ADAPTIVE MODEL TRACKING WITH MITIGATED PASSIVITY CONDITIONS

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Abstract: Feasibility of nonlinear and adaptive control methodologies in multivariable linear time-invariant systems with state space realization \( \{ A, B, C \} \) has apparently been limited by the standard strict passivity (or positive realness) conditions that imply that the product \( CB \) must be positive definite symmetric. A recent paper has managed to mitigate the symmetry condition, requiring instead that the positive definite and not necessarily symmetric matrix \( CB \) be diagonalizable. Although the mitigated conditions were useful in proving pure stabilizability with Adaptive Controllers, the Model Tracking question has remained open. This paper further extends the previous results, showing that the new passivity conditions can be used to guarantee stability of the adaptive control system and asymptotically perfect model tracking. Copyright © 2007 IFAC.

Keywords: Control systems, stability, passivity, uncertain systems, almost strict passivity (ASP), adaptive control

1. INTRODUCTION

Consider the square system

\[
\dot{x}(t) = Ax(t) + Bu(t) \quad (1)
\]
\[
y(t) = Cx(t) \quad (2)
\]

Here, \( x \) is the \( n \)-dimensional state vector, \( u \) is the \( m \)-dimensional input vector and \( y \) is the \( m \)-dimensional output vector, and \( A, B, \) and \( C \) are matrices of corresponding dimensions. Because in various methodologies of nonstationary control the stability analysis concerns both the state and the dynamical gains, stability of the control system has been treated with positive definite quadratic Lyapunov functions of the form

\[
V(t) = x^T(t)Px(t) + \text{tr}[(K(t) - \tilde{K})\Gamma^{-1}(K(t) - \tilde{K})^T]. \quad (3)
\]

Here, \( K(t) \) is the adaptive gain used with the controller \( u(t) = K(t)y(t) \) and \( \tilde{K} \) represents an ideal output feedback gain. Define

\[
A_K = A - B\tilde{K}C \quad (4)
\]

Although the proofs of stability using (3) do not require the original system to be strictly positive real (SPR), they require the existence of a constant output feedback gain \( \tilde{K} \) (unknown and not needed for implementation) that could make the fictitious closed-loop system \( \{ A_K, B, C \} \) is SPR. The common state-space definition of the strictly positive-realness property in linear time invariant systems is:

**Definition 1.** A linear time-invariant system with a state-space realization \( \{ A_K, B, C \} \), where \( A_K \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n} \), with the \( m \times m \) transfer function \( T(s) = C(sI - A_K)^{-1}B \), is called ‘strictly passive (SP)’ and its transfer function ‘strictly positive real (SPR)’ if there exist two positive definite symmetric (PDS) matrices, \( P \) and \( Q \), such that the following two relations are simultaneously satisfied:
\[ PA_K + A_K^T P = -Q \]  \hspace{1cm} (5)

\[ PB = C^T. \]  \hspace{1cm} (6)

Relations (5)-(6) have been shown to be very useful in nonlinear control applications and in particular in control with uncertainty or in adaptive control (Fradkov, 1976), (Sobel et al., 1982), (Steinberg and Corless, 1985), (Barkana and Kaufman, 1985), (Zeheb, 1986), (Fradkov and Hill, 1998). The original system that only needs a constant output feedback to become strictly positive real has been called ‘almost strictly positive real’ (ASPR) (Barkana and Kaufman, 1985), (Barkana, 1987), also called ‘feedback passive’ or ‘passifiable’. For quite a long time, the meaning and practical implications of ASPR systems has remained rather obscure within the adaptive control community, although even as early as 1976 has been shown (Fradkov, 1976) that any minimum-phase system with a positive definite symmetric matrix product \( CB \) can be rendered SPR via constant output feedback, and many other works have re-invented and further developed the idea since (see Barkana, 2004a) and the references therein for a brief history and direct proof of this important statement. The importance of this specific class of systems has gradually gained more and more acceptance in the control community (Kokotovic and Arcak, 2001). Moreover, the class of ASPR systems proved to be more general that initially thought when Huang et al. (Huang et al., 1999) showed that if a system cannot be made SPR via constant output feedback, no dynamic feedback can render it SPR.

