Speed-gradient Adaptive High-gain Observers for Synchronization of Chaotic Systems

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Abstract

We address the problem of output feedback synchronization of certain chaotic systems, under parameter uncertainty. That is, given a *master* system, the objective is to design a *slave* system that copies the dynamics of the master and reconstructs both the state and the values of the constant parameters of the master system. Hence, the synchronization problem that we address enters in the framework of Pecora and Carroll and relies on adaptive observer theory. In particular, the conditions that we impose take the form of *persistency of excitation*.

1 Introduction

Since the celebrated paper [?] master-slave synchronization of chaotic systems has gained an increasing interest, specifically but not only, due to the applications of this problem into secured communication; see for instance [?, ?, ?] to cite a few. Using chaotic systems to transmit and receive information has several advantages as opposed to more conventional methods relying on periodic carrier signals: 1) chaotic modulation offers a better performance since the correlation of waves is lower than in the case of conventional periodic carriers; 2) it may out-perform conventional methods in the case of noisy channels; 3) chaotic modulation presents robust wide-band communications; etc.

In the classic master-slave or transmitter-receiver scheme, a master circuit is tunned to transmit information using a chaotic carrier signal. The signal is received by a "slave" circuit which, if it can be constructed identically to the master, the information may be decoded out of the chaotic carrier. In practice, it is impossible to repeat the master circuit with the exact values of its components even when these values are known. To this, we add the fact that the information is transmitted through a non-ideal channel. All this uncertainty stymies considerably the faculty of reconstructing the useful information.

In this paper, we present an adaptive approach to synchronization which relies on adaptive observer design. As it has been shown in the important paper [?] the synchronization problem may be recasted in a problem of observer-design, well known in the literature of control systems theory. Different observer-based synchronization schemes have been proposed in the literature, *e.g.* relying on sliding modes: [?]; high-gain: [?]; Luenberger-based observers: [?], *etc.* We propose an adaptive observer for a class of systems that covers certain chaotic systems. Then, we give sufficient conditions to achieve master-slave synchronization in the event of parameter uncertainty and assuming that only an output – possibly part of the master's state – is available for measurement.

The rest of the paper is organized as follows. In coming section we introduce some notation and definitions of stability that set the framework for our main results. In Section 3 we present an adaptive observer for a class of detectable systems and give examples of chaotic systems that fit in our framework. In Section 5 we present the proofs of our findings, before concluding with some remarks.

2 Preliminaries

Notation. We say that a function $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathcal{A}$ with $\mathcal{A}$ a closed, not necessarily compact set, satisfies the basic regularity assumption (BRA) if $\phi(t, \cdot)$ is locally Lipschitz uniformly in t and $\phi(\cdot, x)$ is measurable. We denote the usual Euclidean norm of vectors by $|\cdot|$ and use the same symbol for the matrix induced norm. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\alpha \in \mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$, $\beta(s, \cdot)$ is strictly decreasing and $\lim_{t\to\infty} \beta(s, t) = 0$. We denote the solution of a differential equation $\dot{x} = f(t, x)$ starting at $x_\circ$ at time $t_\circ$ by $x(\cdot, t_\circ, x_\circ)$; furthermore, if the latter are defined for all $t \geq t_\circ$ we say that the system is forward complete.

[Uniform global stability] The origin of

$$\dot{x} = f(t, x) \tag{1}$$

where $f(\cdot, \cdot)$ satisfies the BRA, is said to be uniformly globally stable (UGS) if there exists $\kappa \in \mathcal{K}_\infty$ such that, for each $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, each solution $x(\cdot, t_\circ, x_\circ)$ of (1) satisfies

$$|x(t, t_\circ, x_\circ)| \leq \kappa(|x_\circ|) \qquad \forall\, t \geq t_\circ\,. \tag{2}$$

[Uniform global asymptotic stability] The origin of (1) is said to be uniformly globally asymptotically stable (UGAS) if it is UGS and uniformly globally attractive, *i.e.*, for each pair of strictly positive real numbers (r, σ), there exists $T > 0$ such that for each solution

$$|x_\circ| \leq r \implies |x(t, t_\circ, x_\circ)| \leq \sigma \qquad \forall\, t \geq t_\circ + T\,.$$

[UES] The origin of the system $\dot{x} = f(t, x)$ is said to be uniformly exponentially stable on any ball if for any $r > 0$ there exist two constants k and $\gamma > 0$ such that, for all $t \geq t_\circ \geq 0$ and all $x_\circ \in \mathbb{R}^n$ such that $|x_\circ| < r$

$$|x(t, t_\circ, x_\circ)| \leq k\, |x_\circ|\, e^{-\gamma(t-t_\circ)}\,. \tag{3}$$

