STABILIZATION OF SPLAY STATE BY EXTERNAL SIGNAL IN MULTIMODE LASERS

Elena V. Grigorieva
Department of mathematics
Belarus State Economic University
Belarus
grigorieva@tut.by

Abstract

Poincare maps are derived to describe antiphase spiking in class B multimode lasers. Analytical conditions are determined for splay states and clustering of spikes associated with different longitudinal laser modes. Stabilization of splay states is proved in presence of small external signal.

Key words

Synchronization, Laser, Relaxation oscillations.

1 Introduction

Many systems like biological or artificial neural networks are characterized by global nonlinear coupling (all-to-all) between oscillators. In this case there can exist phase-synchronized periodic states – splay states – discussed firstly in connection with N globally coupled Josephson junctions [Strogatz and Mirollo, 1993]. In splay states all elements oscillate with the same waveform but each mode has its phase shifted by $1/N$ of a period from its neighbor.

In laser physics, splay states have been observed in multimode lasers with intracavity second-harmonic generation [Bracikovski and Roy, 1991; Wiesenfeld, 1990], in deeply modulated lasers [Bielawski, 1992; Otsuka, 1991], and in distributed laser arrays [Silber, 1993]. The stable splay states of at least five modes have been experimentally observed. In addition, grouping states (or clusters) in which modes oscillate still out of phase but with different waveforms and periods has been numerically found in deeply modulated multimode lasers [Otsuka and Sato, 1996].

Such attractors appear with a large multiplicity. There coexist $(N-1)!$ splay states as any renumbering of equivalent oscillators is possible. Stability of the splay states near the threshold of oscillatory instability has been discussed in the framework of the bifurcation theory and in the case of spiking as well. The splay state has been shown to be marginally stable for finite pulse width while strictly stable for δ-like pulses [Zimmler, 2007; Calamai, 2009]. The number of splay states grows much faster than the dimension $2N$ of the phase space. Hence, it was supposed the basins of attraction shrinks rapidly and noise induced switching between attractors should be [Wiesenfeld and Hadley, 1989], that was called attractor crowding.

In this paper we show that splay states can be stabilized by small external lighting or by sufficiently high level of spontaneous emission contrary to predicted attractor crowding. Stabilization can be achieved for both finite-width and short-width pulsing. To study such pulsing we derive asymptotically Poincare maps from the original differential equations. The fixed points of the maps can be used for analytical reconstruction of limit cycles. As a result, we find initial conditions leading to splay states and clustered states, determine their stability and explain mechanism of switching between them as dynamical instability. We consider models of multimode solid-state lasers representing two types of spiking behavior: (i) pulses of duration comparable with the period of oscillations and (ii) spikes of asymptotically small duration. The first type of oscillations appears in Nd:YAG laser with intracavity second harmonic generation under relatively high pumping rate. The second type has been observed in the dynamics of a semiconductor laser with periodically driven injection current.

2 Multimode laser with intracavity second harmonic generation

Dynamics of N longitudinal modes is governed by the equations given in [Bracikovski and Roy, 1991]:

$$\eta \dot{I}_k = I_k \left[G_k - \alpha + \epsilon I_k - 2\epsilon \sum_{r=1}^{N} I_r \right],$$

$$\dot{G}_k = \gamma - G_k \left[1 + (1+\beta) I_k + \beta \sum_{r=1}^{N} I_r \right], \quad (1)$$

where $I_k(G_k)$ is the intensity (gain) of the mode k, $r \neq k$, η is the ratio of the cavity round trip and fluorescent lifetime, α, γ, β are proportional to the cavity loss, the gain, and the cross-saturation parameter, respectively, ϵ describes the conversion efficiency of the fundamental harmonic into the second harmonic.

With substitutions

$$\eta' = \left(\frac{\eta}{\alpha}\right)^{1/2}, \; g = \frac{\epsilon}{\alpha\bar{\eta}}, \; q = \frac{\gamma}{\alpha} - 1,$$
$$z_k = \frac{G_k/\alpha - 1}{\eta' q}, \; u_k = \frac{I_k}{q}, \; t' = \frac{t}{\eta'} \qquad (2)$$

we rewrite Eq.(1) in the normalized form

$$\dot{u}_k = q u_k \left[z_k - g u_k - 2g \sum_{r=1}^{N} u_r \right] + \mu,$$
$$\dot{z}_k = 1 - u_k - \beta \sum_{r=1}^{N} u_r - \eta' f(z_k, u_k), \qquad (3)$$

