CONTROL OF LIMIT CYCLE BIFURCATIONS IN THE KUKLES CUBIC SYSTEM

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Abstract
We carry out the global bifurcation analysis of the Kukles system representing a planar polynomial dynamical system with arbitrary linear and cubic right-hand sides and having an anti-saddle at the origin. Using our geometric approach, we control all possible limit cycle bifurcations and solve the problem on the maximum number and distribution of limit cycles in this system. Numerical experiments are done to illustrate the obtained results.

Key words
Kukles cubic system; field rotation parameter; bifurcation; limit cycle.

1 Introduction

In this paper, we continue studying the Kukles cubic system

$$\begin{gathered}\dot{x}=y,\\ \dot{y}=-x+\delta y+a_1x^2+a_2xy+a_3y^2\\ +a_4x^3+a_5x^2y+a_6xy^2+a_7y^3.\end{gathered} \tag{1.1}$$

I. S. Kukles was the first who began to study (1.1) solving the center-focus problem for this system in 1944: he gave the necessary and sufficient conditions for $O(0,0)$ to be a center for (1.1) with $a_7=0$ [Kukles, 1944]. Later, system (1.1) was studied by many mathematicians. In [Lloyd and Pearson, 1992], for example, the necessary and sufficient center conditions for arbitrary system (1.1), when $a_7\neq 0$, were conjectured. In [Rousseau et al., 1995], global qualitative pictures and bifurcation diagrams of a reduced Kukles system $(a_7=0)$ with a center were given. In [Wu at al., 1999], the global analysis of system (1.1) with two weak foci was carried out. In [Ye and Ye, 2001], the number of singular points under the conditions of a center or a weak focus for (1.1) was investigated. In [Zang at al., 2008], new distributions of limit cycle for the Kukles system were obtained. In [Robanal, 2014], an accurate bound of the maximum number of limit cycles in a class of Kukles type systems was provided.

In [Gaiko and van Horssen, 2004; Gaiko, 2008b], we constructed a canonical cubic dynamical system of Kukles type and carried out the global qualitative analysis of a special case of the Kukles system corresponding to a generalized cubic Liénard equation. In particular, it was shown that the foci of such a Liénard system could be at most of second order and that such system could have at most three limit cycles in the whole phase plane. Moreover, unlike all previous works on the Kukles type systems, global bifurcations of limit and separatrix cycles using arbitrary (including as large as possible) field rotation parameters of the canonical system were studied. As a result, a classification of all possible types of separatrix cycles for the generalized cubic Liénard system was obtained and all possible distributions of its limit cycles were found.

In [Gaiko, 2003, 2005, 2008a, 2009b], we also presented a solution of Hilbert's sixteenth problem in the quadratic case of polynomial systems proving that for quadratic systems four is really the maximum number of limit cycles and $(3:1)$ is their only possible distribution. We established some preliminary results on generalizing our ideas and methods to special cubic, quartic and other polynomial dynamical systems as well.

In [Gaiko, 2008b, 2009a, 2011b, 2012a], e. g., we presented a solution of Smale's thirteenth problem [Smale, 1998] proving that the classical Liénard system with a polynomial of degree $2k+1$ could have at most k limit cycles and we could conclude that our results agree with the conjecture of [Lins et al., 1977] on the maximum number of limit cycles for the classical Liénard polynomial system. In [Gaiko, 2012b, 2012c, 2014], under some assumptions on the parameters, we found the maximum number of limit cycles and their possible distribution for the general Liénard polynomial system. In [Gaiko, 2011a], we studied multiple limit cycle bifurcations in the well-known FitzHugh–Nagumo neuronal model. In [Broer and Gaiko, 2010; Gaiko, 2016], we completed the global qualitative analysis of quartic dynamical systems which model the dynamics of the populations of predators and their prey in a given ecological system.

System (1.1) can be considered as a generalized Liénard cubic system. There are many examples in the natural sciences and technology in which such and related systems are applied; see [Gaiko, 2012b, 2012c, 2014]. Such systems are often used to model either mechanical or electrical, or biomedical systems, and in the literature, many systems are transformed into Liénard type to aid in the investigations. They can be used, e. g., in certain mechanical systems with damping and restoring (stiffness), when modeling wind rock phenomena and surge in jet engines. Such systems can be also used to model resistor-inductor-capacitor circuits with non-linear circuit elements. Recently, e. g., a Liénard system has been shown to describe the operation of an optoelectronics circuit that uses a resonant tunnelling diode to drive a laser diode to make an optoelectronic voltage controlled oscillator. There are also some examples of using Liénard type systems in ecology and epidemiology [Gaiko, 2012b, 2012c, 2014]. To control natural processes occurring in such systems, especially related to periodicity and oscillations, we use so-called field rotation parameter; see [Gaiko, 2003].