[Uniform Semiglobal Practical Asymptotic Stability] The origin of (1) is said to be uniformly semiglobally practically asymptotically stable (USPAS) if for each positive real numbers $\Delta > \delta > 0$ and $\sigma > 0$ there exist $T > 0$ and $\kappa \in \mathcal{K}_\infty$ such that $|x(t, t_\circ, x_\circ)| \leq \kappa(|x_\circ|)$ for all $t \geq t_\circ \geq 0$ and

$$|x_\circ| \leq \Delta \implies |x(t, t_\circ, x_\circ)| \leq \sigma + \delta \qquad \forall\, t \geq t_\circ + T\,.$$

3 Adaptive Observers With Persistency of Excitation

Consider a nonlinear system of the form

$$\dot{x} = A(y)x + \Psi(x)\theta + B(t, x) \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector; $\theta \in \Theta$ is a vector of unknown constant parameters and Θ is a compact of $\mathbb{R}^m$; $y = Cx$ is a measurable output; the pair $(A(y(t)), C)$ is detectable, that is $y(t) \equiv 0$ implies that $x(t) \to 0$; the functions Ψ and B are globally Lipschitz, *i.e.* there exist ψ_M and b_M such that, for any vectors $\zeta \in \mathbb{R}^m$, with $|\zeta| = 1$, x_1, $x_2 \in \mathbb{R}^n$ and all $t \geq 0$,

$$|\Psi(x_1)\zeta - \Psi(x_2)\zeta| \leq \psi_M |x_1 - x_2| \qquad (5a)$$

$$|B(t, x_1) - B(t, x_2)| \leq b_M |x_1 - x_2| \,. \qquad (5b)$$

Moreover, there exists $\psi_0 \geq 0$ such that

$$\max_{|\zeta|=1} \left|\zeta^\top \Psi(0)\zeta\right| \leq \psi_0 \,. \qquad (6)$$

Under these conditions we propose for systems of the form (4), the adaptive observer

$$\dot{\hat{x}} = A(y)\hat{x} - L(t, y)C(x - \hat{x}) + B(t, \hat{x}) + \Psi(\hat{x})\hat{\theta} \qquad (7)$$

where $L(\cdot, \cdot)$ satisfies the basic regularity assumption. Using (4), and defining $\bar{x} := \hat{x} - x$, $\bar{\theta} := \hat{\theta} - \theta$ the estimation error dynamics is given by

$$\dot{\bar{x}} \;=\; [\,A(y) - L(t, y)C\,]\bar{x} + \Psi(\bar{x} + x(t))\bar{\theta} + \Phi(t, \bar{x}, x(t), \theta) \qquad (8a)$$

$$\Phi(t, \bar{x}, x(t), \theta) \;:=\; [\,\Psi(\bar{x} + x(t)) - \Psi(x(t))\,]\theta + B(t, \bar{x} + x(t)) - B(t, x(t)) \qquad (8b)$$

Conditions (5) and the assumption that $\theta \in \Theta$ where Θ is a compact of appropriate dimension imply that there exists $\theta_M > 0$ such that

$$|\Phi(t, \bar{x}, x(t), \theta)| \leq \psi_M \theta_M |\bar{x}| + b_M |\bar{x}| =: \phi_M |\bar{x}| \,. \qquad (9)$$

The following assumption on the observer gain L guarantees that the state estimation errors tend uniformly to zero; roughly the condition is that the gain L, through the measurable output $y(t)$, makes the error dynamics persistently excited. Define $y_t := y(t)$ for each t. There exists a globally bounded positive definite matrix function $P(\cdot)$ such that $p_M \geq |P|$ and, defining $\bar{A}(t, y_t) := A(y_t) - L(t, y_t)C$, $-Q(t, y_t) := \bar{A}(y_t)^\top P(t) + P(t)^\top \bar{A}(y_t) + \dot{P}(t)$ we have the following for all $t \geq 0$ and all $y_t \in \mathbb{R}^m$

1. $Q(t, y_t) \geq 0$
2. There exist μ and $T > 0$ such that
$$\int_t^{t+T} Q(\tau, y_\tau) d\tau \;\geq\; \mu I > 0 \,, \qquad \forall\, t \geq 0 \qquad (10)$$
3. There exists $q_M > 0$ such that $q_M \geq |Q(t, y_t)|$.

