The final part deals with the limit case, when the time allowed for controlling the system tends to infinity.

[1] This work is supported by the Russian Foundation for Basic Research (grant 06-01-00332). It has been realized within the programs "State Support of the Leading Scientific Schools" (5344.2006.1) and "Development of Scientific Potential" (2.1.1.1714).

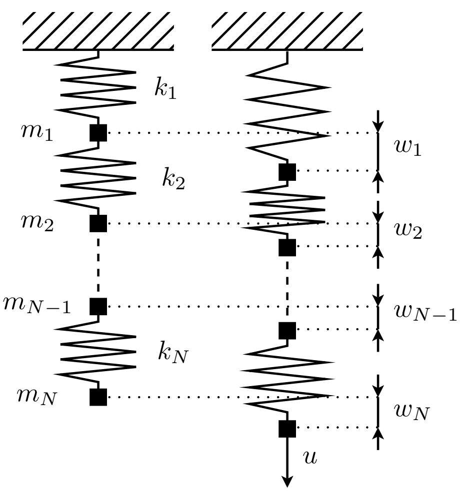

Fig. 1. The chain of springs to be controlled in the equilibrium state (left) and in an arbitrary state (right)

2. THE PROBLEM

We consider the problem of fully stopping the oscillations of a suspended chain of a finite number of loaded springs by applying only an impulse control force to a particular link (Fig. 1 shows control applied to the lower end of the chain). The chain must be brought to an equilibrium in given finite time, so that this is *not* a problem of asymptotic stabilization.

Apart from the springs, the chain also includes given loads attached in between the springs. We assume that the masses of springs are negligibly small as compared to those of the loads. The upper end of the chain is rigidly attached to a fixed suspension. Then the oscillations of the chain could be described by the following system of second-order ODEs:

$$\begin{cases} m_1\ddot{w}_1 = k_2(w_2 - w_1) - k_1 w_1, \\ m_i\ddot{w}_i = k_{i+1}(w_{i+1} - w_i) - k_i(w_i - w_{i-1}) \\ m_\nu\ddot{w}_\nu = k_{\nu+1}(w_{\nu+1} - w_\nu) - \\ \qquad\qquad k_\nu(w_\nu - w_{\nu-1}) + \dfrac{\mathrm{d}U}{\mathrm{d}t}, \\ m_N\ddot{w}_N = -k_N(w_N - w_{N-1}), \end{cases} \tag{1}$$

when $t > t_0$. Here N is the number of springs which are numbered from top to bottom. The loads are numbered similarly, so that the i-th load is attached to the lower end of the i-th spring. w_i is the displacement of the i-th load from the equilibrium, m_i is the mass of the i-th load, k_i is the stiffness coefficient of the i-th spring. The gravity force enters (1) implicitly through determining the lengths of the springs at the equilibrium.

Here dU/dt is the generalized derivative of the generalized control $U(\cdot) \in BV[t_0, t_1]$ which is applied to the ν-th load. Symbol $BV[t_0, t_1]$ stands for the space of functions of bounded variation.

Throughout this paper we assume that the trajectories and their derivatives are left-continuous.

The initial state of the chain at time t_0 is given by the displacements w_i^0 and the velocities of the loads $\dot{w}_i^0$:

$$w_i(t_0) = w_i^0, \quad \dot{w}_i(t_0) = \dot{w}_i^0. \tag{2}$$

The equations (1) may be also interpreted as a spatial discretization of a one-dimensional wave equation for a string with fixed left end and a control force applied to the free right end:

$$\rho(\xi)w_{tt}(t,\xi) = \big[E(\xi)w_\xi(t,\xi)\big]_\xi, \quad t > t_0, \quad 0 < \xi < L;$$
$$w(t,0) = 0, \quad w_\xi(t,L) = E^{-1}(L)\frac{\mathrm{d}U}{\mathrm{d}t}, \quad t \geqslant t_0;$$
$$w(t_0,\xi) = w^0(\xi), \quad w_t(t_0,\xi) = \dot{w}^0(\xi), \quad 0 \leqslant \xi \leqslant L.$$

(Here $w(t,\xi)$ is the displacement of point ξ at time t; each point ξ of the string is characterized by the value of the Young modulus $E(\xi)$ and the mass density $\rho(\xi)$.) Therefore, the presented approach may be also useful for investigating problems of *impulse boundary control for the wave equation.*

Problem 1. (Of Open-Loop Control). Find generalized control $U^*(\cdot) \in BV[t_0, t_1]$, such that functional

$$J(U(\cdot)) = \operatorname*{Var}_{[t_0,t_1]} U(\cdot) \tag{3}$$

attains its minimum at $U = U^*$, under condition that the trajectory of the system (1), emanated from initial conditions (2), satisfies the terminal conditions $w_i(t_1+0) = 0$, $\dot{w}_i(t_1+0) = 0$.