In this paper, we will use the obtained results and our bifurcational geometric approach for studying limit cycle bifurcations of Kukles cubic system (1.1). In Section 2, we construct new canonical systems with field rotation parameters for studying global bifurcations of limit cycles of (1.1). In Section 3, using these canonical systems and geometric properties of the spirals filling the interior and exterior domains of limit cycles, we give a solution of the problem on the maximum number and distribution of limit cycles for Kukles system (1.1). This is related to the solution of Hilbert's sixteenth problem on the maximum number and distribution of limit cycles in planar polynomial dynamical systems [Gaiko, 2003]. Numerical experiments are also done to illustrate the theoretical results.

2 Canonical Systems

Applying Erugin's two-isocline method [Gaiko, 2003] and studying the rotation properties [Bautin and Leontovich, 1990; Gaiko, 2003; Perko, 2002] of all parameters of (1.1), we prove the following theorem.

Theorem 2.1. *Kukles system* (1.1) *with limit cycles can be reduced to the canonical form*

$$\begin{aligned} \dot{x} &= y \equiv P(x,y), \\ \dot{y} &= q(x) + (\alpha_0 - \beta + \gamma + \beta\, x + \alpha_2\, x^2)\, y \\ &\quad + (c + dx)\, y^2 + \gamma\, y^3 \equiv Q(x,y), \end{aligned} \tag{2.1}$$

where
1) $q(x) = -x + (1+1/a)\, x^2 - (1/a)\, x^3,\ a = \pm 1, \pm 2$
or
2) $q(x) = -x + b\, x^3,\ b = 0, -1,$
or
3) $q(x) = -x + x^2;$
$\alpha_0, \alpha_2, \gamma$ *are field rotation parameters and* β *is a semi-rotation parameter.*

Proof. System (1.1) has two basic isoclines: the cubic curve

$$-x + \delta y + a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x^3 + a_5 x^2 y + a_6 x y^2 + a_7 y^3 = 0$$

as the isocline of "zero" and the straight line $y = 0$ as the isocline of "infinity".

These isoclines intersect at least at one point: at the origin which is an anti-saddle (a center, a focus or a node). Besides, (1.1) can have two more finite singularities (two saddles or a saddle and an anti-saddle) or one additional finite singular point (a saddle or a saddle-node), or no other finite singularities at all. All these singular points lie on the x-axis $(y = 0)$, and their coordinates are defined by the equation

$$q(x) \equiv -x + a_1 x^2 + a_4 x^3 = 0 \tag{2.2}$$

depending just on the parameters a_1 and a_4.

Without loss of generality, $q(x)$ as given by (2.2) can be written in the following forms:
1) $q(x) \equiv -(1/a)x(x-1)(x-a)$
$= -x + (1+1/a)\, x^2 - (1/a)\, x^3,\ a = \pm 1, \pm 2$ or
2) $q(x) \equiv -x(1 - bx^2) = -x + b\, x^3,\ b = 0, -1,$ or
3) $q(x) \equiv -x(1-x) = -x + x^2.$

It means that, together with the anti-saddle in $(0, 0)$, we can have also:
1) two saddles: at $(1, 0)$ and $(-2, 0)$ for $a = -2$ or at $(1, 0)$ and $(-1, 0)$ for $a = -1$; or a saddle at $(1, 0)$ and an anti-saddle at $(2, 0)$ for $a = 2$; or a saddle-node at $(1, 0)$ for $a = 1$;
2) no other finite singularities;
3) one saddle at $(1, 0)$.

At infinity, system (1.1) has at most four singular points: one of them is in the vertical direction and the others are defined by the equation

$$a_7 u^3 + a_6 u^2 + a_5 u + a_4 = 0,\ u = y/x. \tag{2.3}$$

Instead of the parameters $\delta,\ a_2,\ a_3,\ a_5,\ a_6,\ a_7,$ also without loss of generality, we can introduce some new parameters $c,\ d,\ \alpha_0, \alpha_2, \beta, \gamma$:

$$\delta = \alpha_0 - \beta + \gamma;\ a_2 = \beta;\ a_3 = c;$$
$$a_5 = \alpha_2;\ a_6 = d;\ a_7 = \gamma$$

to have more regular rotation of the vector field of (1.1) [Gaiko, 2003].

