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CONTROL OF STABILITY OF NONLINEAR ELASTIC PENDULUM

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Abstract

Simple condition for the stability of vertical large amplitude oscillations of nonlinear heavy spring elastic pendulum is given that is valid for a wide range of control parameter values. The condition is tested and the bifurcation responsible for the lost of stability is explained and illustrated.

Key words

Elastic pendulum, stability, bifurcation.

1 Introduction

Elastic pendulum is a simple model that exhibits a wide and surprising range of dynamic phenomena [Anicin, Davidovic and Babovic, 1993], [Breitenberger, Mueller, 1981], [Cayton, 1977], [Davidovic, Anicin and Babovic, 1996], [Lai, 1984], [Olsson, 1976], [Rusbridge, 1980].

The first known publication about the elastic pendulum is the paper [Vitt and Gorelik, 1933] by Vitt and Gorelik. They consider small amplitude oscillation of a planar elastic pendulum in the 2:1 resonance. Connection to the Fermi resonance of CO_2 is mentioned. This paper has been translated from Russian by Lisa Shields and published by Peter Lynch on his web page.

Deterministic chaos has been observed in numerical simulation and presented in the form of Poincare section, auto-correlation function, Lyapunov exponent and power spectrum in [Carretero-Gonzalez, Nunez-Yepez and Salas-Brito, 1994]. [Cuerno, Ranada and Ruiz-Lorenzo, 1992], [Nunez-Yepez, Salas-Brito, Vargas and Vicente, 1990], The bifurcation diagram of a plane elastic pendulum is sketched in [Kuznetsov, 1999]. The most thorough treatment of small amplitude oscillation of both plane and space elastic pendulum can be found in the works by Peter Lynch [Holm and Lynch, 2002], [Lynch, 2002a], [Lynch, 2002b], [Lynch and Houghton, 2004].

2 Definition of the system

Consider a pendulum consisting of a point bob of mass m_B suspended on a homogeneous elastic spring of mass m_S with the elasticity constant k, see Fig. 1. Assume a homogeneous gravitational field with intensity $(0, -g)$.

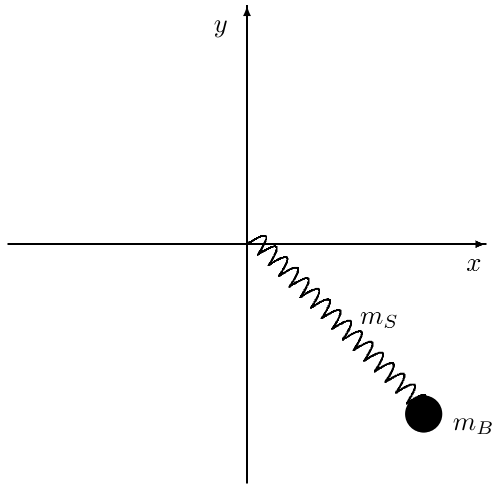

Figure 1. Heavy spring elastic pendulum consists of a point mass m_B attached to a spring with mass m_S which is fixed at the other end point.

In [Pokorny, 2008] we derive the equations of motion. In 2-dim they are

$$\begin{aligned} \ddot{x} &= \left(\frac{1}{\sqrt{x^2+y^2}} - 1\right)x \\ \ddot{y} &= \left(\frac{1}{\sqrt{x^2+y^2}} - 1\right)y - p. \end{aligned} \tag{1}$$

This system has the only parameter

$$p = \frac{(m_B + \frac{m_S}{2})g}{k\ell_0}$$

where ℓ_0 is the length of the unloaded spring.

3 Stability condition

In [Pokorny, 2008] we also derive the stability condition for the vertical oscillations

$$x = 0, \qquad y = -1 - p + a \sin t \tag{2}$$

where a is the amplitude of the vertical oscillation. The vertical oscillation of the heavy spring elastic pendulum with the relative amplitude a is stable for

$$|a| < |a_C(p)|$$

where

$$a_C(p) \doteq (3p)^{\frac{2}{3}} - 1. \tag{3}$$

The error of this estimate of $a_C(p)$ is less than 0.005 for $0 < p < 0.6$.

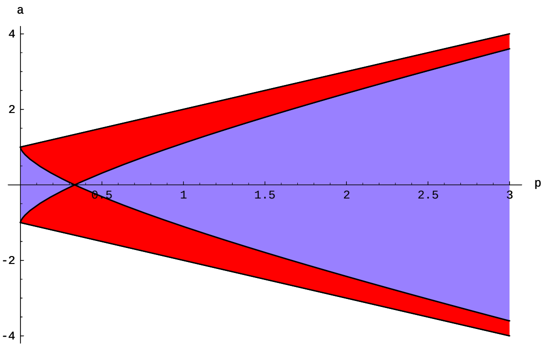

Figure 2. The stable region (blue) and the unstable region (red) for the vertical oscillation with amplitude a and the external field p.