Note that here time t_1 is fixed.

The open-loop control problem 1 (see Krasovski (1968))is well studied. In particular, it was proved that there exists an optimal control with number of impulses not exceeding $2N$ (Neustadt 1964).

In this paper we will be interested in finding the control in feedback form. Denote $x = (w, \dot{w}) \in \mathbb{R}^{2N}$.

Definition 2. A closed-loop impulse control is the pair $\mathscr{U} = (\mathbb{J}, v)$, where $\mathbb{J} \subseteq [t_0, t_1] \times \mathbb{R}^{2N}$ is the *jump set*, and $v : \mathbb{J} \to \mathbb{R}$ is the *jump amplitude.* The latter should satisfy the following condition:

$$(t, x + B(t)v(t,x)) \notin \mathbb{J}, \quad \forall (t,x) \in \mathbb{J}. \tag{4}$$

The new terms introduced here are clarified by the next definition.

Definition 3. For a given initial condition (2), an open-loop control $U(\cdot) \in BV[t_0, t_1]$,

$$\mathrm{d}U(t) = \sum_{i=1}^{s} v_i \delta(t - t_i)\,\mathrm{d}t,$$

is *consistent* with the closed-loop control $(\mathbb{J}, v)$, if it satisfies the following conditions (assuming that $x(t_i - 0) = x(t_i)$):

- $(t_i, x(t_i)) \in \mathbb{J}$, $i = \overline{1,s}$;
- $v_i = v(t_i, x(t_i))$, $i = \overline{1,s}$;
- $(t_i, x(t_i + 0)) \notin \mathbb{J}$ when $i = \overline{1,s}$.

Here $x(\cdot)$ is the trajectory generated by control $U(\cdot)$ and initial condition (2). Condition (4) prohibits successive jumps at the same time moment, i.e. $t_i \neq t_j$ when $i \neq j$.

Problem 4. (Of Closed-Loop Control). Find a closed-loop impulse control $\mathscr{U} = (\mathbb{J}, v)$, such that for any initial condition (2), any open-loop control $U(\cdot)$ consistent with $\mathscr{U}$ solves Problem 1.

In problems 1 and 4 the functional J and the terminal constraint may be replaced with a functional of generalized Meier–Bolza type:

$$J(U(\cdot)) = \operatorname*{Var}_{[t_0,t_1]} U(\cdot) + \varphi(x(t_1+0)).$$

Here $\varphi : \mathbb{R}^{2N} \to \mathbb{R} \cup \{+\infty\}$ is a proper closed convex terminal functional. The latter allows to formulate the optimality principle later. A particular choice of $\varphi(x) = \mathscr{I}(x \mid \{0\})$ leads to the problems of the above[2].

3. THE SYSTEM

Before proceeding with solution, we provide some details on the system.

First, we rewrite the original system (1) in the normalized matrix form. To do this, we introduce an extended state vector $x \in \mathbb{R}^n$, $n = 2N$, defined by $(x_1, \dots, x_N) = w$, $(x_{N+1}, \dots, x_{2N}) = \dot{w}$. Then

$$\mathrm{d}x(t) = Ax(t)\,\mathrm{d}t + b\,\mathrm{d}U(t), \qquad (5)$$
$$x(t_0) = x^0 = \begin{pmatrix} w^0 \\ \dot{w}^0 \end{pmatrix},$$
$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & 0 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_N \end{pmatrix},$$
$$K = \begin{pmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{N-1} & k_{N-1}+k_N & -k_N \\ & & & -k_N & k_N \end{pmatrix},$$
$$b = (0, \dots, 0, m_\nu^{-1}, 0, \dots, 0)^T.$$

The singular value decomposition for symmetric matrix $A_0 = M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$ is

$$A_0 = Q\Lambda Q^T,\ QQ^T = I,\ \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N).$$

where the eigenvalues λ_i are positive (since A_0 is positive definite) and different (since A_0 is tridiagonal). The columns of Q are eigenvectors of A_0: $q^{(1)}$, …, $q^{(N)}$, $A_0 q^{(j)} = \lambda_j q^{(j)}$.

