

# Delay-induced multistability near a global bifurcation

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**Abstract**—We study the effect of a time-delayed feedback within a generic model for a saddle-node bifurcation on a limit cycle. Without delay the only attractor below this global bifurcation is a stable node. Delay renders the phase space infinite-dimensional and creates multistability of periodic orbits and the fixed point. Homoclinic bifurcations, period-doubling and saddle-node bifurcations of limit cycles are found in accordance with Shilnikov’s theorems.

## I. INTRODUCTION

Time-delayed feedback was originally proposed in the context of chaos control [1] as an alternative to the famous OGY method developed earlier by Ott *et al.* [2]. The idea was to achieve stabilization of unstable periodic orbits (UPOs) by adding, to a chaotic system, a control force in the form of the difference between a system variable at time  $t$  and at a delayed time  $t - \tau$ . This method proved to be very powerful and has been successfully applied to various physical systems since then [3]. The scheme was improved by Socolar *et al.* [4], and other variants have been elaborated [5], [6], [7], [8], [9], and applied also to stochastic systems [10], [11], [12]. Moreover, elegant analytical theories [13] were developed supporting, thus, numerical findings. Apart from the practically relevant application of time-delayed feedback, e.g. to lasers [14], the interest lies highly on the mathematical aspect of the problem. Delay differential equations are difficult to handle. The delay renders the system infinite-dimensional and the interplay with nonlinearity uncovers complex dynamic behaviour. Delay-induced multistability was already predicted in the first paper by Pyragas [1]. The idea that time-delayed feedback may not only be used for controlling a system but also for creating new dynamics is not new. Nevertheless, the investigation of delay-induced bifurcations and multistability is still a growing field [15], [16] with applications to the logistic map as well as to laser equations [17], [18].

## II. THE MODEL

Here we study the influence of time-delayed feedback in a generic model representative for excitable dynamics [19], [20], [21]. The behaviour of the system without delay is governed by a global bifurcation, namely a saddle-node bifurcation on a limit cycle (*saddle-node infinite period bifurcation*, *SNIPER*), that takes place when a certain parameter exceeds a threshold. Such a bifurcation was first observed experimentally in a semiconductor device [22], and also

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encountered in various semiconductor models, e.g. for Gunn domains [23], superlattices [24], [25], or lasers [26], [27], [28]. Excitability, naturally, yields the system highly sensitive to fluctuations and therefore this system has served perfectly as an example for *coherence resonance* [29] shown in the seminal paper by Hu *et al.* [30] and [20] over ten years ago. In this paper we do not consider the effect of noise but extend the generic model by incorporating time-delayed feedback according to the Pyragas scheme. The equations are the following:

$$\dot{x} = x(1 - x^2 - y^2) + y(x - b) - K(x - x_\tau) \quad (1)$$

$$\dot{y} = y(1 - x^2 - y^2) - x(x - b) - K(y - y_\tau). \quad (2)$$

Here  $x$  and  $y$  denote the variables at time  $t$ , while  $x_\tau$  and  $y_\tau$  the delayed ones at time  $t - \tau$ , with  $\tau$  and  $K$  being the delay and control strength, respectively. This kind of control is called *diagonal* because the control force may be written in the form of a unity matrix. In the absence of delay, i. e.  $K = 0$ ,  $b$  plays the role of the bifurcation parameter. In polar coordinates  $x = r \cos \varphi$ ,  $y = r \sin \varphi$  Eq. (1) with  $K = 0$  reads

$$\dot{r} = r(1 - r^2) \quad (3)$$

$$\dot{\varphi} = b - r \cos \varphi \quad (4)$$

When  $b < 1$  there are three fixed points: an unstable focus at the origin and a pair of a saddle-point and a stable node on the unit circle with coordinates  $(b, +\sqrt{1 - b^2})$  and  $(b, -\sqrt{1 - b^2})$ , respectively. The latter collide at  $b = 1$  and a limit cycle  $r = 1$  is born. Above but close to the bifurcation, the frequency  $f$  of this limit cycle obeys a characteristic square-root scaling law  $f \sim (b - 1)^{1/2}$ .