Then, taking into account $q(x)$, equation (2.3) is written in the form

$$\begin{gathered}\gamma\, u^3 + d\, u^2 + \alpha_2\, u + s = 0,\\ u = y/x,\ s = -1/a, b.\end{gathered} \tag{2.4}$$

Thus, we have reduced (1.1) to canonical system (2.1).

If $c = d = \alpha_0 = \alpha_2 = \beta = \gamma = 0$, we obtain the following Hamiltonian systems:

$$\dot{x} = y,\ \dot{y} = -x + (1 + 1/a)\, x^2 - (1/a)\, x^3, \tag{2.5}$$
$$a = \pm 1, \pm 2;$$

$$\dot{x} = y,\ \dot{y} = -x + b\, x^3,\ b = 0, -1; \tag{2.6}$$

$$\dot{x} = y,\ \dot{y} = -x + x^2. \tag{2.7}$$

All their vector fields are symmetric with respect to the x-axis, and, besides, the fields of system (2.5) with $a = 2, -1$ and system (2.6) with $b = 0, -1$ are symmetric with respect to the straight line $x = 1$ and centrally symmetric with respect to the point $(1, 0)$. Systems (2.5)–(2.7) have the following Hamiltonians, respectively:

$$H(x, y) = x^2 - (2/3)\,(1 + 1/a)\, x^3 + (1/(2a))\, x^4 + y^2,$$
$$a = \pm 1, \pm 2;$$
$$H(x, y) = x^2 - (b/2)\, x^4 + y^2,\ b = 0, -1;$$
$$H(x, y) = x^2 - (2/3)\, x^3 + y^2.$$

If $\alpha_0 = \alpha_2 = \beta = \gamma = 0$, we will have the system

$$\dot{x} = y, \quad \dot{y} = q(x) + (c + dx)\, y^2 \tag{2.8}$$

and the corresponding equation

$$\frac{dy}{dx} = \frac{q(x) + (c + dx)\, y^2}{y} \equiv F(x, y). \tag{2.9}$$

Since $F(x, -y) = -F(x, y)$, the direction field of (2.9) (and the vector field of (2.8) as well) is symmetric with respect to the x-axis. It follows that system (2.8) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, without loss of generality, the parameters c and d in system (2.1) can be fixed.

To prove that the parameters α_0, α_2, γ and β rotate the vector field of (2.1), let us calculate the following determinants:

$$\Delta_{\alpha_0} = P\, Q'_{\alpha_0} - QP'_{\alpha_0} = y^2 \geq 0,$$
$$\Delta_{\alpha_2} = P\, Q'_{\alpha_2} - QP'_{\alpha_2} = x^2 y^2 \geq 0,$$
$$\Delta_{\gamma} = P\, Q'_{\gamma} - QP'_{\gamma} = y^2(1 + y^2) \geq 0,$$
$$\Delta_{\beta} = P\, Q'_{\beta} - QP'_{\beta} = (x - 1)\, y^2.$$

By definition of a field rotation parameter [Bautin and Leontovich, 1990; Gaiko, 2003], for increasing each of the parameters α_0, α_2 and γ, under the fixed others, the vector field of system (2.1) is rotated in positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (2.1) is rotated in negative direction (clockwise). For increasing the parameter β, under the fixed others, the vector field of system (2.1) is rotated in positive direction (counterclockwise) in the half-plane $x > 1$ and in negative direction (clockwise) in the half-plane $x < 1$ and vice versa for decreasing this parameter. We will call such a parameter as a semi-rotation one.

Thus, for studying limit cycle bifurcations of (1.1), it is sufficient to consider canonical system (2.1) containing the field rotation parameters α_0, α_2, γ and the semi-rotation parameter β. The theorem is proved. □

3 Global Bifurcations of Limit Cycles

By means of our bifurcational geometric approach [Gaiko and van Horssen, 2004; Gaiko, 2008b, 2009a, 2011b, 2012a, 2012b, 2012c, 2014, 2015, 2016], we will consider now the Kukles cubic system in the form (when $a = 2$):

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = -(1/2)x(x - 1)(x - 2)\\ + (\alpha_0 - \beta + \gamma + \beta\, x + \alpha_2\, x^2)\, y\\ + (c + dx)\, y^2 + \gamma\, y^3.\end{gathered} \tag{3.1}$$

All other Kukles systems can be considered in a similar way. Using system (3.1), we will prove the following theorem.