where

$$f(z_k, u_k) = z_k \left(1 + q u_k + \beta \sum_{r=1}^{N} q u_r \right),$$

and we add μ to model the external lighting, in particular, spontaneous emission. Let us specify the values of parameters. From the experimental work [Bracikovski and Roy, 1991] we take the ratio of the cavity round trip and fluorescent lifetime $\eta = 8 \times 10^{-7}$, the cavity loss $\alpha = 10^{-2}$, the gain $\gamma \sim 0.08$, the cross-saturation parameter $\beta \sim 0.6$, and the conversion efficiency of the fundamental intensity into doubled intensity $\epsilon = 5 \times 10^{-5}$. That gives normalized parameters $\eta' \sim 10^{-3} \ll 1, \; q \sim 10 \gg 1, \; g \sim 0.5, \; \beta < 1$. The physical situation, therefore, implies a studying of Eqs.(3) in the limit $\eta' g^{-1} \ll (gq)^{-1} \ll 1$, i.e. it is reasonable to consider the case of sufficiently high conversion g and large pumping rate q while η' is a very small parameter. Under such parameters antiphase relaxation oscillations are observed.

Numerical simulations show that the dynamics of system (3) with $\eta' \ll 1$ is very similar to the dynamics with $\eta' = 0$. Contrary to that the relaxation solutions with small $\mu \ll 1$ (i.e. with external lighting) and $\mu = 0$ appear to be quite different. Thus we will study the system (3) in two cases

$$q \gg 1, \; \eta' = 0, \; \mu = 0 \text{ or } \mu \ll 1. \qquad (4)$$

In order to investigate relaxation oscillations we apply here the special asymptotic method developed in our previous works [Grigorieva and Kashchenko, 1993]. Following this method we fix the Poincaré section and choose the set of initial conditions as the vector $\xi \in S$, $S \subset R^{N-1}$. Then we construct uniform asymptotic approximations of solutions and show that after a certain time the solution again falls within S. The Poincare operator of the shifting along the trajectories which maps ξ from S onto $\bar{\xi}$ that is also from S, is thereby analytically defined. To a fixed point of the operator there corresponds a periodic solution of Eqs.(3) of the same stability. Here we restrict ourselves to a study of only the main terms of the asymptotic expansion.

2.1 Two coupled modes

Dynamics of two coupled modes is governed by the system (3) where $N = 2$. We consider below inhomogeneous solutions for which during the pulse of one mode $t \in (t_i, t_{i+1})$ with $u_k(t) > 1$, another mode (modes) is suppressed, $u_{j \neq k}(t) \ll 1$. It is convenient, therefore, to choose the Poincaré section so that $u_k(t_i) = 1, \; \dot{u}_k(t_i) > 0$. The notations are shown in Fig.1. The particular form of oscillations reproduces well the experimentally observed form of pulses [Bracikovski and Roy, 1991].

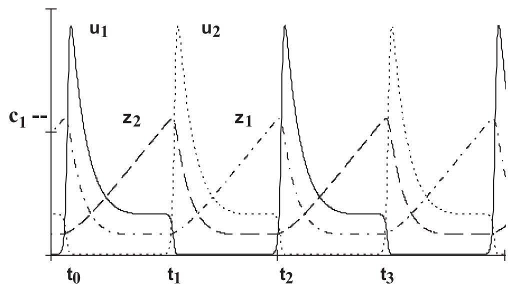

Figure 1. Two-modes antiphase relaxation solution $u_1(t), u_2(t)$ to Eqs.(3) for $v = 10^4, g = 0.5, \beta = 0.1$ and $\mu = 0$. Temporal moments at which $u_1(t_0) = u_2(t_1) = u_1(t_2) = ... = 1$ correspond to the moments where $z_1(t_0) = c_1, z_2(t_1) = \bar{c}_1, ...$.