## III. LINEAR STABILITY ANALYSIS

We prepare the system slightly below the bifurcation ( $b = 0.95$ ) and switch on the control. The first question that arises concerns the stability of the three fixed points and how this changes, or not, due to delay. For this, we perform a standard linear stability analysis and derive the characteristic equation for the roots,  $\Lambda$ , which determine the stability of the fixed points. For the unstable focus, the characteristic equation is:

$$(1 - K + Ke^{-\Lambda\tau} - \Lambda)^2 + b^2 = 0. \quad (5)$$

Due to the presence of the delay, Eq. (5) has infinitely many solutions. However, the stability of the fixed points is determined by a finite number of critical roots with largest real parts. Using the Lambert function  $W$ , which is defined

as the inverse function of  $g(z) = ze^z$  for complex  $z$  [31], [32], the solution of Eq. (5) can be expressed as:

$$\Lambda = \frac{1}{\tau} W[K\tau e^{\tau(K-1 \pm b)}] - K + 1 \mp ib. \quad (6)$$

In the case of the saddle and the node, the characteristic equation can be factorized into two equations:

$$\Lambda + K + 2 - Ke^{-\Lambda\tau} = 0 \quad (7)$$

$$\Lambda^{s,n} + K \mp \sqrt{1-b^2} - Ke^{-\Lambda^{s,n}\tau} = 0, \quad (8)$$

with solutions:

$$\Lambda_1 = \frac{1}{\tau} W[K\tau e^{\tau(2+K)}] - 2 - K \quad (9)$$

$$\Lambda_2^{s,n} = \frac{1}{\tau} W[K\tau e^{\tau(K \mp \sqrt{1-b^2})}] - K \pm \sqrt{1-b^2} \quad (10)$$

The superscripts 's' and 'n' denote the saddle (upper sign) and the node (lower sign), respectively.

Figure 1 shows the real parts of the eigenvalues  $\Lambda$  as a function of  $\tau$  for a fixed value of  $K$  for all three fixed points. One may see the eigenvalues of the uncontrolled system at  $\tau = 0$ , and their interaction with the delay-induced modes (blue) with increasing  $\tau$ . In all three cases, control is unable to change the stability of the fixed point: in the case of the unstable focus, the mode with the largest real part (red) tends to very small values with increasing delay, remaining however positive. The same holds for the unstable mode of the saddle. Symmetric behaviour is observed for the stable modes of both saddle and node: they tend to zero as a function of  $\tau$  but remain negative. The picture does not change qualitatively even for other values of  $K$  and therefore one might conclude that no delay-induced bifurcations of fixed points take place.

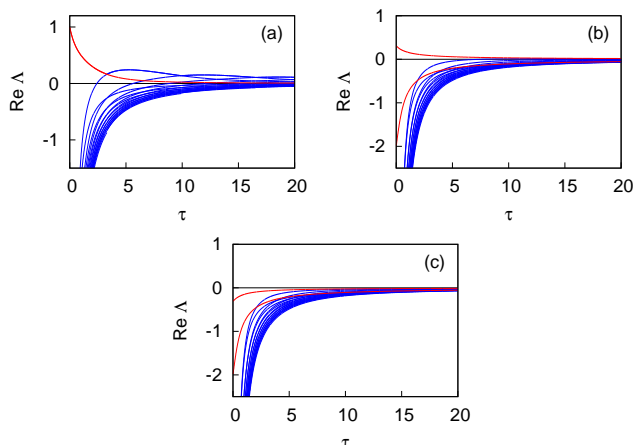


Fig. 1. Real parts of the complex eigenvalues  $\Lambda$  as a function of  $\tau$ , for fixed  $K = 1$  and  $b = 0.95$ . For the (a) unstable focus, (b) the saddle point and (c) the stable node. The modes emerging from the uncontrolled system and the delay-induced modes are marked red and blue, respectively.