However, from relation (6) and its transpose one gets

\[ B^T PB = B^T C^T = CB > 0. \]  \hspace{1cm} (7)

The non-singularity of \( CB \) implies that the transfer function \( T(s) \) has \( n \) poles and \( n - m \) zeros, yet (7) also implies that the customary SPR relations can be applied only to systems where the product \( CB \) is positive definite symmetric (PDS).

While the implied positivity of \( CB \) could be expected and understood as a natural extension of the sign condition in SISO systems, the symmetry condition seemed to limit the applicability of adaptive control techniques, as its satisfaction in uncertain systems may be difficult to guarantee, in general. Although both requirements seemed to be needed for the proof of stability with adaptive controllers, a recent publication (Barkana, Teixeira and Hsu, 2006) showed that the symmetry condition can be mitigated and that the positive definite matrix \( CB \) must only be diagonalizable.

2. MITIGATION OF THE SYMMETRY CONDITION

While investigating ways that would possibly mitigate the symmetry assumption on the product \( CB \) and thus extend the feasibility of the SPR concept to larger classes of systems, it was intuitive (Barkana, Teixeira and Hsu, 2006) to try the new Lyapunov function

\[ V(t) = x^T(t)Px(t) + tr[S(K(t) - \bar{K})\Gamma^{-1}(K(t) - \bar{K})^TS^T]. \]  \hspace{1cm} (8)

Note that any nonsingular matrix factor \( S \) (unknown and not needed for implementation) would allow the matrical product in the second term in (8) to be positive definite symmetric and thus the trace to be positive definite. Borrowing a definition that has been introduced by Fradkov and his colleagues (Fradkov, 2003), (Peaucelle, Fradkov and Andrievski, 2005), the applicability of passivity conditions was extended (Barkana, Teixeira and Hsu, 2006) via the following definition:

**Definition 2.** Under the assumption of Definition 1, the state-space realization \( \{ A_K, B, C \} \) is called W-Strictly Passive (WSP) and its transfer function \( T(s) = C(sI - A_K)^{-1}B \) is called W-Strictly Positive Real (WSPR) if there exist three positive definite symmetric matrices, \( P, Q, \) and \( W = S^TS \) such that the following two relations are simultaneously satisfied:

\[ PA_K + A_K^T P = -Q \]  \hspace{1cm} (9)

\[ PB = C^TW \]  \hspace{1cm} (10)

It is important to note that the symmetry condition for \( W = S^TS \) has initially originated in the requirement that \( S \) in the Lyapunov function (8) be nonsingular and the second term in (8) be positive definite.

Furthermore, it was shown (Barkana, Teixeira and Hsu, 2006) that a system can become WSP via constant output feedback if it is minimum-phase and if the positive definite and not necessarily symmetric product \( CB \) is diagonalizable. Finally, it was also shown that the WSP conditions (9)-(10) are sufficient conditions that can guarantee stability with adaptive output feedback controllers (Barkana, Teixeira and Hsu, 2006).

The development in (Barkana, Teixeira and Hsu, 2006) had thus finally ended with a straightforward result that managed to mitigate a symmetry condition that had been around for more than 40 years.

Note: Although it mitigates a long established symmetry condition, Definition 2 still excludes those systems where the positive definite \( CB \)}
The output given by (fictitious) system associated WSP relations are equivalent to requiring that the output of a ‘model’. Still, it is interesting to mention that these control the output to render the fictitious closed-loop system WSP. As most systems are not WSP, we called WASP out requiring the customary applications of the useful passivity properties with-

We will show that this simple result allows the application of the new definition, we again emphasize here the necessity because, ultimately, the Lyapunov derivative does require W to be Positive Definite Symmetric.

To avoid any eventually misleading interpretation of the new definition, we again emphasize here that the (fictitious) matrix W is not necessarily symmetric. This could apparently lead to further relaxation of the passivity conditions, because one can show that the existence of such a W that is PD and not necessarily symmetric is then guaranteed if the product CB has just all eigenvalues in the right half-plane, even if CB is neither symmetric nor Positive Definite. However, as we show in this paper, although some examples may show stability, at least at this stage this attempt finally proved to be an exercise in futility because, ultimately, the Lyapunov derivative does require W to be Positive Definite Symmetric.

