RANDOM-PHASE INITIAL FUNCTION AND THE ORGANIZATION AND IDENTIFICATION OF SOLUTIONS IN THE MACKEY-GLASS DELAYED MODEL

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Abstract

Time-delayed systems are crucial in many fields of Physics and Engineering. One paradigmatic example of time-delayed system is the well-known Mackey-Glass (MG) equation, which models physiological processes, mainly respiratory and hematopoietic (i.e. formation of blood cellular components) diseases. In this presentation, multistability in the long term dynamics of the MG delayed model is analyzed by using a recently proposed electronic circuit capable of controlling the initial conditions. New approaches for both the nonlinear function and the delay block of the circuit are made. In practice, in spite of using a finite set of capacitors, an excellent agreement between the experimental observations and the numerical simulations is manifested. In the continuous limit, the equations of the circuits exactly corresponds to the MG model. The dynamics of the system exhibits a remarkable richness and as the delay is increased different periodic or aperiodic solutions appear. The system's phase-space is explored by varying the parameter values of different families of initial functions. In particular we consider here families of random-phase sinusoidal functions. By means of a symbolic method aimed at classifying solutions, we confirm the existence of abundant periodic solutions that have the same period and the same quantity of maximums but a different alternation of peaks of dissimilar amplitudes.

Key words

time-delay, multistability, Mackey-Glass model, delay differential equations

1 Introduction

Chaotic systems are characterized by disorder and unpredictable behavior. The discovery of strange attractors [Ruelle and Takens, 1971], in the 1970s, revealed the existence of certain regularities in the phase space of chaotic systems. Nonlinear systems often display multistability, that is, the coexistence of different attractors for the same set of parameters. From observed noisy time-series that display similar oscillatory patterns, identifying and distinguishing different coexisting attractors is a challenging task, in particular when noise induces switching among different attractors. Recently, the phenomenon of extreme multistability has been predicted numerically in systems of coupled oscillators [Hens et al., 2012] and experimental observations in a system of two coupled Rossler-like oscillators were found to be consistent with the numerical predictions: by constructing an electronic circuit representing the system, Patel et al [Patel et al., 2014] demonstrated a controlled switching to different attractor states through a change in initial conditions only, keeping fixed the system's parameters. While in most multi-stable systems chaotic attractors are rare [Feudel and Grebogi, 1997], systems with time delays are an exception to this rule. A time-delay renders the phase space of a system infinite-dimensional, as one needs to specify, as initial condition, the value of a function, F_0, over the time interval (-τ,0), with τ being the delay time. This initial condition is referred to as the initial function (IF).

One paradigmatic example of delayed system is that proposed by Mackey and Glass [Mackey and Glass, 1977] dealing with physiological processes, mainly respiratory and hematopoietic (*i.e.* formation of blood cellular components) diseases. In effect, in the production of blood cells is a considerable delay between the initiation of cellular production in the bone narrow and the release into the blood. Generally, in these processes, the evolution of the system at a given time not only depends on the state of the system at the current time but also on the state of the system at *previous* times.

In their pioneering paper, Mackey and Glass (MG) showed that a variety of physiological systems can be adequately described in terms of simple nonlinear delay-differential equations. The model proposed by MG exhibits a wide range of behaviors including periodic and chaotic solutions. The importance of the MG model lies in the fact that the onset of some diseases are associated with alterations in the periodicity of certain physiological variables, for example, irregular breathing patterns or fluctuations in peripheral blood cell counts [De Menezes and Dos Santos, 2000].

The dynamics of processes involving time delays, as those studied by MG, is far more complex than that of non-delayed, *i.e.* instantaneous, systems. Actually, if the dynamics of a system at time t depends on the state of the system at a previous time $t - \tau$, the information needed to predict the evolution is contained in the entire interval $(t - \tau, t)$. Thus, the evolution of a delayed system depends on *infinite* previous values of the variables. Mathematically, delayed systems are modelled in terms of delayed differential equations (DDEs) and one single DDE is equivalent to infinite ordinary differential equations (ODEs). Due to their infinite dimensionality, the accuracy of numerical simulations of DDEs is specially delicate. In practice, this problem is avoided considering large transients. However, there persist doubts about the stability and accuracy of the methods used to numerically integrate DDEs.