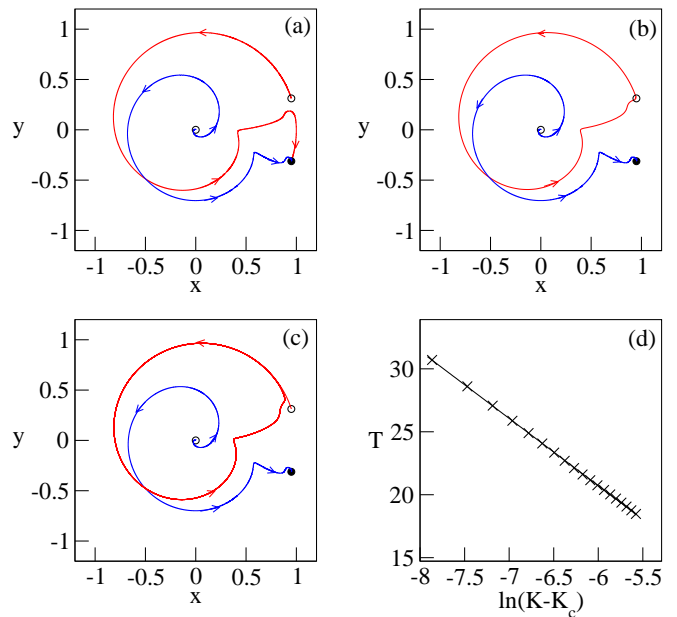


Fig. 2. (a) Two dimensional projection of the phase space below the homoclinic bifurcation ( $K = 0.335$ ). (b) Homoclinic orbit (red) achieved at  $K_c = 0.3401$ . (c) Delay-induced limit cycle (red) above the homoclinic bifurcation ( $K = 0.3438$ ). (d) Scaling of the oscillation period  $T$  above but close to the critical point  $K_c$  (crosses: simulation data, solid line: linear fit). Full and open circles mark stable and unstable fixed points, respectively. Parameters:  $b = 0.95$ ,  $\tau = 3$ .

#### IV. GLOBAL BIFURCATION ANALYSIS

However, the above local analysis gives no information on the global changes in phase space that delay potentially induces. A numerical investigation shows, in fact, that there exists bistability in a certain parameter regime in the  $K$ - $\tau$  plane: trajectories starting close to the saddle point are attracted by a delay-induced limit cycle, whereas trajectories starting elsewhere end up in the stable node. Keeping  $\tau = 3$  fixed we find the critical value  $K_c$  of  $K$  for which this delay-induced limit cycle is born and observe a scaling  $T \sim \ln|K - K_c|$  in the period  $T$  of the corresponding oscillations, typical for the case of a homoclinic bifurcation. In Fig. 2 phase portraits of the system below, at and above the bifurcation are shown. Trajectories with different initial conditions are shown: one starting from the vicinity of the unstable focus in the origin (blue) and one from the vicinity of the saddle (red). Bistability is revealed in Fig. 2(c) where two attractors (the stable node and the delay-induced limit cycle) coexist. The 'kink' in the trajectory shortly before the loop closes is due to the control: the control force starts acting at  $t = 3$  when the system is still moving on the slow part of the unit circle. Therefore, its effect is not so noticeable. As the system moves faster the control force attains higher values and the trajectory starts deviating from its deterministic path at  $t = 13$ . This deviation becomes large at  $t = 18$  where the trajectory, shortly before settling in the stable node, appears to be "attracted" to the saddle, resulting in this 'kink' in the  $x - y$  projection. Also, as  $K$  approaches the critical value, the trajectory passes closer and

closer to the saddle on its way to the stable node. This ends in a homoclinic orbit at  $K = K_c$ , (Fig. 2(b)) from which a periodic orbit is generated (Fig. 2(c)). The period of the born limit cycle scales according to  $T \approx -\Lambda_u^{-1} \ln|K - K_c|$ , where  $\Lambda_u$  is the real part of the least unstable eigenvalue of the saddle point (i. e., the one closest to the imaginary axis). One may calculate this from Eq. (9) and find  $\Lambda_u^{-1} = 0.1739^{-1} = 5.75$ . This is in rather good agreement with the slope of the solid line in Fig. 2(d) which equals 5.35.