After a change of variables $w = M^{-\frac{1}{2}} Q\bar{w}$ (and, respectively, $\bar{x} = (\bar{w}, \dot{\bar{w}})$), the system (5) becomes

$$\mathrm{d}\bar{x}(t) = \bar{A}\bar{x}(t)\,\mathrm{d}t + \bar{b}\,\mathrm{d}U(t), \qquad (6)$$
$$\bar{x}(t_0) = \bar{x}^0 = \begin{pmatrix} \bar{w}^0 \\ \dot{\bar{w}}^0 \end{pmatrix},$$
$$\bar{A} = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix},$$
$$\bar{b} = m_\nu^{-\frac{1}{2}} (0, \dots, 0, q_\nu^{(1)}, q_\nu^{(2)}, \dots, q_\nu^{(N)})^T.$$

That is, the original system is reduced to N harmonic oscillators with same control input:

$$\begin{cases} \dot{\bar{x}}_j = \bar{x}_{j+N}, \\ \mathrm{d}\bar{x}_{j+n} = -\lambda_j \bar{x}_j\,\mathrm{d}t + b_j\,\mathrm{d}U(t), \end{cases} \quad j = \overline{1, N}.$$

Next we check the *complete controllability* (Kalman 1960) property of the system.

[2] By $\mathscr{I}(x \mid A) = 0$ in A and $+\infty$ elsewhere we denote the indicator function of convex set A.

For matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ denote $\Phi(A, B) = \big(B\ AB\ \dots\ A^{n-1}B\big)$. We need the following auxiliary statement.

Lemma 5. System $\ddot{x}(t) = Ax(t) + Bu(t)$ is completely controllable if and only if the system $\dot{x}(t) = Ax(t) + Bu(t)$ is completely controllable.

Theorem 6. System (5) is completely controllable if and only if $q_\nu^{(i)} \neq 0$, $i = \overline{1, N}$.

PROOF. After applying Lemma 5 to (6), we need to find the rank of $\Phi(\Lambda, \bar{b})$. We have

$$\Phi(\Lambda, \bar{b}) = m_\nu^{-\frac{1}{2}} \begin{pmatrix} q_\nu^{(1)} & \lambda_1 q_\nu^{(1)} & \dots & \lambda_1^{N-1} q_\nu^{(1)} \\ \dots & \dots & \dots & \dots \\ q_\nu^{(N)} & \lambda_N q_\nu^{(N)} & \dots & \lambda_N^{N-1} q_\nu^{(N)} \end{pmatrix}.$$

Using the formula for determinant of Vandermonde matrix, we have

$$\det \Phi(\Lambda, \bar{b}) = m_\nu^{-\frac{1}{2}} \left[\prod_{i=1}^{N} q_\nu^{(i)}\right] \left[\prod_{1\leqslant i<j\leqslant N} (\lambda_j - \lambda_i)\right].$$

Since all λ_j are different, we get the complete controllability criterion: all $q_\nu^{(i)}$ are non-zero.

Corollary 7. In the case of equal masses and stiffness coefficients — $m_i \equiv m$, $k_i \equiv k$, the system (5) is completely controllable if and only if $\gcd(\nu, 2N + 1) = 1$. In particular, this is always the case when $\nu = 1, 2, N$.

PROOF. Here the matrix A_0 is

$$A_0 = \frac{k}{m}\hat{A}_0, \quad \hat{A}_0 = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots\ -1 & 2 & -1 \\ 0 & 0 & 0 & \dots\ 0 & -1 & 1 \end{pmatrix}$$

It is straightforward to check that the eigenvectors of $\hat{A}_0$ are $q_j^{(i)} = \sin\frac{j(2i-1)}{2N+1}\pi$, $i, j = \overline{1, N}$, and the corresponding eigenvalues are $\lambda_i = 2\Big(1 - \cos\frac{2i-1}{2N+1}\pi\Big)$. Taking into account that $q_j^{(i)} = \sin(\nu - \frac{j(2i-1)}{2N+1})\pi = \sin\frac{2\nu(N+1-i)}{2N+1}\pi$ we see that the coordinates $q_\nu^{(i)}$ span the entire set $\sin\frac{\nu i}{2N+1}\pi$, $i = \overline{1, 2N}$. None of these values is zero if and only if $\nu i \neq 0(\mathrm{mod} 2N + 1)$, $i \in \overline{1, 2N}$.