Thank to its richness in behaviors, the Mackey-Glass model has acquired relevance of its own [Junges and Gallas, 2012; Sano et al., 2007; Wan and Wei, 2009]. One frequent application is to use the MG model as a simple way to generate a high-dimensional chaotic signal (see for example [Grassberger and Procaccia, 1983]) which can be helpful to characterize strange attractors using the Kolmogorov entropy for instance and, at the end, as a way to distinguish between deterministic chaos and random noise. Other application could be the employment of the output of MG model to check the effectiveness of a control or stabilization scheme [Namajūnas et al., 1995]. MG model was also proposed in the context of forecasting chaotic data [Farmer and Sidorowich, 1987], or nonlinear estimation problems [Wan and Van Der Merwe, 2000].

Recently, a novel electronic circuit has been proposed a highly precise implementation of the Mackey-Glass delay differential equation [Amil et al., 2015a]. Under this approach, the discrete equations governing the dynamics of the circuit are exact, in spite of the fact that in the electronic circuit the infinite phase space of the MG system is discretized via a finite set of values. Using this experimental setup [Amil et al., 2015b], the system's phase-space was explored by varying the parameter values of two families of initial functions. It is shown that the evolution equation of the electronic circuit, in the continuous limit, exactly corresponds to the MG model. In practice, when using a finite set of capacitors, an excellent agreement between the experimental observations and the numerical simulations is manifested. As the delay is increased different periodic or aperiodic solutions appear. Abundant periodic solutions that have the same period but a different alternation of peaks of dissimilar amplitudes are observed and classified. As the parameter space of the initial functions is infinite dimensional and it is not possible to explore it completely, it is natural to consider pseudo-random initial functions. Here, we extend the previous studies to the case in which the IFs presents a random phase. We found a remarkable richness of structures in the parameter space.

2 The exactly integrable discrete model

The original version of the Mackey-Glass delay-differential equation, is [Mackey and Glass, 1977]

$$\frac{dP}{dt} = \frac{\beta_0 \Theta^n P_\tau}{\Theta^n + P_\tau^n} - \gamma P \tag{1}$$

where P is the density of mature circulating white blood cells, τ is the delay time and $P_\tau = P(t - \tau)$. The parameters Θ and β_0 and the exponent n are related to the production of white blood cells while γ represents the decay rate.

The number of parameters can be reduced by re-scaling the variables $x = P/\theta$ and $t' = t\gamma$. After the re-scaling, a simplified equation for $x(t')$ for the MG model is obtained

$$\frac{dx}{dt'} = \alpha \frac{x_\Gamma}{1 + x_\Gamma^n} - x \tag{2}$$

where $\Gamma = \gamma\tau$ is the normalized delay time, $\alpha = \beta_0/\gamma$, and $x_\Gamma = x(t' - \Gamma)$.

The electronic implementation was divided in two main parts: the delay block, which presents only a time shift between its input and its output; and the function block, which implements the nonlinear term of the equation. The complete circuit looks as in Fig. 1. In this scheme the function block implements the production term in the Eq. 2 without delay

$$f(v) = \beta \frac{v}{\theta^n + v^n}, \tag{3}$$

and the delay block approximates the transfer function:

$$v_{out}(t) = v_{in}(t - \tau). \tag{4}$$

Assuming ideal behavior of both blocks, the equation for the potential at the capacitor terminals is given by

$$\frac{dv_c(t)}{dt} = \frac{1}{RC}\left[f\left(v_c\left(t-\tau\right)\right) - v_c\left(t\right)\right] \quad (5)$$

which can be identified with Eq. 2 by setting $t' = t/RC$, $x = v_c/\theta$, $\Gamma = \tau/RC$, and $\alpha = \beta/\theta^n$.

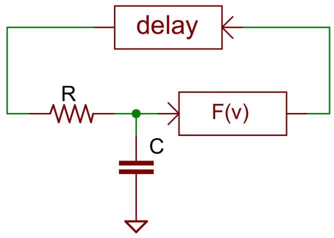

Figure 1. Block schematic of the circuit.