In the following we use a bifurcation continuation tool [33], [34] and follow the homoclinic bifurcation in the  $K - \tau$  plane. The produced bifurcation curve can be seen in Fig. 3 (left). It consists of two main curves: one running through points A – E and a second tongue-like curve. In the white area the system is monostable (stable fixed point) while in the yellow area a delay-induced periodic attractor is born via a homoclinic bifurcation marked by the red curves.

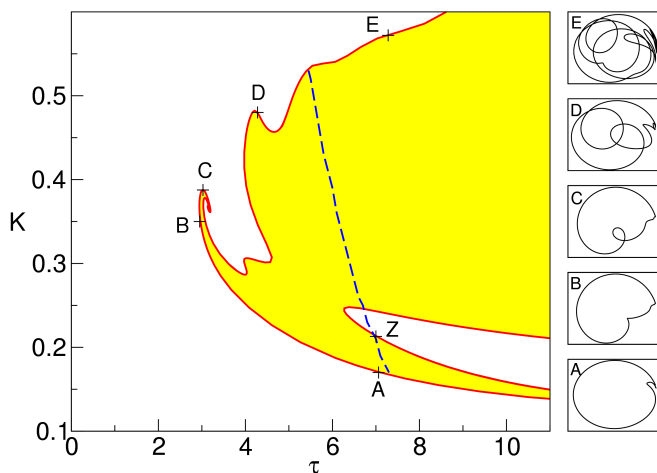


Fig. 3. Curve of homoclinic bifurcations (red) in the  $K - \tau$  plane (left). A - E labels various points with homoclinic orbits, which are shown in the  $x - y$  phase plane in the panel on the right. Delay-induced limit cycles exist, in addition to the stable fixed point, in the yellow area. The blue dashed curve separates the regions  $\sigma_0 < 0$  (left) and  $\sigma_0 > 0$  (right)

At this point one should emphasize the role of the saddle-point: due to the delay, the saddle possesses no longer two distinct eigenvalues (one positive, i. e. unstable, and one negative, i. e. stable) but infinitely many. Moreover, complex eigenvalues come into play as well. The eigenvalues, however, that determine the behaviour of the colliding homoclinic orbit, are the leading ones, i. e. those closest to the imaginary axis. In Fig. 4 one can see the eigenvalue spectrum for two parameter values on the homoclinic curve and notice that the leading eigenvalues of the saddle (red) are a positive real eigenvalue, as in the original uncontrolled system, and a complex conjugate pair with negative real parts, generated by the delay. This means that the saddle may turn into a *saddle-focus* for certain values of  $K$  and  $\tau$ . Homoclinic orbits attached to a saddle-focus approach the fixed point in an oscillating manner. This explains the phase portraits in Fig. 3 (right) which become more and more complicated as  $K$  increases.

From the above, it is clear that the two basic ingredients

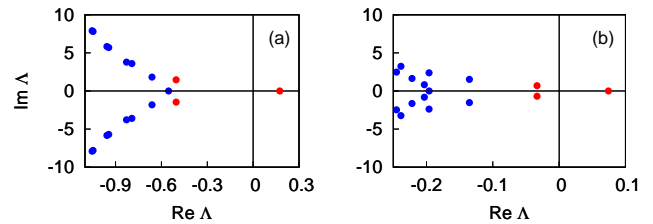


Fig. 4. Spectra of the eigenvalues  $\Lambda$  of the saddle for two points on the homoclinic bifurcation line: (a)  $K = 0.3401$ ,  $\tau=3$  (B in Fig. 3) and (b)  $K = 0.57$ ,  $\tau = 7.28$  (E in Fig. 3). The saddle quantity  $\sigma_0 = Re(\Lambda_s) + Re(\Lambda_u)$  is negative in (a) whereas in (b) it is positive. ( $b = 0.95$ )

responsible for the delay-induced dynamics in our system are the homoclinic orbits and the saddle-foci. The theory of homoclinic bifurcations for ordinary differential equations has been well developed [35], [36], [37]. As already mentioned, global bifurcations are strongly related to excitability and therefore one expects to encounter them in excitable systems. Various physical systems such as modulation-doped semiconductor heterostructures [38], semiconductor lasers [27], [28], neuron models [39] and chemical systems [40] have been studied in this respect, both theoretically and experimentally. On the other hand, less work has been carried out for systems with delay undergoing such nonlocal bifurcations [41]. It is therefore appropriate to analyze a generic system like the one studied here which, despite its simplicity, exhibits rich delay-induced dynamics with a homoclinic bifurcation as key component.