Fig. 2 shows the regions in the p-a plane with stable (blue) and unstable (red) vertical oscillation with amplitude a and the parameter p.

To illustrate the stability condition assume we want to find the minimal value of the parameter p where the vertical oscillations with the amplitude $a = 0.1$ become unstable. The two positive solutions of the equation

$$|(3p)^{\frac{2}{3}} - 1| = 0.1$$

give estimates for two critical values

$$p_1 \doteq 0.2846$$

and

$$p_2 \doteq 0.3846.$$

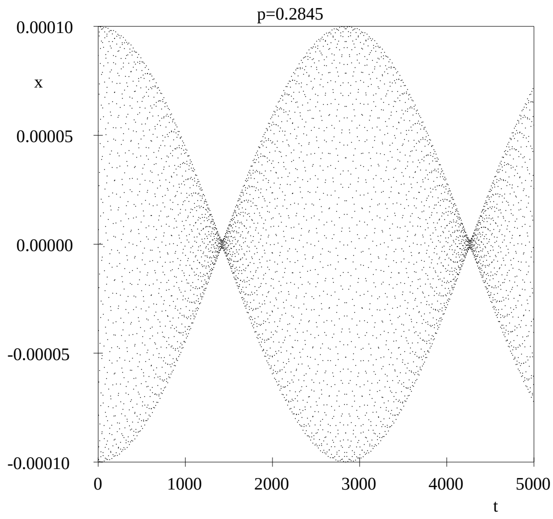

Figure 3. When the vertical oscillations are stable, the horizontal displacement x of the bob remains small in time if starting from a small value. Here $x(0) = 0.0001$ and $x'(0) = 0$ was used. In this picture $x(t)$ is sampled in equidistant time intervals $\Delta t = 1$ and the points are not joined.

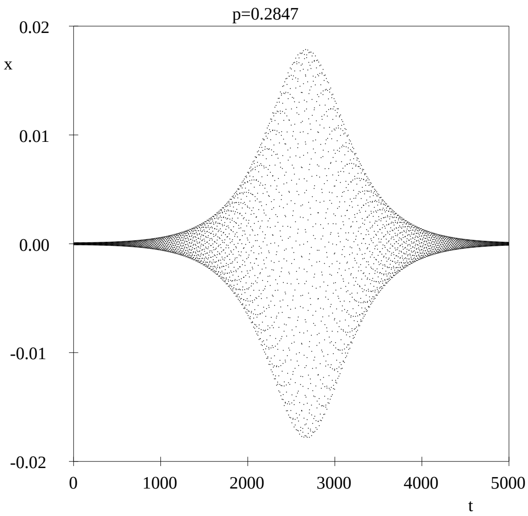

Figure 4. When the vertical oscillations are unstable, the horizontal displacement x of the bob grows even for arbitrarily small initial value $x(0)$. Here $x(0) = 0.0001$ and $x'(0) = 0$ was used.

Remark: for a going to zero, both p_1 and p_2 go to $\frac{1}{3}$ exactly.

Thus the vertical oscillations with the amplitude $a = 0.1$ are stable for $p < p_1 \doteq 0.2846$. What happens when p crosses this critical value? First we show $x(t)$ for p slightly below and slightly above this critical value.

In Fig. 3 the time dependence of x is plotted for the parameter value $p = 0.2845$ and the initial condition $x(0) = 0.0001$, $x'(0) = 0$, $y(0) = -1 - p - a = -1.3845$, $y'(0) = 0$. The motion is quasi-periodic and $x(t)$ is never greater in absolute value than the initial value $x(0)$.