Let us fix the initial moment $t_0 = 0$ at the Poincare section $u_1(0) = 1, \; \dot{u}_1 > 0$, corresponding to the onset of the pulse of the first mode and determine the initial conditions $\xi = (c_1, c_2, m_2)$ as follows:

$$c_1 = z_1(0), \; c_2 = z_2(0), \; m_2 = u_2(0), \qquad (5)$$

We integrate Eqs.(3) on the interval $t \in (0, t_1)$ taking into account conditions $q \gg 1$ and $u_1(t) > 1, \; u_2(t) \ll 1$. At the moment $t = t_1$ one gets $u_2(t_1) = 1, \; \dot{u}_2(t_1) > 0$, and $u_1(t_1) = O(1), \; \dot{u}_1(t_1) < 0$. Hence, the initial situation (5) appears again with replacing $u_1 \leftrightarrow u_2, \; z_1 \leftrightarrow z_2$ and

$$\bar{c}_1 = z_2(t_1), \; \bar{c}_2 = z_1(t_1), \; \bar{m}_2 = u_1(t_1), \qquad (6)$$

where $t_1 = t_1(m_2, c_1, c_2)$ is the root of the transcendental equation. Details of the derivation of the map is given in [Grigorieva, 2004].

Note, the map (6) is valid if for any iteration the conditions $c_1 > g(1+2m_2),\ c_2 < g(m_2+2)$ are fulfilled. The fixed point of this map corresponds to the limit cycle – antiphase periodic solution to the original system (3). Without loss of generality three-dimensional nonlinear map (6) can be reduced to the one-dimensional map

$$\bar{c}_1 = g(1+\beta) - \beta c_1 + (1-\beta)T, \tag{7}$$

where $T = T(c_1) = t_1$ means the duration of the pulse.

In the case of $\mu = 0$ the pulse duration $T_0(c_1)$ can be found as the positive root of the equation

$$T_0^2 + 2T_0\frac{\beta(g-c_1)-g}{1-\beta} - 2g\frac{(2-\beta)(c_1-g)}{1-\beta} = 0 \tag{8}$$

In presence of small external lighting $\mu \neq 0$ the pulse width essentially decreases. However the width weakly depends on μ if μ is small but not exponentially small, $\exp(-qA) \ll \mu \ll 1$. Then $T_\mu(c_1)$ can be found as the positive root of the equation

$$T_\mu^2(1-\beta)/2 - T_\mu[g - \beta(c_1-g)] = 0. \tag{9}$$

From Eq.(8) and Eq.(9) we conclude that $T_0 < T_\mu$. It follows from the fact that the minimal intensity of the suppressed mode $u_2(t) \sim \exp(-qA)$ is exponentially small in the system without external lighting, $\mu = 0$, while in the system with external lighting $0 < \mu \ll 1)$ the minimal value is of the order $u_2(t) \sim \mu/q$.

Both maps are shown in Fig.2. For the system without external lighting, $\mu = 0$, map (7),(8) has the fixed point c_1^* and the corresponding pulse width is

$$T_0^* = 2g\left(1 + \frac{2}{(1-\beta)^2}\right). \tag{10}$$

In the case $0 < \mu \ll 1$ the fixed point of the map (7),(9) reads as $c_{1\mu}^* = g(3-\beta)/(1-\beta)$ and the corresponding pulse width is

$$T_\mu^* = 2g\frac{1+\beta}{(1-\beta)^2}. \tag{11}$$

The pulse amplitude as well as full description of relaxation pulse form are given in [Grigorieva, 2004]

Linear stability analysis of the fixed point shows that such limit cycles are stable for any $\beta < 1$ because its Floquet multiplier is $\lambda_0 = \frac{2}{5} + O(\beta)$ in the case of $\mu = 0$ or $\lambda_\mu = \beta + O(\beta^2)$ in the case of $\mu \ll 1$.

Note, the condition $q \gg 1$ for relaxation oscillations is not very strict. In instance, analytically given by

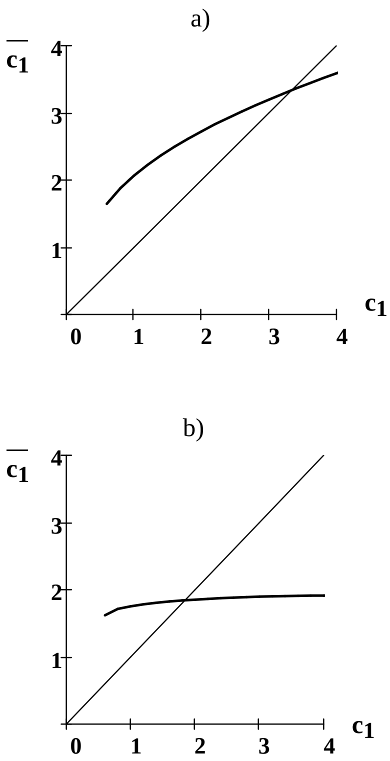

Figure 2. (a) Map (7),(8) for $\mu = 0$ and (b) map (7),(9) for $\mu = 0.01$ under $g = 0.5, \beta = 0.1$.