## V. DELAY-INDUCED MULTISTABILITY

In what follows we will apply the theorems on homoclinic orbits connecting to saddle-foci as developed by Shilnikov [35]. According to them, the so-called saddle quantity is crucial for the homoclinic bifurcations occurring in high-dimensional systems. The saddle quantity is defined as  $\sigma_0 = Re(\Lambda_s) + Re(\Lambda_u)$ , where  $\Lambda_s$  and  $\Lambda_u$  are the leading stable and unstable eigenvalues, respectively. Shilnikov proved, among other, that negative  $\sigma_0$  results in the birth of a unique stable limit cycle from a homoclinic orbit. On the other hand, for  $\sigma_0 > 0$ , a wide variety of homoclinic bifurcations may occur, some of which involve infinitely many periodic orbits in the vicinity of the homoclinic orbit. The green dashed curve in Fig. 3 shows the condition  $\sigma_0 = 0$ . Along the homoclinic bifurcation line in Fig. 3 the saddle quantity changes sign, thereby allowing for both scenarios to take place. Figure 4 shows the eigenvalue spectra for two different points on the bifurcation line corresponding to negative and positive saddle quantities, respectively.

In the following, we restrict ourselves to a fixed value of  $\tau = 7$  and reveal multistability beyond the homoclinic bifurcation. For  $\tau = 7$ , a homoclinic bifurcation takes place at  $K = 0.17145$  (point A in Fig. 3). In this case  $\sigma_0 = -0.0116$ , and the bifurcation creates one stable limit cycle, with Floquet multipliers within the unit circle (inset of Fig. 5(a)). The period  $T$  of this limit cycle increases monotonically as the

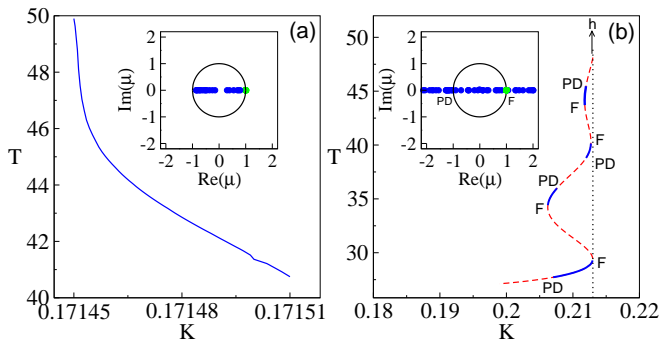


Fig. 5. (a) Period  $T$  of limit cycle born in a homoclinic bifurcation at  $(K, \tau) = (0.17145, 7)$  (point A in Fig. 3,  $\sigma_0 < 0$ ). (b) Period  $T$  of limit cycles in the multistable regime at  $(K, \tau) = (0.213, 7)$  (point Z in Fig. 3,  $\sigma_0 > 0$ ), undergoing infinitely many fold (F) and period-doubling (PD) bifurcations, before ending in a homoclinic orbit  $h$  for  $T \rightarrow \infty$  at  $K = 0.213$ . Solid blue and red dashed lines denote stable and unstable limit cycles, respectively. The insets show the two leading Floquet multipliers of the periodic orbit  $\mu_1 = 1$  (green) and  $\mu_2$  (blue) with  $K$  as a parameter in (a), and  $T$  as a parameter in (b), in the complex plane.  $b = 0.95$