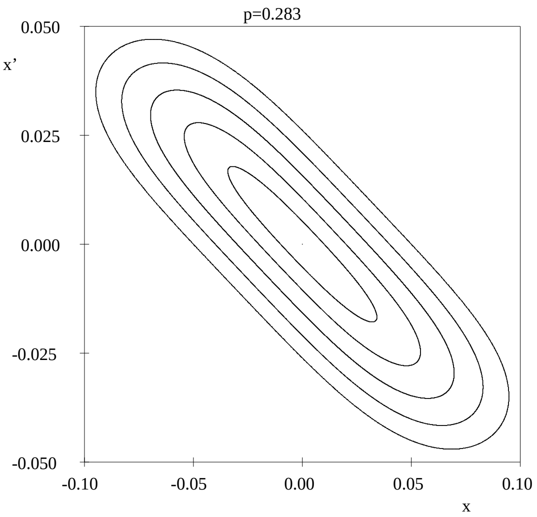

Figure 5. The Poincare cut of the trajectory with the horizontal plane $y = -1 - p$ in the stable regime. The vertical periodic oscillation results in a single point $(x, x') = (0, 0)$. Around this periodic solution (a closed loop in the state space) there is a one parameter family of quasi-periodic solutions (invariant tori in the state space) resulting in a one parameter family of invariant closed curves in the Poincare section. Only 5 curves are depicted.

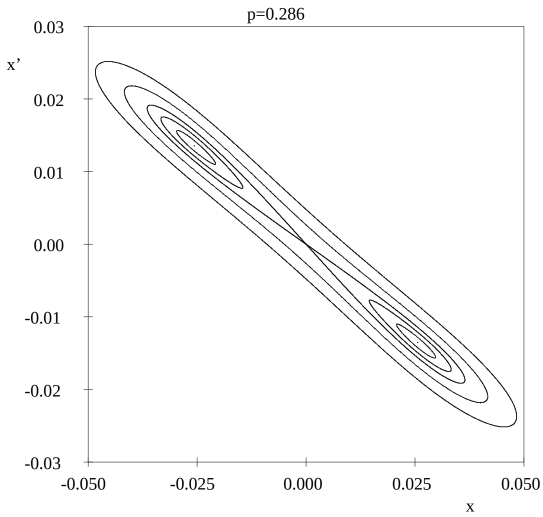

Figure 6. The Poincare cut for supercritical value of the parameter p. The figure-eight curve going from the origin shows the homoclinic orbit dividing the plane into regions of two types. Those inside the loops of the figure-eight contain invariant curves corresponding to invariant tori with two intersections with the horizontal plane. And the region outside of the figure-eight is filled with the invariant curves corresponding to tori with just a single intersection with the horizontal plane.

A small change of the parameter p results in a different type of behavior. In Fig. 4 the same variables are plotted for the parameter value $p = 0.2847$ and the initial condition $x(0) = 0.0001$, $x'(0) = 0$, $y(0) = -1 - p - a = -1.3847$, $y'(0) = 0$. The variable $x(t)$ grows approximately exponentially for a certain time period, reaching an amplitude greater than the initial value by more than two orders of magnitude.

The qualitative change in the phase portrait can be observed in the Poincare cut. Consider the horizontal plane $y = -1 - p$ containing the lower equilibrium point of the system and consider those intersections of the trajectory with this plane where the bob goes up, i.e. the points such that $y = -1 - p$ and $y' > 0$. The projections of these points to the x-x' plane are presented in the following two pictures. In Fig. 5 the projection of the Poincare cut into the x-x' plane is shown for $a = 0.1$ and $p = 0.283$. These parameters correspond to stable vertical oscillation. This stable vertical periodic solution corresponds to a closed loop in the state space and to a single point in the Poincare cut (the point (0,0) in our projection). Around this point there is a one parameter family of invariant closed curves covered densely by the intersections of the trajectory with the plane. Five of them are shown in Fig. 5 for five different initial conditions $(x(0), x'(0))$, namely: $(0.01, 0)$, $(0.02, 0)$, $(0.03, 0)$, $(0.04, 0)$, $(0.05, 0)$.

Increasing the parameter p above the critical value $p_1 \doteq 0.2846$ the vertical oscillations lose their stability via a period doubling bifurcation giving rise to a periodic solution with a period two times greater. This new periodic solution intersects the horizontal plane $y = -1 - p$ in four points, two of them with $y' > 0$ and two of them with $y' < 0$.

This is illustrated in Fig. 6 where for the supercritical value of the parameter $p = 0.286$ the Poincare cut is shown for five different initial conditions $(x(0), x'(0))$, namely: $(0.009, 0)$, $(0.005, 0)$, $(0.00001, 0)$, $(0.0146, -0.00804)$, $(0.0208, -0.0112)$. Only points with $y' > 0$ are shown.

Emanating from the origin there is a figure-eight curve. This corresponds to a homoclinic orbit. Inside each of its two lobes there is a point corresponding to the new periodic solution. This point is surrounded by a one parameter family of closed loops corresponding to invariant tori. Outside this figure-eight curve there is the one parameter family of closed loops corresponding to invariant tori that survived the bifurcation.