The purpose of the delay block is to copy the input to the output after some time delay. The implementation of this block with analog electronic is possible using a Bucket Brigade Device (BBD), which is a discrete-time analog device. Internally it consists of an array of N capacitors in which the signal is moved along one step at each clock cycle. In our implementation we used the integrated circuits MN3011 and MN3101 as BBD and clock signal generator respectively.

In this approach to implement the time delay approximates the desired transfer function given, in this case by Eq. 4, by sampling the input signal and outputting their samples N clock periods later. The effective transfer equation read as

$$v_{out}\left(t\right) = g_d v_{in}\left(T_s\left\lfloor\frac{t}{T_s} - N + 1\right\rfloor\right) + V_d \quad (6)$$

where T_s, g_d and V_d stand, respectively, for the sampling period, gain and offset voltage introduced by the BBD. In the MN3011 the sampling period can vary between $5\mu s$ and $50\mu s$ and N can be selected among the values provided by the manufacturer $(396, 662, 1194, 1726, 2790, 3328)$. The accuracy of the implementation is reported in [Amil et al., 2015a] where several input and output signals are plotted.

Since the delay block we used approximates an ideal delay, the effective equation of the implemented circuit will also approximate the original Mackey-Glass equation.

The output of the delay block remains constant in each clock period, so it seems natural to solve Eq. 6 in steps. Let us solve it, then, for $jT_s \le t < (j+1)\,T_s$ and let $v_i = v_c\left(iT_s\right)$, then

$$\frac{dv_c}{dt}\left(t\right) = \frac{1}{RC}\left[f\left(v_{j-N+1}\right) - v_c\right]. \quad (7)$$

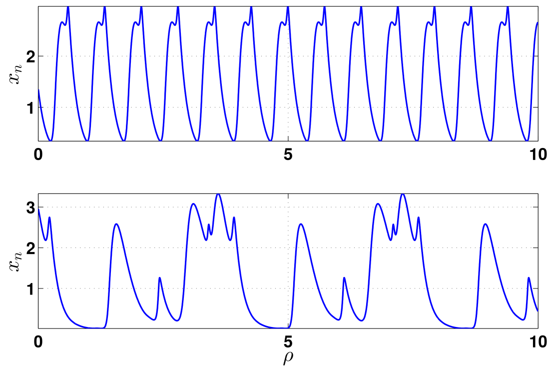

Figure 2. Two examples of temporal evolution of the system variable simulated using the discrete scheme.

Since $f\left(v_{n-N+1}\right)$ is constant, this equation can be readily solved, the solution knowing the value of v_c in $t = jT_s$ is

$$v_c\left(t\right) = \left(v_j - f\left(v_{j-N+1}\right)\right)e^{\frac{nT_s - t}{RC}} + f\left(v_{j-N+1}\right). \quad (8)$$

Setting $t = (j+1)T_s$ and substituting the expression for $f(v)$ given in Eq. 3 it results

$$v_{j+1} = v_j e^{\frac{-T_s}{RC}} + (1 - e^{\frac{-T_s}{RC}})\beta\frac{v_{j-N+1}}{\theta^n + v_{j-N+1}^n}. \quad (9)$$

This discrete time effective equation approaches to the original continuous time equation, Eq. 5, when N grows to infinity and the delay time $\tau = NT_s$ is kept constant.

3 Results

In Figs. 2 and 3, several examples, both empirical and numerical, time-traces are shown. In these figures, the parameter values of the electronic circuit or the numerical simulations are kept constant while only the initial conditions are varied. Looking at these solutions we observe different sequence of maxima with different amplitudes.