In other words, given the system (1)-(2), if one can assume that the (possibly unstable) plant is minimum-phase and that all eigenvalues of CB are located in the right half-plane, the fictitious control

\[ u(t) = \tilde{K}_c y(t) \]  

would result in the closed-loop system

\[ \dot{x}(t) = [A - B\tilde{K}_c C]x(t) + Bv(t) \]  

\[ y(t) = Cx(t) \]

that satisfies the WSP relations

\[ P[A - B\tilde{K}_c C] + [A - B\tilde{K}_c C]^TP = -Q \]  

\[ PB = C^TW^T \]

The new conditions and definitions are important only if one can show that they are as useful as the customary SPR conditions for the proofs of stability with adaptive controllers. Here we note that with the publication of (Barkana, Teixeira and Hsu, 2006), some colleagues have expressed their concern that the uncertainty in W may actually eliminate the usefulness of the WASP relations in the tracking case. Therefore, instead of the simple adaptive stabilizing illustration of (Barkana, Teixeira and Hsu, 2006), in this paper we decided to present a full adaptive model tracking case that uses the Simple Adaptive Control (SAC) methodology (Kaufman, Barkana and Sobel, 1998), (Barkana, 2007). Specifically, we assume that the plant output is required to track the output of a ‘model’.

### 3.1 Model following with SAC

In SAC methodology, the so-called ‘model’ is simply a stable plant that only serves to generate the trajectory that the plant should follow, and for this reason it is also called ‘command generator’ and the methodology is sometimes called ‘command generator tracking’. Otherwise, the ‘model’ is not required to reproduce the plant or to use any prior knowledge about the plant and can also be of any (lower or larger) order insofar as it generates the desired trajectory. As usual in adaptive control, one first assumes that the underlying fully deterministic output model tracking problem is solvable. A recent publication (Barkana, 2005) shows that if the Model Reference uses a step input in order to generate the desired trajectory, the underlying tracking problem is always solvable. If, instead, the model input command is itself generated by an unknown system of order \( n_a \), the model is required to be sufficiently large.
to accommodate this command (Barkana, 1983),
(Kaufman, Barkana and Sobel, 1998), or
\[ n_m + m \geq n_u \] (17)
The model is:
\[
\hat{x}_m(t) = A_m x_m(t) + B_m u_m(t) \\
y(t) = C_m x_m(t)
\] (18)
Here, \( x_m \) is the \( n_m \)-dimensional state vector, \( u_m \) is the \( m \)-dimensional input vector and \( y_m \) is the \( m \)-dimensional output vector, and \( A_m, B_m, \) and \( C_m \) are matrices of corresponding dimensions.
The simple adaptive control (SAC) algorithm (Sobel et al., 1982), (Barkana and Kaufman, 1985) monitors the tracking error
\[ e_y(t) = y_m(t) - y(t) \] (20)
and the available model variables, \( x_m \) and \( u_m \), and uses the following reference vector
\[ r^T(t) = [e_y(t), x_m(t), u_m(t)]^T \] (21)
to generate the adaptive control gains
\[ K(t) = [K_e(t), K_x(t), K_u(t)] \] (22)
through the procedure
\[ \dot{K}(t) = e_y(t)r^T(t)\Gamma \] (23)
and the adaptive control signal
\[
\hat{u}(t) = \dot{K}(t) \hat{r}(t) = K_e(t)e_y(t) + K_x(t)\hat{x}_m(t) + K_u(t)u_m(t).
\] (24)
Here, \( \Gamma \) is a positive definite scaling matrix that regulates the rate of adaptation. The underlying deterministic tracking problem assumes that there exists an ideal control
\[ \hat{u}^*(t) = \hat{K}_x x_m(t) + \hat{K}_u u_m(t). \] (25)
that could keep the plant along an ideal trajectory \( \hat{x}^*(t) \) that would asymptotically perform perfect tracking. In other words, the ideal plant
\[
\hat{x}^*(t) = A\hat{x}^*(t) + B\hat{u}^*(t) \\
\hat{y}^*(t) = C\hat{x}^*(t)
\] (26)
(27)
moves along "ideal trajectories" such that
\[ \hat{y}^*(t) = y_m(t) \] (28)
A recent work (Barkana, 2005) has given a thorough treatment to the existence of the underlying ideal control gains. It was shown that such ideal control gains always exist under the minimum-phase assumption. Therefore, here we can assume that the underlying problem is solvable and thus, that some ideal gains \( \hat{K}_x \) and \( \hat{K}_u \) exist. Because the plant and the model can have different dimensions, the ‘following error’ \( e_x(t) \) is defined to be the difference between the ideal and the actual plant state
\[ e_x(t) = x^*(t) - x(t) \] (29)
and correspondingly
\[ e_y(t) = y_m(t) - y(t) = y^*(t) - y(t) = Ce_x(t) \] (30)
Differentiating (29) gives:
\[
\dot{e}_x(t) = \dot{x}^*(t) - \dot{x}(t) = Ax^*(t) + Bu^*(t) - Ax(t) - Bu(t) \]
(31)
\[
\dot{e}_y(t) = \dot{y}^*(t) - \dot{y}(t) = A\hat{y}^*(t) - B(\hat{u}(t) - u(t))
\] (32)
Adding and subtracting \( B\hat{K}_x e_y(t) \) above gives
\[ \dot{e}_x(t) = (A - B\hat{K}_x C)e_x(t) - B \begin{bmatrix} \hat{K}_x & -\hat{K}_u \end{bmatrix} r(t) \] (33)
where for convenience we denoted
\[ \hat{K} = [\hat{K}_e \hat{K}_x \hat{K}_u] \]
This long introduction allows the proof of the following theorem of stability:

**Theorem 1.** Under the WASP conditions and the assumptions of this subsection, all gains and state variables of the Adaptive Control system represented by (23) and (32) are bounded and the system performs asymptotically perfect tracking.

**PROOF.** The positive definite Lyapunov function (11) applied to the adaptive system (23) and (32) is
\[ V(t) = e_x^T(t) Pe_x(t) + tr[W(K(t) - \hat{K})\Gamma^{-1}(K(t) - \hat{K})^T]. \] (34)
At this stage, we do not require \( W \) to be symmetric. The derivative of \( V(t) \) is
\[ \dot{V}(t) = e_x^T(t) Pe_x(t) + e_x^T(t)\dot{P}e_x(t) \]
\[ + e_x^T(t)(A - B\hat{K}C)^T Pe_x(t) \\
- e_x^T(t)PB[K(t) - \hat{K}]r(t) \\
- e_x^T(t)\dot{r}(t)[K(t) - \hat{K}]^TB^T Pe_x(t) \\
+ tr[W\dot{e}_x(t)e_x^T(t)\Gamma^{-1}(K(t) - \hat{K})^T] \\
+ tr[W(K(t) - \hat{K})\Gamma^{-1}\Gamma r(t)e_y^T(t)] \] (35)
Recalling that $tr(AB) = tr(BA)$, $x^T y = y^T x$, and $tr(x^T y) = x^T y$ and using the WASP relations gives

$$
\dot{V}(t) = e_x^T(t)[P(A - BK) + (A - BK)^T P]e_x(t)
- e_x^T(t)C^T W^T[\dot{K}(t) - \dot{K}]r(t)
- r^T(t)[\dot{K}(t) - \dot{K}]W^T e_x(t)
+ e_x^T(t)C^T W[K(t) - K]r(t)
+ r^T(t)[K(t) - K]W^T e_x(t)]
$$

(37)

One of the last two terms in (37), originating in the derivative of the adaptive gain terms in $V(t)$, cancels a previous, possibly troubling, nonpositive term and thus lead to the Lyapunov derivative

$$
\dot{V}(t) = e_x^T(t)[P(A - BK) + (A - BK)^T P]e_x(t)
+ e_x^T(t)C^T W[K(t) - K]r(t)
$$

At this stage, one can see that the symmetry of $W$ is obviously needed to finally get

$$
\dot{V}(t) = -e_x^T(t)Qe_x(t).
$$

(39)

The Lyapunov derivative $\dot{V}(t)$ in (39) is thus negative definite with respect to $e_x(t)$, although only negative semidefinite with respect to the entire state-space $\{e_x(t), K(t)\}$. Although the direct result of Lyapunov stability theory is only that all dynamic values are bounded, according to LaSalle’s invariance principle (Kaufman, Barkana and Sobel, 1998), all state-variables and adaptive gains are bounded and the system ultimately ends within the domain defined by $\dot{V}(t) \equiv 0$. Because $\dot{V}(t)$ is negative definite in $e_x$, the system thus ends with $e_x(t) \equiv 0$, that in turn implies $e_y(t) \equiv 0$. In other words, the adaptive control system demonstrates asymptotic convergence of the state and output error and boundedness of the adaptive gains. Furthermore, it has been recently shown (Barkana, 2005) that the adaptive control gains ultimately reach a set of stabilizing constant values at the end of a steepest descent minimization of the tracking error. QED.