How do the eigenvalues of the linearization matrix of the Poincare map in $(0, 0)$ change? For $p < p_1$ we have a pair of complex conjugate eigenvalues on the unit circle in the complex plane. For p approaching p_1 these eigenvalues approach -1. When the parameter p crosses the critical value p_1 the eigenvalues cross the point -1 and then stay real, one of them less than -1, the other greater than -1. As one eigenvalue leaves the unit circle in the complex plane, the solution becomes unstable. The eigenvector corresponding to the eigenvalue outside of the unit circle gives the unstable direction (more precisely is tangent to the unstable manifold of the origin).

Even when the vertical oscillations lose their stability, initial conditions with $x(0) = 0$ and $x'(0) = 0$ exactly lead to pure vertical oscillations. However, almost any small deviation from zero, either in x or in x' (or in both of them) leads to the orbit with points that depart from the origin in the Poincare section. If the initial deviation from zero was small it takes a long time before the deviation grows to observable values. This means the pendulum behaves for a long time similarly to the case of stable vertical oscillations. After some time the point in the Poincare section goes away from the origin, travels around the loop of the figure-eight (this is manifested by horizontal swinging of the pendulum) and then it goes back close to the origin where it stays for a long time to depart again.

Initial conditions not close to the origin lead to quasi periodic oscillations on the invariant tori, either inside the figure-eight loop or outside of it.

To summarize, for a fixed (not necessarily close to zero) amplitude a of vertical oscillations when the parameter p is increased, then the following three types of behavior can be observed depending on the parameters a and p.

1. For small positive p the vertical oscillations are stable.
2. When crossing the critical value
$$p_1 \doteq \frac{1}{3}(1-a)^{\frac{3}{2}}$$
a period doubling bifurcation occurs and the vertical oscillations are unstable for
$$p_1 < p < p_2.$$
A branch of periodic solutions appears for $p > p_1$.
3. When
$$p > p_2 \doteq \frac{1}{3}(1+a)^{\frac{3}{2}}$$
the vertical oscillations are again stable. When decreasing the parameter p then crossing the critical value p_2 a similar period doubling bifurcation occurs.

It is important to note that in the unstable region for

$$p_1 < p < p_2$$

the origin is a hyperbolic homoclinic fixed point of the Poincare map. For initial conditions corresponding to points $(x(0), x'(0))$ in a small neighborhood of the origin, we can observe the following four types of qualitatively different behavior. The figure-eight intersects a small neighborhood of the origin in two curves that cross each other in the origin. One of them is tangent to the eigenvector of the linearization matrix of the Poincare map corresponding to the eigenvalue less than -1. This curve is the unstable manifold. The other curve is tangent to the eigenvector corresponding to the eigenvalue greater than -1. This curve is the stable manifold. The four different types of behavior depending on the initial condition are as follows.

1. When starting in the origin we get unstable periodic vertical oscillations.
2. Starting on the stable manifold results in orbit points approaching the origin. The corresponding trajectory of the continuous time system approaches the unstable vertical periodic trajectory. This type of behavior is also unstable thus experimentally difficult to observe.
3. Initial conditions corresponding to points close to the origin and on the unstable manifold produce the orbit of points lying on the figure-eight curve. They depart from the origin initially, and then they approach the origin asymptotically again, according to the previous scenario.
4. However, the most typical case is as follows: we start in a point close to the origin which is not located on the figure-eight curve exactly, it is only close to it. Then initially the points of the orbit depart from the origin, the amplitude of the horizontal oscillations grows temporarily, reaches its maximum and then decreases again to small values close to the initial one. In this phase the points of the orbit approach the origin for a certain number of iterations. Then they depart again. In the course of time, in a typical case, they fill densely a closed invariant curve in the Poincare section, the trajectory fills densely an invariant torus. In the limit of initial condition approaching the origin, the period of the envelope (which was approximately 5000 in the case shown in Fig. 4) goes to infinity and the ratio of the maximum value of $x(t)$ to the initial value $x(0)$ (which was more than 100 in the case shown in Fig. 4) goes to infinity too.

4 Conclusion

The surprising behavior of the elastic pendulum when the vertical oscillations are sometimes stable but sometimes interrupted by intermittent short time horizontal oscillations (as observed also in [Havranek and Certik, 2006], [Dvorak, 2006]) was explained using the dynamical systems approach. The model of a heavy spring elastic pendulum was given and the condition for the stability of vertical oscillations was tested. The lost of stability of vertical oscillations was explained by the period doubling bifurcation leading to the birth of a family of periodic solutions with twice the period.

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