Eqs.(11) values $c_1^* = 3.338,\ T^* = 3.47$ for the parameters $q = 7.5,\ g = 0.5,\ \beta = 0.1$ approximate well numerical ones, $c_1^* = 3.35,\ T^* = 3.71$. It is satisfactory even in the case of $q \sim 3$.

2.2 Three coupled modes

Let us consider the dynamics of three coupled modes governed by Eqs.(3) with $N = 3$. The initial conditions read

$$\begin{aligned} u_1(0) &= 1, & z_1(0) &= c_1, \\ u_2(0) &= e^{qm_2}, & z_2(0) &= c_2, \\ u_3(0) &= m_3, & z_3(0) &= c_3, \end{aligned} \tag{12}$$

where $m_2 < 0,\ m_3 > 1,\ c_1 > c_2 > c_3$ are arbitrary values from the region providing the onset of the first mode pulse.

We integrate Eqs.(3) on the interval $t \in (0, t_1)$ where $u_1(t) > 1,\ u_2(t) \ll 1,\ u_3(t) \ll 1$. The moment $t_1 = min\{T_1, T_2\}$ corresponds to the conditions $u_2(T_1) = 1, u_3(T_2) = 1$. Obviously, two ways are then possible:

1. If $T_1 < T_2$ the intensity of the second mode becomes: $u_2(t_1) = 1,\ \dot{u}_2(t_1) > 0$, of the first one: $u_1(t_1) = O(1),\ \dot{u}_1(t_1) < 0$, and of the third one: $u_3(t_1) = o(1)$. The initial situation appears again with (right) shift of mode indexes

$(1,2,3) \rightarrow (2,3,1)$ and replacing

$$\bar{m}_2 = f(t_1, c_3),\ \bar{m}_3 = u_1(t_1),$$
$$\bar{c}_1 = z_2(t_1),\ \bar{c}_2 = z_3(t_1),\ \bar{c}_3 = z_1(t_1). \quad (13)$$

2. If $T_1 > T_2$ the intensity of third mode becomes: $u_3(t_2) = 1,\ \dot{u}_3(t_2) > 0$, of the first one: $u_1(t_2) = O(1),\ \dot{u}_1(t_2) < 0$, and of the second one: $u_2(\tilde{t}_2) = o(1)$. The initial situation appears again with (left) shift of mode indexes $(1,2,3) \rightarrow (3,1,2)$ and replacing

$$\bar{m}_2 = m_2 + f(t_1, c_2),\ \bar{m}_3 = u_1(t_1),$$
$$\bar{c}_1 = z_3(t_1),\ \bar{c}_2 = z_2(t_1),\ \bar{c}_3 = z_1(t_1). \quad (14)$$

Thus the condition $T_1 = T_2$ divides the phase space $(c_1, c_2, c_3, m_2, m_3)$ into regions with different dynamical rules.

Without loss of generality Eqs.(13),(14) can be reduced to the three-dimensional map

$$\begin{pmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{m}_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} f_1(T_1) \\ f_2(T_1) \\ f_3(T_1) \end{pmatrix}, & T_1 < T_2 \\ \begin{pmatrix} f_2(T_2) \\ f_1(T_2) \\ -f_3(T_2) \end{pmatrix}, & T_2 < T_1 \end{cases} \quad (15)$$

where $f_1(T_i) = c_2 + (1-\beta)T_i - \beta(c_1 - g),\ f_2(T_i) = g + (1-\beta)T_i - \beta(c_1 - g),\ f_3(T_i) = (g - c_2)T_i - m_2$ and values of the functions $T_{1,2}(c_1, c_2, m_2)$ can be found as the positive roots of quadratic equations

$$d_2 T_1^2 + T_1(c_2 - c_3 + d_1) + (d_0 + m_2) = 0,$$
$$d_2 T_2^2 + d_1 T_2 + d_0 = 0,$$

with $d_0 = g(\beta - 2)(c_1 - g),\ d_1 = c_3 + \beta(g - c_1) - 2g,\ d_2 = (1-\beta)/2$.