Finally, it was shown that, while its assumed existence indeed facilitates the proof of stability and asymptotically perfect tracking of the adaptive control system without requiring the symmetry of $CB$, the fictitious symmetric matrix $W$ and its assumed uncertainty play no role in implementation and have no (negative) effect whatsoever on the asymptotic tracking properties of the adaptive control system.

4. COUNTEREXAMPLES TO STANDARD MODEL REFERENCE ADAPTIVE CONTROL

For an illustration of the stability properties of Simple Adaptive Control, in this section we use some "counterexamples" that lead to divergence when standard gradient-based MRAC techniques are applied (Hsu and Costa, 1999). In these examples, a 2*2 stable plant with CB positive definite is required to follow the behavior of a stable model of same order. Both the plant and the model have diagonal system matrices and same negative eigenvalues, and only the input-output matrix differentiates between the two. The plant, a 2D adaptive robotic visual servoing with uncalibrated camera, is defined by the system matrices

$$
A = \begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix} \quad B = \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

(40)

The simple model is defined by the matrices

$$
A_m = \begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix} \quad B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

(41)

It is shown (Hsu and Costa, 1999) that the standard MRAC system becomes unstable with $a = \varphi = 1, h = 0.5$, and $u_m = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ or with $a = 9, \varphi = 1, h = 0.5$, and $u_m = \begin{bmatrix} 10 \sin \varphi \\ 10 \cos \varphi \end{bmatrix}$. The divergence is treated in detail in (Hsu and Costa, 1999) and we only mention it here because instability occurred in both cases although there was no "unmodeled dynamics," there was "sufficient excitation," and the required "sufficient" passivity conditions were also satisfied. It is worth mentioning that this was a theoretical analysis that resulted in a diverging equation. Here, we revisit the examples and attempt to use the simplicity of SAC, so we first use the same slow adaptation rate as (Hsu and Costa, 1999), $\gamma = 1$, with all adaptive gains. However, because the rate of adaptation is theoretically unlimited with SAC, we will later show that higher adaptation rates not only do not affect the stability but they also definitely result in superior performance.

1) First case: $h = 0.5$, $a = 1$, $\varphi = 1$, $u_m(t) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Note that CB is positive definite, yet not symmetric. One can see that the SAC system indeed shows a stable behavior. Because all initial adaptive gains are zero and the rate of adaptation is slow, one can see an expected large transient before all values converge and the tracking error vanishes.

2) A second case was run with two sinusoidal input commands: $a = 9$, $u_m(t) = \begin{bmatrix} 10 \sin \varphi \\ 10 \cos \varphi \end{bmatrix}$. SAC again
3) Third case: Here, we run the second case after we eliminate any input command in order to avoid any impression that the SAC might need input excitation: $a = 9$, $u_m(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note that if the plant starts at zero initial conditions, it would remain there. Therefore, we just gave it some initial conditions $\begin{bmatrix} 1 \\ 1.2 \end{bmatrix}$. With no input command, while all model states are zero, $K_x$ and $K_u$ remain zero, while $K_e$ reaches some stabilizing value. Here, one can see that the SAC system remains stable with no dependence on the existence of a model or input command.

4) For an illustration of SAC performance, we again run the second case, yet the step input becomes a square wave input. Here we use high adaptation coefficients $\gamma_e = 1e4$, $\gamma_x = 1e2$, $\gamma_u = 1e2$.

As seen below, the plant follows the model so closely that most of the time, except for an initial adaptation transient, the model and plant positions practically coincide.
CB of the (basically unknown) augmented system cannot be guaranteed, this paper facilitates the use of parallel feedforward by eliminating a fundamental limitation of the approach. In various design environments, one can use available prior knowledge to either devise a stabilizing controller first (Barkana, 1987), (Kaufman, Barkana and Sobel, 1998), or directly the ‘parallel feedforward configuration (PFC)’ or ‘shunt’ (Iwai and Mizumoto, 1992), (Fradkov, 1994), (Betser and Zeheb, 1995).

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