Corollary 8. For $\nu = 1$ and $\nu = N$ the system (5) is completely controllable for arbitrary $m_i > 0$, $k_i > 0$.

PROOF. Consider the case $\nu = 1$ (for $\nu = N$ the proof is similar). Let $A_0 q = \lambda q$, $q \neq 0$, $q_1 = 0$. Then $(A_0 q)_1 = -k_2 q_2 = 0$, hence $q_2 = 0$. Continuing this process, we get $q_i = 0$, $i = \overline{1, N}$, which contradicts our assumptions.

4. THE DYNAMIC PROGRAMMING APPROACH

In this section, we recall some facts from on dynamic programming approach to impulse control problems (see Daryin, Kurzhanski and Seleznev, 2005).

The *value function* $V(t_0, x_0)$ of problem 4 is the optimal value of $J(U(\cdot))$ under given fixed initial position (t_0, x_0). An extended notation $V(t_0, x_0; t_1, \varphi(\cdot))$ will be also used to emphasize the dependence of the optimal value $V(t_0, x_0)$ on terminal time t_1 and terminal function $\varphi(\cdot)$.

Notation $W(t_0, x_0) = W(t_0, x_0; t_1, x_1)$ will be used for the minimal variation of problem (1) with fixed right end $x(t_1 + 0) = x_1$. As discussed above, $W(t_0, x_0; t_1, x_1) = V(t_0, x_0; t_1, \mathscr{I}(\cdot \mid \{x_1\}))$. It may be expressed as (Krasovski 1968, Kurzhanski and Osipov 1969)

$$W(t_0, x_0; t_1, x_1) = \sup_{p \in \mathbb{R}^n} \frac{\left\langle p, x_1 - \mathrm{e}^{(t_1 - t_0)A} x_0 \right\rangle}{\max_{t \in [t_0, t_1]} \left| b^T \mathrm{e}^{(t_1 - t)A} p \right|}.$$

Using the last formula, we have

$$V(t_0, x_0) = \inf_{x_1 \in \mathbb{R}^n} \{\varphi(x_1) + W(t_0, x_0; t_1, x_1)\}.$$

Here the infimum is attained, and the corresponding optimal control exists.

The following statement is true:

Statement 9. The value function $V(t_0, x_0)$ is convex in x and its conjugate is given by

$$V^*(t_0, p) = \varphi^*(\mathrm{e}^{(t_0 - t_1)A^T} p) + \\ + \mathscr{I}\left(\mathrm{e}^{(t_0 - t_1)A^T} p \;\middle|\; \mathscr{B}_{\|\cdot\|_{[t_0,t_1]}}\right). \quad (7)$$

Here $\mathscr{B}_{\|\cdot\|_{[t_0,t_1]}}$ is the unit ball in the semi-norm

$$\|p\|_{[t_0,t_1]} = \max_{t \in [t_0, t_1]} \left| b^T \mathrm{e}^{(t_1 - t)A^T} p \right|,$$

$\varphi^*(p)$ is the Fenchel conjugate of $\varphi(x)$ (Rockafellar 1970).

In particular, $W(t_0, x_0; t_1, 0) = \rho(x_0 \mid \mathscr{X}[t_0])$, where $\mathscr{X}[t_0] = \mathrm{e}^{(t_0 - t_1)A} \mathscr{B}_{\|\cdot\|_{[t_0,t_1]}}$ is the *polar set*, ρ stands for the support function of a convex set. Another representation of the value function is

$$W(t, x) = \min \left\{ \alpha > 0 \;\middle|\; \alpha^{-1} x \in \mathscr{X}_1[t] \right\}, \quad (8)$$

where $\mathscr{X}_1[t]$ is the *backward reachability domain* of the system (5) from point $x(t_1 + 0) = 0$ under constraint $\operatorname*{Var}_{[t,t_1]} U(\cdot) \leqslant 1$.