If $T_1 < T_2$ for any iteration then the map has the fixed point (c_1^*, c_2^*, m_2^*). This scenario corresponds to the pulses of the duration

$$T_1^* = g[2 + 3(1-\beta)^{-2}] \quad (16)$$

for modes in following sequence:

$$S^{1,2,3} : 1, 2, 3, 1, 2, 3, ... \quad (17)$$

i.e. one of the splay states is realized. This fixed point is unstable, hence, a few iterations later the phase trajectory crosses the surface $T_1 = T_2$. The second scenario in the map given by Eqs.(15) provides phase trajectory returns into the first region as shown in Fig.3. In such a way different pulse sequences can be created. In particular, if conditions $T_1 < T_2$ and $T_1 > T_2$ strictly alternate then pulsing of modes is observed in the sequence

$$C^1_{2,3} : 1, 2, 1, 3, 1, 2, 1, 3, ... \quad (18)$$

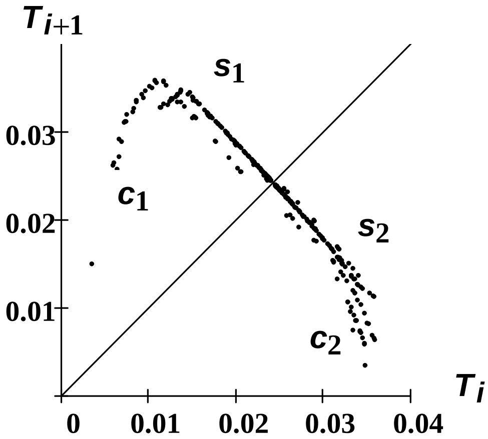

Figure 3. Attractor of the map (15) presented as $T_{i+1}(T_i)$ for $g = 0.5,\ \beta = 0.1$. The part $s_1 s_2$ corresponds to the splay state $S^{1,2,3}$, the parts marked by c_1, c_2 – to the cluster state $C^1_{2,3}$.

Such a regime can be interpreted as a cluster state because modes are divided in two groups: pulses of mode 1 oscillate with twice frequency and smaller maximal intensity than pulses of modes $2, 3$. This state is also unstable, hence, few iterations later the system falls again into the vicinity of the fixed point (16). Doing so the switching to another splay state $S^{1,3,2}$ can occur due to possible permutation of mode indexes. One can see, therefore, a typical picture of intermittency as shown in Fig.4. There can also exist various alternative states due to permutation of mode indexes.

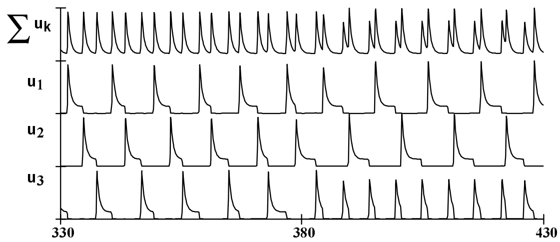

Figure 4. Results of the numerical integration of Eqs.(3) for $N = 3,\ \eta' = 10^{-2},\ q = 9.5,\ g = 0.5,\ \beta = 0.1$. Switching between splay states $S^{1,2,3}$ occurs through cluster state $C^1_{2,3}$.

Thus the obtained map predicts dynamical instability in the form of intermittency between the splay states divided by grouping states in the case of $\mu = 0$.

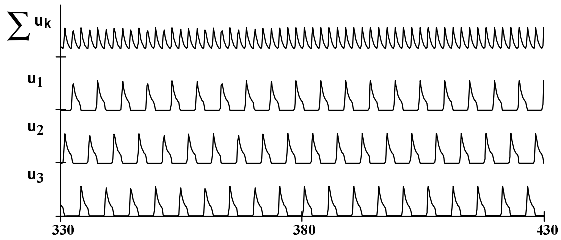

Figure 5. Results of the numerical integration of Eqs.(3) for $N = 3,\ \eta' = 10^{-2},\ q = 9.5,\ g = 0.5,\ \beta = 0.1.$ Stable splay state occurs due to external lighting $\mu = 0.01$.

Let us consider the effect of small external lighting $\mu \neq 0$. In this case the moment t_1 of the mode jump is determined by the equation $t_1 = min\{T_{\mu 1}, T_{\mu 2}\}$ where

$$T_{\mu i} = \frac{\beta(c_1 - g) - c_i}{1 - \beta}.$$

As far as $c_2 > c_3$ have been chosen then the inequality $T_1 < T_2$ is valid for any iteration. Hence the only scenario leading to the splay state is possible. The numerical example is shown in Fig.5. Thus small external lighting can stabilize the phase-synchronous behavior. This conclusion does not change if we apply noisy term with nonzero mean value instead of constant μ to model spontaneous emission. Namely, the sequence of mode pulsing is ordered while pulse amplitudes can slightly vary.