Theorem 3.1. *Kukles cubic system* (1.1) *can have at most four limit cycles in* $(3\!:\!1)$*-distribution.*

Proof. According to Theorem 2.1, for the study of limit cycle bifurcations of system (1.1), it is sufficient to consider canonical system (2.1) containing the field rotation parameters α_0, α_2, γ and the semi-rotation parameter β. We will work with system (3.1) which has three finite singularities: a saddle $S(1, 0)$ and two anti-saddles, $O(0, 0)$ and $A(2, 0)$.

Vanishing all of the rotation parameters α_0, α_2, γ and also the parameter β, we will get the system

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = -(1/2)x(x - 1)(x - 2)\\ +(c + dx)\, y^2\end{gathered} \tag{3.2}$$

which is symmetric with respect to the x-axis and has centers as anti-saddles at the points $O(0, 0)$ and $A(2, 0)$. Its center domains are bounded by separatrix loops of the saddle $S(1, 0)$.

Let us input successively the field rotation parameters into (3.2). Begin with the parameter α_0 supposing that $\alpha_0 > 0$:

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = -(1/2)x(x - 1)(x - 2)\\ + \alpha_0\, y + (c + dx)\, y^2.\end{gathered} \tag{3.3}$$

On increasing α_0, the vector field of (3.3) is rotated in positive direction (counterclockwise) and the centers O and A turn into unstable foci.

Fix α_0 and input the parameter $\beta > 0$ into (3.3):

$$\begin{gathered}\dot{x} = y, \\ \dot{y} = -(1/2)x(x-1)(x-2) \\ +(\alpha_0 - \beta + \beta x)\, y + (c + dx)\, y^2.\end{gathered} \tag{3.4}$$

Then, in the half-plane $x > 1$, the vector field of (3.4) is rotated in positive direction again and the focus A remains unstable; in the half-plane $x < 1$, the vector field is rotated in negative direction and, when $\beta = \alpha_0 > 0$, the focus O becomes weak. Fix this value of the parameter $\beta = \beta^{AH}$ (the Andronov–Hopf bifurcation value).

Fix the parameters $\alpha_0 > 0$, $\beta = \beta^{AH} > 0$ and input the third parameter, $\alpha_2 < 0$, into this system:

$$\begin{gathered}\dot{x} = y, \\ \dot{y} = -(1/2)x(x-1)(x-2) \\ +(\alpha_0 - \beta + \beta x + \alpha_2 x^2)\, y + (c + dx)\, y^2.\end{gathered} \tag{3.5}$$

The vector field of (3.5) is rotated in negative direction (clockwise) and a big stable limit cycle appears immediately from infinity. Denote this cycle by Γ_1^{bc}.

On decreasing α_2, the cycle Γ_1^{bc} will contract and, for some value $\alpha_2 = \alpha_2^{8l}$, a separatrix eight-loop of the saddle S will be formed around the points O and A. On further decreasing α_2, two stable limit cycle, Γ_1^O and Γ_1^A, will appear from the eight-loop surrounding O and A, respectively. These cycles will contract and, finally, will disappear at the foci O and A.

Suppose that on decreasing α_2, the limit cycle Γ_1^O and Γ_1^A still exist and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding the points O and A.

Denote the domains outside the cycle Γ_1^O and Γ_1^A by D_1^O and D_1^A, the domains inside the cycles by D_2^O and D_2^A, respectively. It is clear that on decreasing α_2, a semi-stable limit cycle cannot appear in the domains D_1^O and D_1^A, since the focus spirals filling these domains will untwist and the distance between their coils will increase because of the vector field rotation in negative direction.

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domains D_2^O and D_2^A. Suppose it appears in a domain for some values of the parameters: $\alpha_0^* > 0$, $\alpha_2^* < 0$, $\beta^{AH} > 0$. Return to initial system (3.2) and change the order of inputting the field rotation parameters.

Input first the parameter $\alpha_2 < 0$:

$$\begin{gathered}\dot{x} = y, \\ \dot{y} = -(1/2)x(x-1)(x-2) \\ + \alpha_2\, x^2 y + (c + dx)\, y^2.\end{gathered} \tag{3.6}$$

Fix it under $\alpha_2 = \alpha_2^*$. The vector field of (3.6) is rotated in negative direction and the points O and A become stable foci.