We remark for further development that Inequality (10), which is known as *persistency of excitation*, is equivalent to

$$\int_t^{t+T} \xi^\top Q(\tau, y_\tau)\xi d\tau \;\geq\; \mu, \qquad \forall\, t \geq 0, \quad |\xi| = 1 \,.$$

Next, consider the adaptation law

$$\dot{\hat{\theta}}(t) = -\gamma \Psi(\hat{x}(t))^\top P(t)\hat{x}(t) \,, \quad \gamma > 0 \qquad (11)$$

which, considering that $\dot{\theta} = 0$, is equivalent to

$$\dot{\bar{\theta}} = -\gamma \Psi(\bar{x} + x(t))^\top P(t)\bar{x} - \gamma \Psi(\bar{x} + x(t))^\top P(t)x(t) \,, \quad \gamma > 0 \,. \qquad (12)$$

In order to guarantee that the parameter errors $\bar{\theta}(t) \to 0$ we shall also impose a persistency-of-excitation condition on the function $\Psi(x(t))$: The function $\Psi(x(t))$ is such that there exist positive numbers μ_ψ and T_ψ such that, for any unitary vector $\zeta \in \mathbb{R}^m$,

$$\int_t^{t+T_\psi} |\Psi(x(\tau))\zeta|\, d\tau \;\geq\; \mu_\psi \,, \qquad \forall\, t \geq 0 \,. \qquad (13)$$

Under these conditions we have the following. The origin of the estimation error dynamics corresponding to $\bar{x}$ and $\bar{\theta}$, *i.e.* equations (8) and (12), is uniformly semi-globally practically asymptotically stable provided that: 1) conditions (5) hold; 2) Assumptions 3 and 3 are satisfied; 3) the solutions $x(t)$ and their derivatives $\dot{x}(t)$ are bounded for all t.

Roughly, Proposition 3 establishes conditions for the state and parameter estimation errors to converge to an arbitrarily small neighborhood of the origin. In the context of master-slave synchronization of chaotic systems, Proposition 3 establishes conditions under which two chaotic systems with unknown constant parameters, synchronize, in the event that only an output of the master system is measurable.

In the present context of synchronization, conditions (5) are mild regularity properties that are satisfied by a number of chaotic systems as we shall illustrate below. The assumption on $x(t)$ is not restrictive either in the present context if we assume that $x(t)$ corresponds to the solutions of an ordinary differential equation $\dot{x} = f(t, x, \theta)$ such that for a particular choice of θ the system exhibits a chaotic behavior and therefore, $x(t)$ is bounded. Boundedness of $\dot{x}(t)$ follows directly from the usual hypotheses imposed on f to guarantee existence and uniqueness of solutions. The only conditions that are, *in general*, hard to verify are the persistency-of-excitation conditions; yet, we stress that this property has been showed to be *necessary* for parameter convergence, in the context of adaptive control (see *e.g.* [**?**], [**?**]).

Assumption 3 is a structural condition on the function $\Psi(\cdot)$ as well as on the *richness* of its trajectories $x(t)$. A particular method to verify Assumption 3 on the PE of the observer gain is using high-gain observers – *cf.* [**?**, **?**, **?**]. According with [**?**], we make the following *detectability* hypothesis: Let $\Phi_x(t, t_\circ)$ denote the transition matrix associated to $A(y_t)$, *i.e.*, the solution of

$$\begin{cases} \dot{\Phi}_x(t, t_\circ) = A(y_t)\Phi_x(t, t_\circ)\,, \\ \Phi_x(t_\circ, t_\circ) = I\,. \end{cases}$$

Assume that there exist positive numbers T_x and μ_x, such that, for all $t \geq 0$

$$\int_t^{t+T_x} \Phi_x(\tau, t)^\top C^\top C \Phi_x(\tau, t) d\tau \geq \mu_x I\,. \quad (14)$$

Next, for any given $\rho_x > 0$, we define the observer gain $L(t, y)$ as

$$\begin{aligned} L(t, y_t) &:= P(t, y_t)^{-1} C^\top \\ \dot{P}(t, y_t) &= 2C^\top C - \rho_x I - P(t, y_t)A(y_t) - A(y_t)^\top P(t, y_t)\,, \quad \forall t \\ P(t, y_{t_\circ}) &= P_\circ = P_\circ^\top > 0 \quad \forall\, t \in [t_\circ, t_\circ + T_x]. \end{aligned}$$

It can be shown that, under Assumption 3, one has $P(t, y_t) \geq \mu_x e^{-\rho_x T_x} I$ for all $t \geq t_\circ + T_x$. On the other hand, a direct calculation yields that the matrix

$$-Q(t, y_t) \;:=\; [A(y_t) - L(t, y_t)C]^\top P(t) + P(t)^\top [A(y_t) - L(t, y_t)C$$

with P and L given by (15) and (16), satisfies $Q(t, y_t) \equiv \rho_x I$, hence Assumption 3 is trivially satisfied.