bifurcation point is approached (Fig. 5(a)). Moving further along  $\tau = 7$  other homoclinic bifurcation curves are crossed, e. g. at  $K = 0.213$  (Fig. 5(b), cf. point Z in Fig. 3). There, the saddle quantity is positive ( $\sigma_0 = 0.0023$ , calculated analytically from Eq. (10)), and the picture is much more complicated: an infinite number of bifurcations take place, which are related to saddle-node (fold) bifurcations of pairs of stable and unstable limit cycles, and additional period-doubling (flip) bifurcations of the stable limit cycles. The dependence of the period  $T$  of the limit cycles upon  $K$  is a nonmonotonic multivalued function, whose turning points are associated with saddle-node bifurcations. In between the fold bifurcations of the stable limit cycles, pairs of forward and inverse period doubling bifurcations occur. In the insets of Fig. 5 the trivial Floquet multiplier  $\mu_1 = 1$  and the leading Floquet multiplier  $\mu_2$  of the periodic orbit are plotted. It can be seen how  $\mu_2$  changes as  $T$  is varied along the multivalued function in the main figure, showing how fold and flip bifurcations occur, at  $\mu_1 = 1$  and  $\mu_1 = -1$ , respectively. One should also expect other bifurcations near the critical point due to secondary homoclinic orbits which are beyond the scope of this paper.

## VI. CONCLUSIONS

In conclusion, we have presented a mechanism for delay-induced multistability in a system near a global bifurcation. In addition to the fixed point attractor which the uncontrolled system already possesses, a time-delayed feedback in the form of Pyragas difference control induces one or more co-existing limit cycle attractors. Depending upon the feedback control strength  $K$  and the delay time  $\tau$ , either a single stable limit cycle is born in a homoclinic global bifurcation, or an infinite number of (stable and unstable) periodic orbits is induced undergoing a rich menagerie of bifurcation scenarios including period doubling and fold bifurcations. We have shown that the key ingredient in the observed dynamics is a homoclinic orbit connected to a saddle-focus created by

delay. A bifurcation continuation in the  $K - \tau$  plane was performed. Moreover, we were able to verify Shilnikov's theory of homoclinic bifurcations in a certain parameter regime. The excitable nature of the system and the infinite-dimensional phase space, due to delay, appear to play a crucial role in the induced homoclinicity.

These results are interesting also from the point of view of applications, since our generic model is representative for a wide range of real-world systems. For instance, the transition from stationary to moving field domains in semiconductor superlattices has been shown to be associated with a saddle-node bifurcation on a limit cycle as described by Eq.(1) at  $K = 0$  [25], and time-delayed feedback control can also be realized in this system [9]. Already without delay, this system has been noted for its high multistability of stationary domain states [42], [43], [44], and bistability or higher multistability has been found in many other semiconductor nanostructures, see e.g. [45], [46].