As it is shown in [Amil et al., 2015b], multistability is manifested in a wide range of parameters. Here we focus on how do the different, periodic or chaotic, solutions appear as a function of the IF. However, as the parameter space defining the IFs is infinite dimensional we consider pseudo-random IFs. the influence of the initial conditions. In order to identify parameter regions where multi-stability occurs, we developed an algorithm for time-series analysis that allows to unambiguously distinguish similar waveforms. This analysis algorithm is based in a symbolic representation of a time-series, and allows to label the different *periodic* solutions. Two symbols were used, which correspond to highest peaks, and to 2nd highest peaks. Once the symbolic string was generated, the algo-

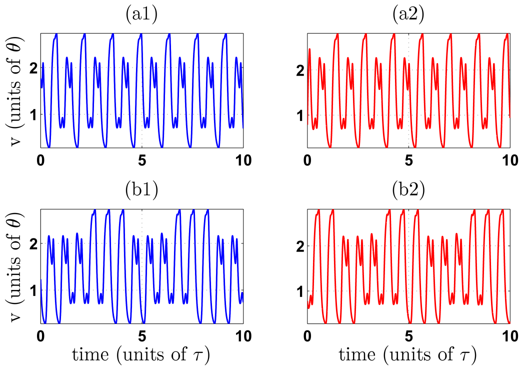

Figure 3. Comparison between simulations using the discretized solution (left column) and experimental results obtained from the electronic circuit (right column). The top and bottom rows display coexisting solutions obtained from different initial functions. The parameter values are: $n = 4, \alpha = 4.9, \Gamma = \tau/RC = 15.7$ and $N = 396$

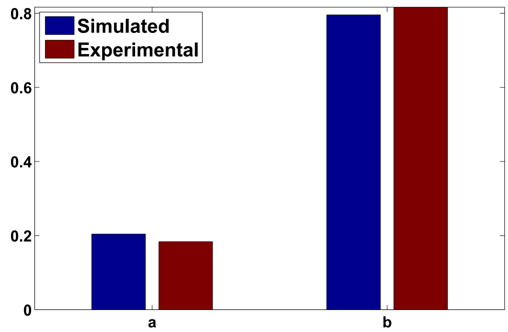

Figure 4. Histograms showing the abundance of solutions given in Fig. 2 in experimental and simulations (discrete time) results. The parameter values are: $n = 4, \alpha = 4.9, \Gamma = \tau/RC = 15.7$ and $N = 396$.

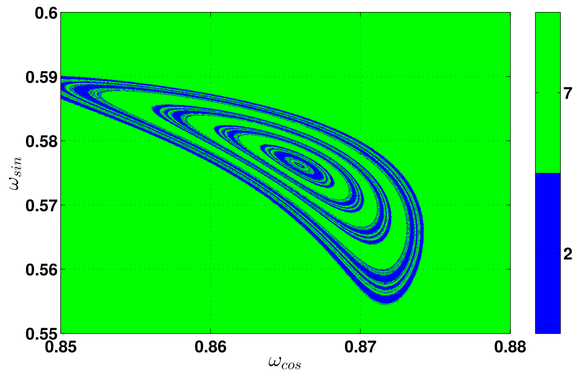

Figure 5. Map of parameters (ω_{sin} and ω_{cos}) that define the initial function given by Eq. (10)], which evolve to a solutions with 2 (blue) or 7 (green) maxima per period. The parameter values are as in Fig. 4

rithm searched for periodicity, and if found, the time-series was labelled with the symbolic string, written in a unique way under cyclic permutations. For example, the symbolic strings $AAABBBAAABBB$ and $BBBAAABBBAAA$ both represent the same periodic solution, which has three consecutive high maxima followed by three consecutive smaller maxima. This way of labelling different solutions allows to distinguish among solutions with the same number of peaks per period. The algorithm can be extended to analyze more complex waveforms.

To investigate multi-stability one needs to consider different initial functions, $v(t - \tau) = F_0(t)$ with $t \in (-\tau, 0)$. Here we consider a family with two parameters ω_{sin} and ω_{cos} given by pseudo-random sinusoidal functions

$$F_0 = 2 + \sin(\omega_{sin} 2\pi t/\tau) + \cos(\omega_{cos} 2\pi t/\tau). \quad (10)$$

Let us proceed as follows, for each pair of values, ω_{sin} and ω_{cos}, a transient time is neglected (about 5000τ in the simulations and 1000τ in the experiments) and time series of length 200τ (simulations) or 100τ (experiments) are recorded. Their periodicity is analyzed with the symbolic algorithm and the solutions are plotted in the ω_{sin} and ω_{cos} space. If the MG system is only two-dimensional these plots would identify the basins of attraction of the different solutions; however, the MG system is a delayed system and thus, these plots only classify the different solutions obtained in terms of the two parameters that determine the initial function.