Inputting the parameter $\beta > 0$ into (3.6), we will have the system

$$\begin{gathered}\dot{x} = y, \\ \dot{y} = -(1/2)x(x-1)(x-2) \\ +(-\beta + \beta x + \alpha_2\, x^2)\, y + (c + dx)\, y^2,\end{gathered} \tag{3.7}$$

the vector field of which is rotated in positive direction in the half-plane $x > 1$ and in negative direction in the half-plane $x < 1$. Fix it under $\beta = \beta^{AH}$.

Inputting the parameter $\alpha_0 > 0$ into (3.7), we will get again system (3.5), where the vector field is rotated in positive direction. Under this rotation, stable limit cycles, Γ_1^O and Γ_1^A, will appear from the foci O and A, when they change the character of stability. These cycles will expand, the focus spirals will untwist and the distance between their coils will increase on increasing the parameter α_0 to the value $\alpha_0 = \alpha_0^*$. It follows that there are no values of $\alpha_0 = \alpha_0^* > 0$, $\alpha_2 = \alpha_2^* < 0$ and $\beta = \beta^{AH} > 0$, for which a semi-stable limit cycle could appear in the domains D_2^O and D_2^A.

Thus, we have proved the uniqueness of limit cycles surrounding the points O and A for $\alpha_0 > 0$, $\alpha_2 < 0$ and $\beta = \beta^{AH} > 0$. Similarly, it can be proved the uniqueness of a big limit cycle surrounding all the finite singularities O, A and S for this set of the parameters.

Consider again system (3.5) for $\alpha_0 > 0$, $\alpha_2 < 0$ and $\beta = \beta^{AH} > 0$ supposing that it has two stable limit cycles, Γ_1^O and Γ_1^A. Change the parameter β : $\beta > \beta^{AH} = \alpha_0 > 0$. On increasing this parameter, the weak focus O will become rough stable one generating an unstable limit cycle, Γ_2^O (the Andronov–Hopf bifurcation). On further increasing β, the limit cycle Γ_2^O will join with Γ_1^O forming a semi-stable limit cycle, Γ_{12}^O, which will disappear in a "trajectory concentration" surrounding the point O. Can another semi-stable limit cycle appear around this point in addition to Γ_{12}^O? It is clear that such a limit cycle cannot appear either in the domain D_3^O bounded by the origin O and Γ_2^O or in the domain D_1^O bounded on the inside by Γ_1^O because of the increasing distance between the spiral coils filling these domains under increasing β.

To prove impossibility of the appearance of a semi-stable limit cycle in the domain D_2^O bounded by the cycles Γ_1^O and Γ_2^O (before their joining), suppose the contrary, i. e., for some set of values of the parameters $\alpha_0^* > 0$, $\alpha_2^* < 0$ and $\beta^* > 0$, such a semi-stable cycle exists. Return to system (3.2) again and input the parameters $\alpha_2 < 0$ and $\beta > 0$ getting system (3.7). In the half-plane $x < 1$, both parameters act in a similar way: they rotate the vector field of (3.7) in negative direction turning the origin O into a stable focus. In the half-plane $x > 1$, they rotate the field in opposite directions generating a stable limit cycle from the focus A.

Fix these parameters under $\alpha_2 = \alpha_2^*$, $\beta = \beta^*$ and input the parameter $\alpha_0 > 0$ into (3.7) getting again system (3.5). Since, by our assumption, this system has two limit cycles for $\alpha_0 < \alpha_0^*$, there exists some value of the parameter, α_0^{12} $(0 < \alpha_0^{12} < \alpha_0^*)$, for which a semi-stable limit cycle, Γ_{12}^O, appears in system (3.5) and then splits into a stable cycle, Γ_1^O, and an unstable cycle, Γ_2^O, on further increasing α_0. The formed domain D_2^O, bounded by the limit cycles Γ_1^O, Γ_2^O and filled by the spirals, will enlarge since, by the properties of a field rotation parameter, the interior unstable limit cycle Γ_2^O will contract and the exterior stable limit cycle Γ_1^O will expand on increasing α_0. The distance between the spirals of the domain D_2^O will naturally increase, what will prevent the appearance of a semi-stable limit cycle in this domain for $\alpha_0 > \alpha_0^{12}$. Thus, there are no such values of the parameters $\alpha_0^* > 0$, $\alpha_2^* < 0$ and $\beta^* > 0$, for which system (3.5) would have an additional semi-stable limit cycle.