Using (7) one may prove the next result:

Theorem 10. The value function $V(t, x; t_1, \varphi(\cdot))$ satisfies the *principle of optimality* in the form of the semigroup property. Namely, for each $\tau \in [t_0, t_1]$ we have:

$$V(t_0, x_0; t_1, \varphi(\cdot)) = V(t_0, x_0; \tau, V(\tau, \cdot; t_1, \varphi(\cdot))).$$

Note that, unlike problems without impulse controls, in the general case $V(t_1, x; t_1, \varphi(\cdot)) \leqslant \varphi(x)$, since from (7) it follows that

$$V^*(t_1, p) = \varphi^*(p) + \mathscr{I}\left(b^T p \;\middle|\; [-1, 1]\right).$$

For example, if $\varphi(x) = \mathscr{I}(x \mid \{0\})$, then

$$V(t_1, x; t_1, \varphi(\cdot)) = \begin{cases} \|x\|/\|b\|, & x \parallel b; \\ +\infty, & \text{otherwise} \end{cases} \leq \varphi(x).$$

Theorem 11. The value function $V(t, x)$ is the viscosity solution (Crandall and Lions 1983) to the Hamilton–Jacobi–Bellman equation:

$$\min \{H_1(t, x, V_t, V_x),\, H_2(t, x, V_t, V_x)\} = 0, \quad (9)$$

$$V(t_1, x) = V(t_1, x; t_1, \varphi(\cdot)).$$

$$H_1(t, x, \xi_t, \xi_x) = \xi_t + \langle \xi_x, Ax \rangle,$$

$$H_2(t, x, \xi_t, \xi_x) = 1 - \left| b^T \xi_x \right|.$$

Due to (9), in any position (t^*, x^*) there are two possibilities for the control. Either $H_1 = 0$, and the control may choose $\mathrm{d}U(t) = 0$, or $H_1 > 0$, in which case it is necessary the $H_2 = 0$, and the control has a jump in direction $-b^T V_x$. The magnitude of the jump is to be selected in such a way that after the jump we again have $H_1 = 0$, i.e.

$$\mathrm{d}U(t) = -\alpha b^T V_x(t^*, x^*) \delta(t - t^*)\, \mathrm{d}t, \quad (10)$$

$$H_2(t^*, x^* - \beta b b^T V_x(t^*, x^*)) = 0, \quad \beta \in [0, \alpha];$$

$$H_1(t^*, x^* - \alpha b b^T V_x(t^*, x^*)) = 0.$$

The solution to Problem 4 is then

- $\mathbb{J} = \{ (t, x) \mid H_1(t, x) > 0 \}$;
- $v(t, x) = -\alpha b^T V_x$, with α as defined above.

Remark 12. All results in this section are true for the system (6), with A, b, x and V replaced by $\bar{A}$, $\bar{b}$, $\bar{x}$ and $\bar{V}$ respectively.

5. A NUMERICAL ALGORITHM

Here we describe numerical algorithm for finding the value function and control in case $\varphi(x) = \mathscr{I}(x \mid \{0\})$.

The polar set $\mathscr{X}[t]$ may be written as

$$\mathscr{X}[t] = \bigcap_{\tau \in [t, t_1]} \left\{ p \;\middle|\; \left| b^T \mathrm{e}^{(t - \tau)A^T} p \right| \leqslant 1 \right\}.$$

Choose $K + 1$ time instants $t = \tau_0 < \tau_1 < \ldots < \tau_K = t_1$, then

$$\mathscr{Z}[t] \subseteq \hat{\mathscr{Z}}[t] = \\ = \bigcap_{\tau \in \{\tau_0,\ldots,\tau_K\}} \left\{ p \;\middle|\; \left| b^T \mathrm{e}^{(t-\tau)A^T} p \right| \leqslant 1 \right\}. \quad (11)$$

The set $\hat{\mathscr{Z}}[t]$ is an external approximation of the polar set $\mathscr{Z}[t]$, therefore the function

$$\hat{V}(t,x) = \sup_{p \in \hat{\mathscr{Z}}[t]} \langle x, p \rangle. \quad (12)$$

is an upper estimate for the value function $V(t,x)$.