In Fig. 4, two histograms, for experimental and simulations results are shown indicating the abundance of solution in the parameter space of the initial function. It is clear that both experimental and numerical experiments exhibit similar frequency of appearance of the different solutions.

In Fig. 5-7 the organization of the solutions in the parameter space of the initial functions is depicted. The color code represent the asymptotic solution obtained as a function of the parameters that define the initial function. The simulation presented corresponds to F_0 given by Eq. 10 and the parameters of the MG model and of the electronic circuit are as in Figs. 3). In these figures it is evidenced the coexistence of periodic and aperiodic solutions in a wide region of the parameter space. Therefore, our study indicates that, at least for the model parameters considered here, the electronic circuit reproduces the main features of the MG system (the shape of the waveforms, the bifurcation diagrams, and the maps of bistable and multistable solutions) and thus, it could be used to investigate other issues, for example, noise-induced switching, or how multi-stability affects synchronization.

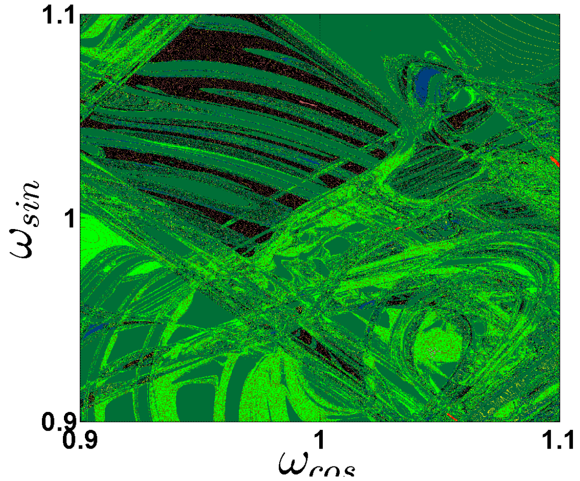

Figure 6. Map of parameters (ω_{sin} and ω_{sin}) that define the initial function given by Eq. (10)], which evolve to different periodic solutions (color) or a chaotic solution (black). The parameter values are: $n = 8$, $\alpha = 8$, and $N = 396$.

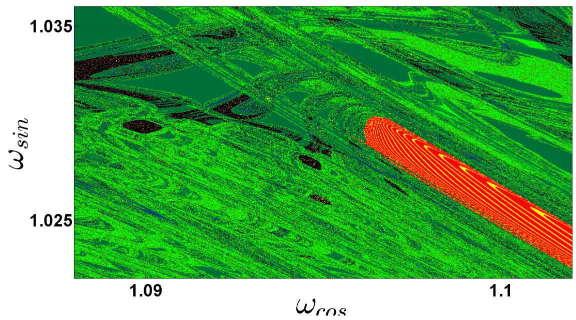

Figure 7. Zoom of the parameter space shown in Fig. 6.

4 Conclusion

Using the electronic implementation proposed in [Amil et al., 2015a], the initial condition space of the Mackey-Glass model studied in [Amil et al., 2015b] was further explored. Using the discrete-time equation that approximates the exact solutions of the MG model and in particular, extended to random-phase initial functions. The maps of initial conditions that result in different periodic solutions were found to exhibit complex structures, which are not uncommon in delayed systems [Shrimali et al., 2008]. In significant parameter regions, different periodic or aperiodic solutions, but with similar waveforms, coexist. These solutions, exhibiting the alternation of peaks of different amplitudes, can be classified distinguished by means of a symbolic algorithm. A relevant consequence is that, in contrast to other systems in which it is sufficient to count the number of peaks per period (see for example [Cabeza et al., 2013; Freire et al., 2013; Freire et al., 2014]), here it is necessary to consider the ordering of the peaks to identify the solutions.

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