The obtained results can be expanded to the case of arbitrary quantity of modes. As N increases and $\mu = 0$ the number increases of subspaces given by conditions $T_1 = T_2,\ T_1 = T_3,\ ...\ T_1 = T_{N-1}$ where $T_k,\ k = 2, ...N-1$, are roots of the equations

$$T_k^2(1-\beta)/2 + T_k[c_{k+1} + \beta(g - c_1) - 2g] + \\ +g(\beta - 2)(c_1 - g) + m_{k+1} = 0.$$

That explains switching between various sequences of modes leading to complex temporal structures. The external lighting $\mu \neq 0$ or spontaneous emission of the sufficient level can stabilize the splay states because it is possible to choose the initial conditions resulting in $T_1 < T_2 < T_3 < ... < T_{N-1}$ for any iteration of the corresponding map.

3 Multi-mode semiconductor laser with periodically driven pumping rate

Dynamics of multi-mode semiconductor laser with periodically driven pumping rate in the case of homogeneous linewidth can be described by following equations [Otsuka, 1991; Otsuka and Sato, 1996]:

$$\begin{aligned} \dot{u}_k &= v u_k \left[n_0 - \frac{n_k}{2} - 1 \right], \\ \dot{n}_k &= n_0 u_k - n_k \left(1 + \beta \sum_{r=1}^{N} u_r \right), \\ \dot{n}_0 &= q - n_0 - \sum_{r=1}^{N} u_r (n_0 - \frac{n_r}{2}), \end{aligned} \tag{19}$$

where u_k is the intensity of k-mode, $k = 1, 2, ..., N$, $r \neq k$, n_0 is the constant term of the spatial Fourier expansion of the population inversion, n_k are first order components of the Fourier expansion of the population inversion, η is the ratio of photon damping time in the cavity to the relaxation time of inversion of population created by the pumping rate $q = q_0 + q_1 \cos(\omega t + \phi)$ with q_0 is the constant part of the pump normalized to the pumping rate at the first threshold of generation, q_1 is the modulation amplitude, and ω is the modulation frequency. The models incorporate cross saturation of population inversion among modes due to spatial hole-burning effect providing global mode coupling. Numerical simulations show that far away the threshold, splay states are interrupted by intervals where modes oscillate still out of phase but with different waveforms and periods.

For class-B lasers including semiconductor lasers, some solid-state lasers and CO_2 lasers the parameter $v \sim 10^3 \gg 1$ that provides regimes of spiking type under deep modulation of the pumping rate, $q_1 \sim q_0$. The duration of spikes $\Delta \to 0$ under $v \to \infty$. Note, in the previous case of high pumping rate, $q \gg 1$, the duration of pulses was comparable with the period of oscillations, $\Delta \sim T/N$.

Let us choose the initial conditions for system (19) in such a way that a pulse of the mode numbered 1 starts at $t = 0$ while the intensities of other modes are exponentially small:

$$u_i(0) = e^{vd_i},\ n_i(0) = c_i,\ \Phi(0) = \varphi, \tag{20}$$

where $c_1 < c_0 - 1 < c_2 < c_3 < ... < c_N,\ \ d_N < ... < d_2 < d_1 = 0$. Integrating Eqs.(19) by the asymptotical method given in [Grigorieva and Kashchenko, 1993], one can get at the moment $t = T(c_0, c_2, d_2, \varphi)$ the system finds itself into the state analogous to the state (20) with replacing of mode indexes $(1, 2, ..., N-1, N)$ to

$(2, 3, ..., N, 1)$ and parameters c, c_i, d_i, φ to

$$\begin{aligned}
\bar{c}_0 &= q_0 + (c - p - q_0 + K\cos\psi)e^{-T} \\
&\quad -K\cos(\omega T + \psi), \\
\bar{\varphi} &= \varphi + \omega T, \quad \mod 2\pi, \\
\bar{c}_k &= c_{k+1}e^{-p-T}, \\
\bar{d}_k &= d_{k+1} - d_k + \frac{(c_2 - c_{k+1})}{2}e^{-p-T}, \\
\bar{c}_N &= [c_1(1-p) + c_0 p]e^{-T}, \\
\bar{d}_N &= -d_{N-1} + \frac{(c_2 - c_1)}{2}e^{-p-T},
\end{aligned} \tag{21}$$

where $k = 1, ..., N-1$, $p = p(c_0, c_1)$ is the pulse energy and $T = T(c_0, c_2, d_2, \varphi)$ means the interval between pulses determined as the first positive root of the transcendental equations.