Since set $\hat{\mathscr{Z}}[t]$ is defined by a finite number of linear inequalities, relation (12) is a problem of linear programming.

If the value $\hat{V}(t,x)$ is finite, then

$$\hat{V}(t,x) = \langle x, p(t,x) \rangle, \quad p(t,x) \in \operatorname*{Arg\,max}_{p \in \hat{\mathscr{Z}}[t]} \langle x, p \rangle.$$

If the maximizer $p(t,x)$ is unique, then the estimate $\hat{V}$ is differentiable at (t,x) and $\hat{V}_x(t,x) = p(t,x)$ (Demyanov 1974).

The control may have a jump at position $\{t,x\}$ if the following condition holds:

$$\left| b^T V_x \right| \geqslant 1 \quad \Longleftrightarrow \quad \left| b^T p(t,x) \right| \geqslant 1.$$

The direction of the impulse is $\hat{u} = -\operatorname{sign} b^T p(t,x)$. The jump amplitude $\hat{\alpha}$ is determined as the maximum value of $\alpha > 0$ such that $p(t,x)$ remains a maximizer after the jump, i.e.

$$p(t,x) \in \operatorname*{Arg\,max}_{p \in \hat{\mathscr{Z}}[t]} \langle x + \alpha b \hat{u}, p \rangle.$$

The value of $\hat{\alpha}$ may be calculated as follows. Let $\xi_1, \ldots, \xi_n$ be the active independent constraints in the problem (12), (11), so that $\langle \xi_i, p \rangle = 1$, $i = \overline{1,n}$. Express vector x through basis $\{\xi_i\}$:

$$x = \sum_{i=1}^{n} \lambda_i \xi_i, \quad \lambda = \Xi^{-1} x, \quad \Xi = \begin{pmatrix} \xi_1 & \cdots & \xi_n \end{pmatrix}.$$

Since $\langle x, p \rangle = \sum_{i=1}^{n} \lambda_i \langle \xi_i, p \rangle$, we should have $\lambda_i \geqslant 0$. Therefore

$$\hat{\alpha} = \max \left\{ \alpha \;\middle|\; \Xi^{-1}(x + \alpha b \hat{u}) \geqslant 0 \right\}.$$

Denote $\mu = \Xi^{-1} b \hat{u}$, then

$$\hat{\alpha} = \min_{i=1,\ldots,n} \left\{ -\lambda_i \mu_i^{-1} \;\middle|\; \mu_i < 0 \right\}.$$

Remark 13. Throughout the numerical simulation of the controlled process, it is not necessary to solve the linear programming problem (12) at each step. Suppose the control had no impulse at time τ_k and vector $p_k = p(\tau_k, x(\tau_k))$ is known. Assume also that $\langle p_k, B(\tau_k) B(\tau_k) u_i \rangle < 1$, $\forall i = \overline{1,M}$, i.e. the constraints corresponding to τ_k are inactive. Then

$$p(\tau_{k+1}, x(\tau_{k+1})) = X^T(\tau_k, \tau_{k+1}) p(\tau_k, x(\tau_k)). \quad (13)$$

If the assumptions above do not hold, then the right-hand side of (13) is usually a good initial guess for finding $p(\tau_{k+1}, x(\tau_{k+1}))$.

6. THE ASYMPTOTIC SOLUTION

When the length of time interval $[t_0, t_1]$ is large (compared to oscillation periods of individual harmonics), most solutions found numerically exhibit a specific behavior. Namely, the impulses occur when the harmonic with largest energy has maximum velocity and zero displacement. In this section, we study such effect by analyzing the properties of solutions as the duration of the time interval tends to infinity.

Here we consider the system in the form (6). The backward reach set $\bar{\mathscr{X}_1}[t]$ here is

$$\bar{\mathscr{X}_1}[t_0] = \operatorname{conv} \bigcup_{t \in [t_0,t_1]} \left\{ \begin{pmatrix} \bar{b}_{N+1} u \lambda_1^{-\frac{1}{2}} \sin \sqrt{\lambda_1} \Delta t \\ \vdots \\ \bar{b}_{2N} u \lambda_N^{-\frac{1}{2}} \sin \sqrt{\lambda_N} \Delta t \\ \bar{b}_{N+1} u \cos \sqrt{\lambda_1} \Delta t \\ \vdots \\ \bar{b}_{2N} u \cos \sqrt{\lambda_N} \Delta t \end{pmatrix} \right\}.$$