4 Adaptive Synchronization via PE Observers

From the previous general developments, we draw the following conclusions in the context of master-slave synchronization.

Consider a chaotic *master* system of the form (4) where θ is such that the solutions $x(t)$ exhibit a chaotic behavior. Let $y = Cx$ be a measurable output of the master system. Construct a slave system according to the dynamics (7), (11). Then, under the conditions of Proposition 3 a slave system synchronizes with the master, in the sense that $\hat{x}(t)$ approaches $x(t)$ arbitrarily close as $t \to \infty$. In particular, we have the following:

1. in the case that the parameters θ are unknown, the errors $|x(t) - \hat{x}(t)|$ and $\left|\theta - \hat{\theta}\right|$ approach an arbitrarily small neighborhood of the origin as $t \to \infty$. Moreover, the size of this neighborhood may be reduced by increasing the persistency of excitation, *i.e.* μ_x and μ;

2. in the case that the parameters θ are unknown but $C = I$, *i.e.* the whole master system's state is measurable, perfect synchronization occurs and the parameters θ may be estimated if the persistency of excitation condition (13) holds;

3. in the case that the parameters θ are known, the slave system will achieve perfect synchronization provided that the persistency of excitation condition imposed in Assumption 3 holds.

4.1 Example: Lorenz system

For illustration we apply our main result in the estimation of on estate and two parameters of the well-known chaotic Lorentz system. The latter is given by

$$\dot{x}_1 = \theta_1(x_2 - x_1) \tag{19}$$

$$\dot{x}_2 = \theta_2 x_1 - x_2 - x_1 x_3 \tag{20}$$

$$\dot{x}_3 = x_1 x_2 - \theta_3 x_3 \,. \tag{21}$$

We assume to measure $y_1 = x_1$, $y_3 = x_3$ and that we know θ_1. Under such conditions the system can be rewritten in the form (4) with $y = x_1$,

$$A(y) := \begin{bmatrix} -\theta_1 & \theta_1 & 0 \\ 0 & -1 & -y_1 \\ 0 & y_1 & 0 \end{bmatrix}, \quad \Psi(x) := \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{bmatrix}. \tag{22}$$

Again, the functions above satisfy the required regularity conditions imposed in Proposition 3.

We tested the proposed algorithm in simulation under the following conditions: 1) For a chaotic behavior we chose $\theta_1 = 16$, $\theta_2 = 45.6$ and $\theta_3 = 4$; 2) the initial states are: $x(0) = [1; 1; 1]$, $\hat{x}(0) = [0; 0; 0]$, $\hat{\theta}_1(0) = 15$, $\hat{\theta}_2(0) = 47$, $\hat{\theta}_3(0) = 25$; the gains are set to $\rho_x = 150$, $\gamma = 0.0001$,

$$P_\circ = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 8 & 3 \\ 2 & 3 & 9 \end{bmatrix}$$