Attractors of $2N$-dimensional map (21) determine dynamical regimes of system (19). The fixed point of the map corresponds to spiking oscillations with a period $T = 2\pi/\omega$ which exists if the inequality $d_2 < (1 - e^{-T})[(c_2 - c_1)(1-p) - c_0]/2$ is valid for any iteration of the map. Such a point corresponds to the splay-state.

Numerical simulations show that basins of such attractors are relatively narrow. They coexists with chaotic attractors. The initial conditions leading to synchronous splay states can be found using the fixed point of the map (21). Also, small external lighting increases the basin of the regular splay states.

4 Conclusion

We have derived the finite-dimensional maps which describe adequately antiphase dynamics of multimode lasers far away the oscillatory threshold. Fixed points of these maps determine spiking oscillations of various forms and of different type of synchronization corresponding to splay states and clustered states. The obtained analytical results can be further applied to the problems of control switching among periodic and cluster states by special injecting signal as it has been done for modulated lasers by [Otsuka and Sato, 1996].

For $N \geq 3$ the obtained maps predict the phase space to be divided into the regions with different dynamical rules. They describe analytically both laminar phase near fixed points and bursting. As far as unstable phase trajectory reaches one of the separating surfaces the switching occurs to another solution. In this way the discrete maps determine alternation between ruins of periodic attractors. Such a scenario of instability is not similar to the attractor crowding [Wiesenfeld and Hadley, 1989]. The last one implies noise-induced switching between $(N-1)!$ coexisting splay states because their attractive basins shrink rapidly as the system size N increases. Contrary to that, our approach elucidates dynamical nature of switching between antiphase solutions of different types. The inclusion of noisy term modelling relatively high spontaneous emission leads to stabilizing of periodic states.

References

Strogatz, Mirollo, S. R. (1993) Splay states in globally coupled Josephson arrays: Analytical prediction of Floquet multipliers. *Phys.Rev.E.* **47**, pp. 220–227.

Bracikovski, C., Roy, R. (1991) Chaos in a multimode solid-state laser system. *Chaos*, **1**, pp. 49–64.

Wiesenfeld, K., Bracikowski, C., James, G., Roy,R. (1990) Observation of antiphase states in a multimode laser. *Phys.Rev.Lett.*, **65** , pp. 1749-1752 .

Bielawski, S., Derozier, D., Glorieux, P. (1992) Antiphase dynamics and polarization effects in the Nd-dopped fiber laser. *Phys.Rev.A.*, **46**, pp. 2811–2822 .

Otsuka, K. (1991) Winner-takes-all dynamics and antiphase states in modulated multimode lasers. *Phys.Rev.Lett.*, **67** , pp. 1090–1093.

Silber, M., Fabiny, L., and Wiesenfeld, K. (1993) Stability results for in-phase and splay-phase states of solid-state laser arrays. *JOSA B*, **10**, pp. 1121–1129.

Otsuka, K., Sato, Y.(1996) Grouping of antiphase oscillations in modulated multimode lasers. *Phys.Rev.A.* **54**, pp. 4464–4472.

Wiesenfeld, K., Hadley, P.(1989) Attractor crowding in oscillator arrays. *Phys.Rev.Lett.*, **62**, pp. 1335-1338.

Zimmler, R., Livi, R., Politi, A., and Torcini, A. (2007) Stability of the splay states in pulse coupled networks. *Phys.Rev.E* , **76**, p. 046102.

Calamai, M., Politi, A., and Torcini, A. (2009) Stability of splay states in globally coupled rotators. *Phys.Rev.E* , **80**, p. 036209. tors

Grigorieva, E.V., and Kashchenko, S.A. Regular and chaotic pulsations in laser diode with delayed feedback. (1993) *Bifur. & Chaos*, **3**, pp. 1515–1528 .

Grigorieva, E.V. (2002) Phase synchronization of spiking oscillations in multimode solid state lasers. *NPCS*, **7**, pp. 97–105.