with $\Delta t = t_0 - t$ and u taking values 1 and -1. From here we see that points $\bar{x} \in \bar{\mathscr{X}_1}[t_0]$ satisfy

$$\lambda_j \bar{x}_j^2 + \bar{x}_{j+N}^2 \leqslant \bar{b}_{N+j}^2, \quad j = 1, \ldots, N,$$

i.e. $\mathscr{X}_1[t_0] \subseteq \bigcap_{j=1}^{N} C_j$, where C_j are cylinders

$$C_j = \left\{ \bar{x} \;\middle|\; \frac{\lambda_j \bar{x}_j^2 + \bar{x}_{j+N}^2}{\bar{b}_{N+j}^2} \leqslant 1 \right\}.$$

Since the eigenvalues λ_i are distinct, the backward reach sets in the limit fill the limiting intersection of cylinders as $t_0 \to -\infty$:

$$\operatorname{cl} \bigcup_{t \leqslant t_1} \mathscr{X}_1[t] = \mathscr{C} = \bigcap_{j=1}^{N} C_j.$$

The corresponding value functions also converge:

$$\lim_{t \to -\infty} W(t,x) = \mathscr{V}(x) = \max_{j=1,\ldots,N} \mathscr{V}_j(x), \quad (14)$$

$$\mathscr{V}_j(x) = \sqrt{\frac{\lambda_j \bar{x}_j^2 + \bar{x}_{j+N}^2}{\bar{b}_{N+j}^2}}.$$

The function $\mathscr{V}(x)$ is a lower estimate for the value function.

In the limit, the HJB quasi-variational inequality turns into $\left| \bar{b}^T \mathscr{V}_x \right| \leqslant 1$, with necessary condition for a jump being $\left| \bar{b}^T \mathscr{V}_x \right| = 1$.

If the maximum in (14) is attained at $j = j_0$, then

$$\left| \bar{b}^T \mathscr{V}_x \right| = \frac{\bar{x}_{j_0+N} / \bar{b}_{N+j_0}}{\mathscr{V}} \leqslant 1,$$

with equality attained when $\bar{x}_{j_0} = 0$. The amplitude of the jump is then $v(t,\bar{x}) = -\alpha \bar{b}^T \mathscr{V}_x$, with maximum possible α chosen from condition

$$\mathscr{V}_{j_0}(\bar{y}) \geqslant \max_{j=\overline{1,\ldots,N}} \mathscr{V}_j(\bar{y}) \quad \beta \in [0, \alpha), \quad (15)$$

where $\bar{y}_j = \bar{x}_j$, $\bar{y}_{j+N} = \bar{x}_{j+N} - \beta \bar{b}_{N+j} \bar{b}^T \mathscr{V}_x(t, \bar{x})$. The latter gives

$$\alpha = \min_{j \neq j_0} \frac{\mathscr{V}_{j_0}^2(\bar{x}) - \mathscr{V}_j^2(\bar{x})}{2(\bar{b}^T \mathscr{V}_x)\left(\frac{\bar{x}_{N+j_0}}{\bar{b}_{N+j_0}} - \frac{\bar{x}_{N+j}}{\bar{b}_{N+j}}\right)}.$$

Note that after the first jump the set J_0 will contain at least two elements.

If the set of maximizers J_0 in (14) has more than one element, it is necessary to apply (15) for each $j_0 \in J_0$ and choose the maximum α. However, in this case the jump is possible only if $\bar{x}_{j_0} = 0$ for all $j_0 \in J_0$, which is not true for any $t \geqslant t_0$ in general case. To overcome this, one may use an ε-optimal strategy, namely, the jump takes place when the condition $\left|\bar{b}^T \mathscr{V}_x\right| \geqslant 1 - \varepsilon$ is satisfied. By integrating this, we get the following estimate:

$$\operatorname{Var} U(\cdot) \leqslant \frac{\mathscr{V}(\bar{x}_0)}{1 - \varepsilon}.$$