## REFERENCES

- [1] K. Pyragas, "Continuous control of chaos by self-controlling feedback," *Phys. Lett. A*, vol. 170, p. 421, 1992.
- [2] E. Ott, C. Grebogi, and J. A. Yorke, "Controlling chaos," *Phys. Rev. Lett.*, vol. 64, p. 1196, 1990.
- [3] E. Schöll and H. G. Schuster, Eds., *Handbook of Chaos Control*. Weinheim: Wiley-VCH, 2007, second completely revised and enlarged edition.
- [4] J. E. S. Socolar, D. W. Sukow, and D. J. Gauthier, "Stabilizing unstable periodic orbits in fast dynamical systems," *Phys. Rev. E*, vol. 50, p. 3245, 1994.
- [5] A. Kittel, J. Parisi, and K. Pyragas, "Delayed feedback control of chaos by self-adapted delay time," *Phys. Lett. A*, vol. 198, p. 433, 1995.
- [6] N. Baba, A. Amann, E. Schöll, and W. Just, "Giant improvement of time-delayed feedback control by spatio-temporal filtering," *Phys. Rev. Lett.*, vol. 89, p. 074101, 2002.
- [7] O. Beck, A. Amann, E. Schöll, J. E. S. Socolar, and W. Just, "Comparison of time-delayed feedback schemes for spatio-temporal control of chaos in a reaction-diffusion system with global coupling," *Phys. Rev. E*, vol. 66, p. 016213, 2002.
- [8] J. Unkelbach, A. Amann, W. Just, and E. Schöll, "Time-delay autosynchronization of the spatiotemporal dynamics in resonant tunneling diodes," *Phys. Rev. E*, vol. 68, p. 026204, 2003.
- [9] J. Schlesner, A. Amann, N. B. Janson, W. Just, and E. Schöll, "Self-stabilization of high frequency oscillations in semiconductor superlattices by time-delay autosynchronization," *Phys. Rev. E*, vol. 68, p. 066208, 2003.
- [10] D. Goldobin, M. Rosenblum, and A. Pikovsky, "Controlling oscillator coherence by delayed feedback," *Phys. Rev. E*, vol. 67, p. 061119, 2003.
- [11] N. B. Janson, A. G. Balanov, and E. Schöll, "Delayed feedback as a means of control of noise-induced motion," *Phys. Rev. Lett.*, vol. 93, p. 010601, 2004.
- [12] B. Hauschildt, N. B. Janson, A. G. Balanov, and E. Schöll, "Noise-induced cooperative dynamics and its control in coupled neuron models," *Phys. Rev. E*, vol. 74, p. 051906, 2006.
- [13] W. Just, T. Bernard, M. Ostheimer, E. Reibold, and H. Benner, "Mechanism of time-delayed feedback control," *Phys. Rev. Lett.*, vol. 78, p. 203, 1997.
- [14] S. Schikora, P. Hövel, H. J. Wünsche, E. Schöll, and F. Henneberger, "All-optical noninvasive control of unstable steady states in a semiconductor laser," *Phys. Rev. Lett.*, vol. 97, p. 213902, 2006.
- [15] J. Xu and P. Yu, "Delay-induced bifurcations in a nonautonomous system with delayed velocity feedbacks," *Int. J. Bifur. Chaos*, vol. 14, p. 2777, 2004.
- [16] A. G. Balanov, N. B. Janson, and E. Schöll, "Delayed feedback control of chaos: Bifurcation analysis," *Phys. Rev. E*, vol. 71, p. 016222, 2005.
- [17] B. E. Martínez-Zérega, A. N. Pisarchik, and L. S. Tsimring, "Using periodic modulation to control coexisting attractors induced by delayed feedback," *Phys. Lett. A*, vol. 318, pp. 102–111, 2003.