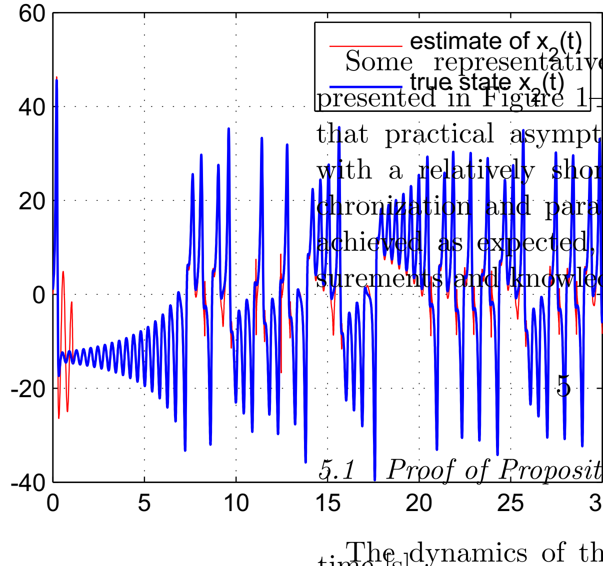

Figure 1. Plots of estimate (slave) and true (master) states $\hat{x}_2$ and x_2

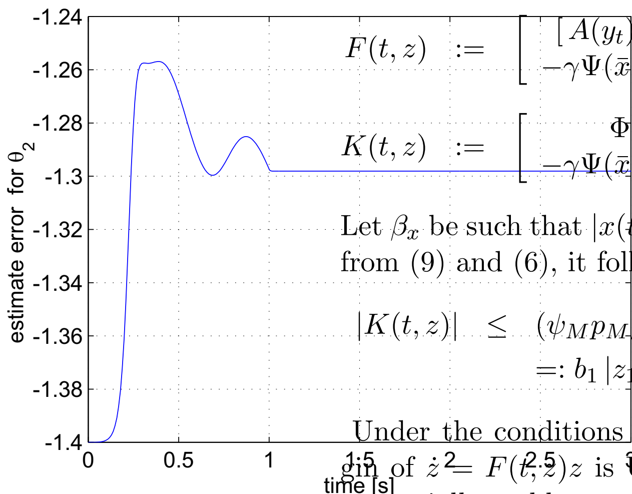

Figure 2. Estimate error for the second parameter, *i.e.* $\bar{\theta}_2$

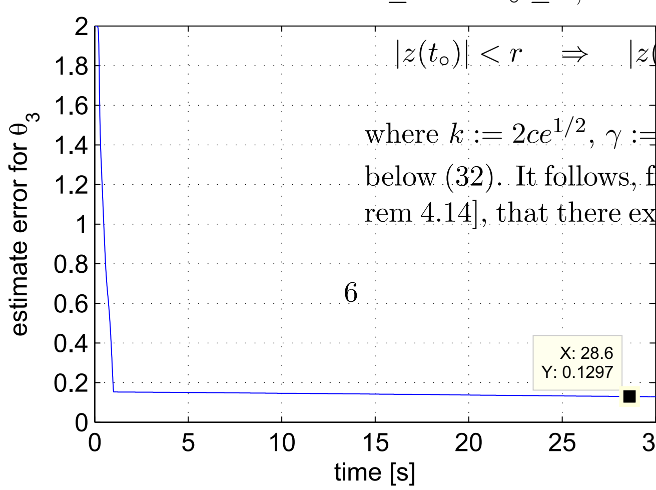

Figure 3. Estimate error for the third parameter, *i.e.* $\bar{\theta}_3$

Some representative simulation results are presented in Figure 1–3. It may be appreciated that practical asymptotic stability is achieved with a relatively short transient. Exact synchronization and parameter estimation are not achieved as expected, due to the lack of measurements and knowledge of the parameters.

5 Proofs

5.1 *Proof of Proposition 3*

The dynamics of the estimation errors $z := \mathrm{col}[\bar{x}, \bar{\theta}]$ is given by

$$\dot{z} = F(t,z)z + K(t,z) \tag{23a}$$

$$F(t,z) := \begin{bmatrix} [A(y_t) - L(t,y_t)C] & -\gamma\Psi(\bar{x}+x(t)) \\ -\gamma\Psi(\bar{x}+x(t))^\top P(t) & 0 \end{bmatrix} \tag{23b}$$

$$K(t,z) := \begin{bmatrix} \Phi(t,\bar{x},x(t),\theta) \\ -\gamma\Psi(\bar{x}+x(t))^\top P(t)x(t) \end{bmatrix}. \tag{23c}$$

Let β_x be such that $|x(t)| \le \beta_x$ for all $t \ge 0$ then, from (9) and (6), it follows that

$$|K(t,z)| \le (\psi_M p_M \beta_x + \phi_M)|z_1| + p_M \psi_M \beta_x \tag{24a}$$

$$=: b_1|z_1| + b_2\,. \tag{24b}$$

Under the conditions of Proposition 3 the origin of $\dot{z} = F(t,z)z$ is UGAS and uniformly exponentially stable on any ball. From the proof of Claim 5.1 – *cf.* Section 5.2 we obtain, for any $r \ge 0$ and $t_\circ \ge 0$,

$$|z(t_\circ)| < r \quad \Rightarrow \quad |z(t)| \le k\,|z(t_\circ)|\,\mathrm{e}^{-\gamma(t-t_\circ)} \tag{25}$$

where $k := 2ce^{1/2}$, $\gamma := \dfrac{1}{2c^2}$ and $c > 0$ is defined below (32). It follows, from the proof of [**?**, Theorem 4.14], that there exists $V_4 : \mathbb{R}_{\ge 0} \times B_R \to \mathbb{R}_{\ge 0}$

with $R := kr$, such that

$$\left(\frac{1-\mathrm{e}^{-2q_M T}}{2q_M}\right)|z|^2 \le V_4(t,z) \le \left(\frac{1-\mathrm{e}^{-2\gamma T}}{2\gamma}\right)|z|^2$$
$$\frac{\partial V_4}{\partial t} + \frac{\partial V_4}{\partial z}F(t,z)z \le -(1-e^{-2\gamma T})|z|^2$$
$$\left|\frac{\partial V_4}{\partial z}\right| \le \frac{2}{\gamma - q_M}\left[1-\mathrm{e}^{-(\gamma-q_M)T}\right].$$