The set $\mathscr{C}$ may be approximated by an external ellipsoid (Kurzhanski and Vályi 1997)

$$\mathscr{C} \subseteq \mathscr{C}_\gamma = \left\{ \bar{x} \;\middle|\; \sum_{j=1}^N \gamma_j \frac{\lambda_j \bar{x}_j^2 + \bar{x}_{j+N}^2}{\bar{b}_{N+j}^2} \leqslant 1 \right\}.$$

Here we assume that $\gamma_j > 0$, $\gamma_1 + \ldots + \gamma_N = 1$. The corresponding lower bound for the value function is

$$\mathscr{V}_\gamma(\bar{x}) = \left(\sum_{j=1}^N \gamma_j \frac{\lambda_j \bar{x}_j^2 + \bar{x}_{j+N}^2}{\bar{b}_{N+j}^2} \right)^{\frac{1}{2}}.$$

The condition of a jump for an ε-optimal strategy, $\left|\bar{b}^T \mathscr{V}_{\gamma x}\right| \geqslant 1 - \varepsilon$, is

$$\sum_{j=1}^N \gamma_j \frac{\bar{x}_{N+j}}{\bar{b}_{N+j}} \geqslant (1 - \varepsilon) \mathscr{V}_\gamma(\bar{x}).$$

The condition (10) for the amplitude of the jump α here turns to be the solution of a quadratic equation.

7. CONCLUSION

This paper presents solutions to the problem of feedback control for a high order oscillating system which describes the motion of a chain of an arbitrary finite number of suspended pendulums. The problem is solved in the class of impulse controls by applying Hamiltonian techniques. New classes of HJB-type variational inequalities relevant for this problem are introduced and solved with respective impulse-type solution strategies being specified. A numerical scheme for the solutions is also indicated.

REFERENCES

Bellman, R. (1957). *Dynamic Programming.* Princeton University Press. Princeton, NJ.

Bensoussan, A. and J.-L. Lions (1982). *Contrôle impulsionnel et inéquations quasi variationnelles.* Paris.

Crandall, Michaël G. and Pierre-Louis Lions (1983). Viscosity solutions of Hamilton–Jacobi equations. *Trans. Amer. Math. Soc.* **277**, 1–41.

Daryin, A. N., A. B. Kurzhanski and A. V. Seleznev (2005). A dynamic programming approach to the impulse control synthesis problem. In: *Proc. Joint 44th IEEE CDC-ECC 2005.* IEEE. Seville.

Demyanov, V. F. (1974). *Minimax: Directional Derivates.* Nauka. Moscow. In Russian.

Dykhta, V. A. and O. N. Sumsonuk (2003). *Optimal impulsive control with applications.* Fizmatlit. Moscow. In Russian.

Gusev, M. I. (1975). On optimal control of generalized processes under non-convex state constraints. In: *Differential Games and Control Problems.* UNC AN SSSR. Sverdlovsk.

Kalman, R. E. (1960). On the general theory of control systems. In: *Proc. 1st IFAC Congress.* Vol. 1. IFAC. Butterworths. London.

Krasovski, N. N. (1957). On a problem of optimal regulation. *Prikl. Math. & Mech.* **21**(5), 670–677. In Russian.

Krasovski, N. N. (1968). *The Theory of Motion Control.* Nauka. Moscow. In Russian.

Kurzhanski, A. B. (1975). Optimal systems with impulse controls. In: *Differential Games and Control Problems.* UNC AN SSSR. Sverdlovsk.

Kurzhanski, A. B. and I. Vályi (1997). *Ellipsoidal Calculus for Estimation and Control.* SCFA. Birkhäuser. Boston.

Kurzhanski, A. B. and Yu. S. Osipov (1969). On controlling linear systems through generalized controls. *Differenc. Uravn.* **5**(8), 1360–1370. In Russian.

Miller, B. M. and E. Ya. Rubinovich (2003). *Impulsive Control in Continuous and Discrete-Continuous Systems.* Kluwer. N.Y.

Motta, Monica and Franco Rampazzo (1995). Space-time trajectories of nonlinear systems driven by ordinary and impulsive controls. *Differential and Integral Equations* **8**, 269–288.

Neustadt, L. W. (1964). Optimization, a moment problem and nonlinear programming. *SIAM J. Control* **2**(1), 33–53.

Rockafellar, R. T. (1970). *Convex Analysis.* Vol. 28 of *Princeton Mathematics Series.* Princeton University Press.