- [18] B. E. Martínez-Zérega and A. N. Pisarchik, "Efficiency of the control of coexisting attractors by harmonic modulation applied in different ways," *Phys. Lett. A*, vol. 340, pp. 212–219, 2005.
- [19] G. Hu, T. Ditzinger, C. Z. Ning, and H. Haken, "Stochastic resonance without external periodic force," *Phys. Rev. Lett.*, vol. 71, p. 807, 1993.
- [20] T. Ditzinger, C. Z. Ning, and G. Hu, "Resonancelike responses of autonomous nonlinear systems to white noise," *Phys. Rev. E*, vol. 50, p. 3508, 1994.
- [21] W. J. Rappel and S. H. Strogatz, "Stochastic resonance in an autonomous system with a nonuniform limit cycle," *Phys. Rev. E*, vol. 50, p. 3249, 1994.
- [22] J. Peinke, U. Rau, W. Clauss, R. Richter, and J. Parisi, "Critical dynamics near the onset of spontaneous oscillations in p-germanium," *Europhys. Lett.*, vol. 9, p. 743, 1989.
- [23] Y. H. Shiau and Y. C. Cheng, "Boundary effect induced nonhysteretic transition caused by saddle-node bifurcation on a limit cycle in n-GaAs," *Solid State Communications*, vol. 99, no. 5, pp. 305–309, 1996.
- [24] M. Patra, G. Schwarz, and E. Schöll, "Bifurcation analysis of stationary and oscillating field domains in semiconductor superlattices with doping fluctuations," *Phys. Rev. B*, vol. 57, pp. 1824–1833, 1998.
- [25] J. Hizanidis, A. G. Balanov, A. Amann, and E. Schöll, "Noise-induced front motion: signature of a global bifurcation," *Phys. Rev. Lett.*, vol. 96, p. 244104, 2006.
- [26] J. L. A. Dubbeldam, B. Krauskopf, and D. Lenstra, "Excitability and coherence resonance in lasers with saturable absorber," *Phys. Rev. E*, vol. 60, no. 6, p. 6580, 1999.
- [27] S. Wiczorek, B. Krauskopf, and D. Lenstra, "Multipulse excitability in a semiconductor laser with optical injection," *Phys. Rev. Lett.*, vol. 88, p. 063901, 2002.
- [28] B. Krauskopf, K. Schneider, J. Sieber, S. Wiczorek, and M. Wolfrum, "Excitability and self-pulsations near homoclinic bifurcations in semiconductor laser systems," *Opt. Commun.*, vol. 215, p. 367, 2003.
- [29] A. Pikovsky and J. Kurths, "Coherence resonance in a noise-driven excitable system," *Phys. Rev. Lett.*, vol. 78, p. 775, 1997.
- [30] G. Hu, T. Ditzinger, C. Z. Ning, and H. Haken, "Stochastic resonance without external periodic force," *Phys. Rev. Lett.*, vol. 71, p. 807, 1993.
- [31] J. K. Hale, *Functional Differential Equations*. New York: Applied Mathematical Sciences Vol. 3, Springer, 1971.
- [32] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert W function," *Adv. Comput. Math.*, vol. 5, p. 329, 1996.
- [33] K. Engelborghs, T. Luzyanina, and D. Roose, "Numerical bifurcation analysis of delay differential equations using dde-biftool," *ACM Transactions on Mathematical Software*, vol. 28, pp. 1–21, 2002.
- [34] K. Engelborghs, T. Luzyanina, and G. Samaey, "Dde-biftool v. 2.00: a matlab package for bifurcation analysis of delay differential equations," Department of Computer Science, K.U.Leuven, Belgium, Tech. Rep. TW-330, 2001.
- [35] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*. New York: Springer, 1995.
- [36] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Berlin: Springer, 1986.
- [37] S. Wiggins, *Global Bifurcations and Chaos*. New York: Springer, 1988.
- [38] R. Döttling and E. Schöll, "Oscillatory bistability of real-space transfer in semiconductor heterostructures," *Phys. Rev. B*, vol. 45, p. 1935, 1992.
- [39] U. Feudel, A. Neimann, X. Pei, W. Wojtenek, H. Braun, M. Huber, and F. Moss, "Homoclinic bifurcations in a hodgkin-huxley model of thermally sensitive neurons," *Chaos*, vol. 10, p. 231, 2000.
- [40] G. Bordyugov and H. Engel, "Creating bound states in excitable media by means of nonlocal coupling," *Phys. Rev. E*, vol. 74, p. 016205, 2006.
- [41] G. C. Sethia and A. Sen, "Excitable dynamics in the presence of time delay," *Phys. Lett. A*, vol. 359, p. 285, 2006.
- [42] F. Prengel, A. Wacker, and E. Schöll, "Simple model for multistability and domain formation in semiconductor superlattices," *Phys. Rev. B*, vol. 50, p. 1705, 1994, *ibid* **52**, 11518 (1995).
- [43] J. Kastrop, H. T. Grahn, K. Ploog, F. Prengel, A. Wacker, and E. Schöll, "Multistability of the current-voltage characteristics in doped GaAs-AlAs superlattices," *Appl. Phys. Lett.*, vol. 65, p. 1808, 1994.
- [44] J. Kastrop, F. Prengel, H. T. Grahn, K. Ploog, and E. Schöll, "Formation times of electric field domains in doped GaAs-AlAs superlattices," *Phys. Rev. B*, vol. 53, p. 1502, 1996.
- [45] M. Meixner, P. Rodin, E. Schöll, and A. Wacker, "Lateral current density fronts in globally coupled bistable semiconductors with S- or Z-shaped current voltage characteristic," *Eur. Phys. J. B*, vol. 13, p. 157, 2000.
- [46] E. Schöll, *Nonlinear spatio-temporal dynamics and chaos in semiconductors*. Cambridge: Cambridge University Press, 2001, Nonlinear Science Series, Vol. 10.