Evaluating the time derivative of $V_4(t,z)$ along the trajectories of (23a) and using (24) we obtain

$$\dot V_4(t,z) \le -(1-e^{-2\gamma T})|z|^2 + \frac{2}{\gamma-q_M}\left[1-\mathrm{e}^{-(\gamma-q_M)T}\right](b_1|z|^2 + b_2|z|)$$

hence if, for any given $\epsilon > 0$, b_1, b_2 and z satisfy

$$b_1 \le \frac{\left(1-\mathrm{e}^{-2\gamma T}-\epsilon\right)(\gamma-q_M)}{4\left[1-\mathrm{e}^{-(\gamma-q_M)T}\right]} \qquad (26)$$
$$|z| \ge b_2\frac{4\left[1-\mathrm{e}^{-(\gamma-q_M)T}\right]}{(1-\mathrm{e}^{-2\gamma T}-\epsilon)(\gamma-q_M)} \qquad (27)$$

we obtain

$$\dot V_4(t,z) \le -\epsilon|z|^2 .$$

It follows that the solutions are uniformly ultimately bounded – *cf.* [**?**, p. 172] for all initial conditions such that $|z_\circ| < r$. On the other hand, the term on the right hand side of (27) may be reduced at will by enlarging γ (*i.e.*, by enlarging c hence, μ and μ_ψ) while the calculations above hold for r arbitrarily large but finite; hence, it follows that the origin is semiglobally uniformly practically asymptotically stable.

5.2 Proof of Claim 5.1

The proof of Claim 5.1 relies on the following: The origin of the system $\dot{\bar x} = [A(y_t) - L(t,y_t)C]\bar x$ is UGES. There exists $c_{z1} < \infty$ such that the function $t \mapsto z_1$ generated by the differential equations $\dot z = F(t,z)z$ where F is defined in (23b), satisfies

$$\int_{t_\circ}^{\infty} |z_1(t)|\,dt \le c_{z1}|z_\circ| \qquad \forall\, t \ge t_\circ \ge 0 \qquad (28)$$

and moreover, the origin of $\dot z = F(t,z)z$ is UGS with $\kappa(s) := c_{z0}s$ – *cf.* Ineq. (2), and

$$c_{z0} := \sqrt{\frac{\max\left\{p_M\,,\,\frac{1}{\gamma}\right\}}{\min\left\{p_m\,,\,\frac{1}{\gamma}\right\}}}. \qquad (29)$$

There exists $c_{z2} < \infty$ such that the function $t \mapsto z_2$ generated by the differential equations $\dot z = F(t,z)z$ where F is defined in (23b), satisfies

$$\int_{t_\circ}^{\infty} |z_2(t)|\,dt \le c_{z2}|z_\circ| \qquad \forall\, t \ge t_\circ \ge 0\,. \qquad (30)$$

From Claims 5.2 and 5.2 above it follows that

$$\int_{t_\circ}^{\infty} |z(t)|\,dt \le c_z|z_\circ| \qquad \forall\, t \ge t_\circ \ge 0 \qquad (31)$$

where $c_z := \max\{c_{z1}\,,\ c_{z2}\}$. It follows from [**?**, Lemma 3] that the origin is uniformly exponentially attractive on any ball, that is, it is uniformly globally attractive and, moreover, for any $r > 0$ we have that

$$|z(t_\circ)| < r \quad\Rightarrow\quad |z(t)| \le 2ce^{1/2}|z(t_\circ)|\,\mathrm{e}^{-\frac{1}{2c^2}(t-t_\circ)} \qquad (32)$$

with $c := \max\{c_z\,,\ c_{z0}\}$. We conclude that the origin of the system is UGAS and uniformly exponentially stable on any ball.

Notice that as c decreases, the rate of convergence $\gamma := -\frac{1}{2c^2}$ increases. As we show in the proof of Claim 5.2 the latter is made possible by enlarging μ and μ_ψ.