Obviously, there are no other values of the parameters α_0, α_2 and β, for which system (3.5) would have more than two limit cycles surrounding the point O and simultaneously more than one limit cycle surrounding the point A (on the same reasons). It follows that system (3.5) can have at most three limit cycles and only in the $(2:1)$-distribution.

Suppose that system (3.5) has two limit cycles, Γ_1^O and Γ_2^O, around the origin O and the only limit cycle, Γ_1^A, around the point A. Fix the parameters $\alpha_0 > 0$, $\alpha_2 < 0$, $\beta > 0$ and input the fourth parameter, $\gamma > 0$, into (3.5) getting system (3.1). On increasing γ, the vector field of (3.1) is rotated in positive direction, the focus O changes the character of its stability, when $\gamma = \beta - \alpha_0$, and a stable limit cycle, Γ_3^O, appears from the origin, since the parameter α_2 is non-rough and negative when $\gamma = \beta - \alpha_0$ (the Andronov–Hopf bifurcation). On further increasing γ, the cycle Γ_3^O will join with Γ_2^O forming a semi-stable limit cycle, Γ_{23}^O, which will disappear in a "trajectory concentration" surrounding the origin O; the other cycles, Γ_1^O and Γ_1^A, will expand disappearing in a separatrix eight-loop of the saddle S.

Let system (3.1) have four limit cycles: Γ_1^O, Γ_2^O, Γ_3^O and Γ_1^A. Can an additional semi-stable limit cycle appear around the origin on increasing the parameter γ? It is clear that such a limit cycle cannot appear either in the domain D_2^O bounded by Γ_1^O and Γ_2^O or in the domain D_4^O bounded by the origin and Γ_3^O because of the increasing distance between the spiral coils filling these domains on increasing γ. Consider two other domains: D_1^O bounded on the inside by the cycle Γ_1^O and D_3^O bounded by the cycles Γ_2^O and Γ_3^O. As before, we will prove impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters, $\alpha_0^* > 0$, $\alpha_2^* < 0$, $\beta^* > 0$ and $\gamma^* > 0$, such a semi-stable cycle exists. Return to system (3.2) again and input first the parameters $\alpha_0 > 0$, $\gamma > 0$ and then the parameter $\alpha_2 < 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -(1/2)x(x-1)(x-2) \\ &+(\alpha_0 + \gamma + \alpha_2\, x^2)\, y + (c + dx)\, y^2 + \gamma\, y^3. \end{aligned} \tag{3.8}$$

Fix the parameters α_0, γ under the values α_0^*, γ^*, respectively. On decreasing the parameter α_2, a big stable limit cycle Γ_1^{bc} appears from infinity and then it contracts forming a separatrix eight-loop of the saddle S around the points O and A. On further decreasing α_2, two stable limit cycle, Γ_1^O and Γ_1^A, will appear from the eight-loop surrounding O and A, respectively. Fix α_2 under the value α_2^* and input the parameter $\beta > 0$ into (3.8) getting system (3.1).

Since, by our assumption, system (3.1) has three limit cycles around the origin O for $\beta < \beta^*$, there exists some value of the parameter, β_{23} $(0 < \beta_{23} < \beta^*)$, for which a semi-stable limit cycle, Γ_{23}^O, appears in this system and then it splits into an unstable cycle, Γ_2^O, and a stable cycle, Γ_3^O, on further increasing β. The formed domain D_3^O bounded by the limit cycles Γ_2^O, Γ_3^O and also the domain D_1^O bounded on the inside by the limit cycle Γ_1^O will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there, i. e., at most three limit cycles can exist around the origin O. On the same reasons, a semi-stable limit cannot appear around the point A on increasing the parameter β, i. e., at most one limit cycle can exist around this point simultaneously with at most three limit cycles surrounding the origin.

All other combinations of the parameters α_0, α_2, β and γ are considered in a similar way. It follows that system (3.1) can have at most four limit cycles and only in the $(3:1)$-distribution. The same conclusion can be made for system (1.1). The theorem is proved. $\square$

We have done also numerical simulations supporting our results based on a Runge-Kutta method using a so-called function of limit cycles introduced in [Gaiko, 2003] which is a function of a field rotation parameter depending on a coordinate of the limit cycle and applying a flow curvature method [Ginoux, 2009] and some other numerical methods [Van 't Wout et al., 2016; Vuik et al., 2015].

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