5.3 Proof of Claim 5.2

Consider Assumption 3. It is a standard result in adaptive control literature $-cf.$ [?] that the condition (10) is equivalent to: (A) for any unitary vector $\xi \in \mathbb{R}^n$ we have

$$\int_t^{t+T} \xi^\top Q(\tau, y_t)\xi d\tau \;\geq\; \mu \qquad \forall\, t \geq 0\,. \tag{33}$$

That is, $\phi(t) := \xi^\top Q(\tau, y_t)\xi$ is PE and satisfies $\phi_M \geq |\phi(t)|$ for all $t \geq 0$.Consider now the function $V_1(t,\bar{x}) := \bar{x}^\top P(t)\bar{x}$; its total derivative along the solutions of $\dot{\bar{x}} = \bar{A}(y_t)\bar{x}$ yields, by assumption, $\dot{V}_1 = -\bar{x}^\top Q(t, y_t)\bar{x} \leq 0$. This implies that, defining p_m and p_M as

$$p_m := \inf_{\substack{|\xi| = 1 \\ t \geq 0}} \xi^\top P(t)\xi \qquad p_M := \sup_{\substack{|\xi| = 1 \\ t \geq 0}} \xi^\top P(t)\xi\,, \tag{34}$$

the solutions of $\dot{\bar{x}} = \bar{A}(y_t)\bar{x}$ satisfy

$$|\bar{x}(t+T)|^2 \leq \frac{p_M}{p_m}\, |\bar{x}(\tau)|^2\,, \qquad \forall\, \tau \in [t, t{+}T]\,, \quad t \geq 0\,. \tag{35}$$

It follows from this, the equation $\dot{V}_1(\tau, \bar{x}(\tau)) = -\bar{x}(\tau)^\top Q(\tau, y_\tau)\bar{x}(\tau)$ and (33) that

$$\begin{aligned} V_1(t, \bar{x}(t)) - V_1(t+T, \bar{x}(t+T)) \;&\geq\; \int_t^{t+T} \bar{x}(\tau)^\top Q(\tau, y_\tau)\bar{x}(\tau) d\tau \\ &\geq\; \int_t^{t+T} |\bar{x}(\tau)|^2 \left(\inf_{|\xi| = 1} \frac{\xi^\top Q(\tau, y_\tau)\xi}{|\xi|^2} \right) d\tau \\ &\geq\; \int_t^{t+T} \frac{p_M}{p_m} \left(\inf_{|\xi| = 1} \frac{\xi^\top Q(\tau, y_\tau)\xi}{|\xi|^2} \right) d\tau\, |\bar{x}(t+T)|^2 \\ &\geq\; \frac{\mu p_M}{p_m}\, |\bar{x}(t+T)|^2 \end{aligned}$$

which implies that

$$\int_{t_0}^{t_0+T} \frac{p_M \mu}{p_m}\, |\bar{x}(t)|^2\, dt + V_1(t_0, \bar{x}(t_0)) - V_1(t_0+T, \bar{x}(t_0+T)) \;\;\geq\;\; \int$$

$$\left(T\frac{p_M}{p_m} + \frac{2p_m}{\mu}\right) |\bar{x}(t_0)|^2 \geq \int_{t_0}^{\infty} |\bar{x}(t)|^2\, dt$$

It follows from [?, Lemma 3] that the origin of $\dot{\bar{x}} = A(y_t)\bar{x}$ is globally exponentially stable, uniformly in y_t. Moreover, defining

$$c := \sqrt{\max\left\{ \left(T\frac{p_M}{p_m} + \frac{2p_m}{\mu}\right)\,,\; \frac{p_M}{p_m} \right\}} \tag{37}$$

we have

$$|\bar{x}(t)| \leq 2ce^{1/2}\, |\bar{x}_\circ|\, \mathrm{e}^{-\frac{1}{2c^2}(t - t_\circ)}\,. \tag{38}$$

5.4 Proof of Claim 5.2

The proof follows naturally from the proof of Claim 5.2. Consider the positive definite function

$$V_2(t, z) := z_1^\top P(t) z_1 + \frac{1}{\gamma}\, |z_2|^2\;; \tag{39}$$

its total derivative along the solutions of $\dot{z} = F(t,z)z$ yields $\dot{V}_2(t,z) = \dot{V}_1(t,z) \leq 0$ which implies that $p_m\, |\bar{x}(t)|^2 + (1/\gamma)\, \left|\bar{\theta}(t)\right|^2 \leq |z(t)|^2 \leq p_M\, |\bar{x}(t_\circ)|^2 + (1/\gamma)\, \left|\bar{\theta}(t_\circ)\right|^2$. It follows that the system is UGS, in particular, it satisfies

$$|z(t)| \leq c_{z0}\, |z(t_\circ)| \quad \forall\, t \geq t_\circ \geq 0$$

with c_{z0} as defined in (29). The first part of the claim follows observing that (36) still holds for the trajectories of $\dot{z} = F(t,z)z$ hence, (28) holds with

$$c_{z1}(\mu, T) := \sqrt{T\frac{p_M}{p_m} + \frac{2p_m}{\mu}}\,.$$

Notice that for each fixed T, $c(\mu, T) \to 0$ as $\mu \to \infty$.

5.5 Proof of Claim 5.2

Let $r > 0$ be an arbitrary number and define $R := c_{z0} r$. Consider the system $\dot{z} = F(t,z)z$ with initial conditions satisfying $|z_\circ| < r$; then, we have that $|z(t)| < R$ for all $t \geq t_\circ$. Consider the function $V_3 : \mathbb{R}_{\geq 0} \times B_R \to \mathbb{R}_{\geq 0}$ defined as

$$V_3(t,z) = V_2(t,z) - \varepsilon \left(\int_t^\infty e^{(t-\tau)} |\Psi(x(\tau))z_2|^2 \, d\tau + z_1^\top \Psi(x(t)) z_2 \right). \tag{40}$$

Under Assumption 3 we have that

$$-\int_t^\infty e^{(t-\tau)} |\Psi(x(\tau))z_2|^2 \, d\tau \leq -\mu_\psi e^{-T_\psi} |z_2|^2$$

hence, in view of the boundedness of $x(t)$ and the Lipschitz property of Ψ we have that V_3 is positive definite for sufficiently small ε; moreover, there exist positive numbers α_1, α_2 such that

$$\alpha_1 |z|^2 \leq V_3(t,z) \leq \alpha_2 |z|^2 .$$

On the other hand, the time derivative of V_3 along the trajectories of $\dot{z} = F(t,z)z$ yields

$$\dot{V}_3(t,z) = \dot{V}_2(t,z) - \varepsilon e^{-T} \frac{\mu_\psi^2}{T_\psi} |z_2|^2 - \varepsilon z_1^\top [\, \dot{\overbrace{\Psi(x(t))}} z_2 - \gamma \Psi(x(t)) P z_1] \tag{41}$$

$$- \varepsilon [\, (A - LC) z_1 + (\, \Psi(z_1 + x(t)) - \Psi(x(t))\,) z_2 \,] \Psi(x(t)) z_2 . \tag{42}$$

Under the regularity assumptions made on $x(t)$, Ψ *etc.*, and considering that $|z(t)| < R$, it follows that there exists a number c_R such that

$$\dot{V}_3(t,z(t)) \leq \; -\varepsilon e^{-T_\psi} \frac{\mu_\psi^2}{T_\psi} |z_2(t)|^2 + \varepsilon c_R [\, |z_1(t)|\, z_2(t) + |z_2(t)|\,] \tag{43}$$

$$\leq - \left(\varepsilon e^{-T_\psi} \frac{\mu_\psi^2}{T_\psi} - \frac{\varepsilon}{2} \right) |z_2(t)|^2 + (c_R^2 + 1) |z_1(t)|^2 \tag{44}$$

which, defining $c_\theta := \left(\varepsilon e^{-T_\psi} \frac{\mu_\psi^2}{T_\psi} - \frac{\varepsilon}{2} \right)$ is equivalent to

$$\int_{t_\circ}^\infty c_\theta |z_2(t)|^2 \, dt \leq V_3(t_\circ, z(t_\circ)) + (c_R^2 + 1) c_{z_1}^2 \int_{t_\circ}^\infty |z_1(t)|^2 \, dt . \tag{45}$$

The result follows with

$$c_{z2}(T_\psi, \mu_\psi) := \sqrt{\frac{\alpha_2 + (c_R^2 + 1) c_{z_1}^2}{c_\theta}} .$$

Notice that $c_{z2}(T_\psi, \mu_\psi, \mu) \to 0$ as $\mu_\psi \to \infty$ and $\mu_\psi \to \infty$.

5.6 Proof of Proposition 4

The proof follows from the developments of the previous section. In the first case, the synchronization error dynamics is given exactly by (8) and (12) whose origin has been showed to be uniformly semiglobally practically asymptotically stable. In the second case, the synchronization error dynamics corresponds to equations (8) and, instead of (12),

$$\dot{\bar{\theta}} = -\gamma \Psi(\bar{x} + x(t))^\top P(t) \bar{x} \quad \gamma > 0 .$$

In this case, b_2 in (24b) is zero and therefore, the calculations involved in the proof of Proposition 3 hold for all $|z| \geq 0$. In the case of the high-gain observer, notice that the synchronization may be achieved from any initial errors. In the third case, the synchronization dynamics is given simply by equation (8) with $\bar{\theta} \equiv 0$ and the result follows from the proof of Claim 5.2 for sufficiently large μ.

6 Conclusion

We presented an adaptive observer scheme for detectable systems which guaranteed uniform semiglobal practical asymptotic stability. In particular, we have shown that under certain persistency of excitation conditions the estimation errors tend to an arbitrarily small neighborhood of the origin. Our scheme applies naturally to the problem of master-slave synchronization in the case of parameter uncertainty of the master system.