LOCALIZING BOUNDS AND THE NONEXISTENCE CONDITIONS FOR COMPACT INVARIANT SETS OF SOME HAMILTONIAN SYSTEMS

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Abstract
In this work we consider three Hamiltonian systems and present results concerning localization bounds and the nonexistence conditions of compact invariant sets. The first two systems are Hamiltonian systems appeared in cosmological studies; they are defined by the conformally/minimally coupled field. Mainly, some nonexistence conditions of compact invariant sets are presented. The third system is a Hamiltonian system with a cubic potential and two degrees of freedom. We give conditions of the existence of a polytope containing all compact invariant sets and, briefly, describe how to compute its bounds.

Key words
Hamiltonian system, conformally coupled field, minimally coupled field, localization, compact invariant set

1 Introduction

Computing a domain which contains all compact invariant sets of a nonlinear multidimensional continuous-time system is a problem studied intensively during the last years, see e.g. papers [Krishchenko, A.P. and Starkov, K.E., 2006],[Starkov, K.E., 2008]. Finding a localization domain, i.e. a domain which contains all compact invariant sets is of essential interest because of the potential application of computer-based methods for its search narrowed in the localization domain and computing the Hausdorff dimension of attractors. The physical significance of compact invariant sets of any differentiable right-side system is related to the fact that they may carry information about a long- time behavior of the system, both in the negative and the positive time. Any globally bounded motion of the system is contained in one of compact invariant sets. Further, the existence or the nonexistence of periodic orbits expresses the fact of the presence or the lack correspondingly of repeatable behavior.

Localization methods elaborated in papers [Krishchenko, A.P. and Starkov, K.E., 2006],[Starkov, K.E., 2008] are formulated in terms of extrema of some differentiable functions called localizing. The goal of this work is to present results of localization analysis of compact invariant sets for some Hamiltonian systems. Some of our results are based on considerations containing in [Starkov, K.E., 2008] for a localization of compact invariant sets of natural polynomial Hamiltonian systems.

The paper is organized as follows. Useful assertions are given in Section 2. In Section 3 we describe how to localize compact invariant sets of a system possessing a first integral. In Sections 4 and 5 we examine Hamiltonian systems formed by the conformally coupled scalar field and by the minimally coupled scalar field correspondingly. Both of these systems have been appeared in cosmological studies, see papers [Maciejewski, A., Radzki, W. and Rybicki, S., 2005],[Maciejewski, A., Radzki, W. , Stachowiak, T., and Szydlowski, M., 2008],[Ma D.-Z., Wu X., and Zhong S.-Y., 2009]. In Section 6 we demonstrate how to solve the localization problem of all compact invariant sets for one class of natural polynomial Hamiltonian systems with a cubic potential and two degrees of freedom which has been examined in several papers, see e.g. in [Maciejewski A J, Przybylska M, Stachowiak T and Szydlowski M , 2008]. We give conditions under which a system from this class possesses a polytope containing all compact invariant sets and briefly describe how to compute its bounds.The complete version of the content of Section 6 is contained in [Starkov, K.E., 2011].

2 Some preliminaries and necessary notations

Let us introduce some objects and recall useful assertions, see in [Krishchenko, A.P. and Starkov, K.E., 2006],[Krishchenko, A.P. and Starkov, K.E., 2007]. We consider a $C^\infty-$ differentiable system

$$\dot{x} = F(x), \tag{1}$$

with $x \in \mathbf{R}^n$, $F(x) = (F_1(x), \ldots, F_n(x))^T$ and $F_i(x) \in C^\infty(\mathbf{R}^n)$, $i = 1, \ldots, n$.

Let $h(x) \in C^\infty(\mathbf{R}^n)$ be a function such that h is not the first integral of the system (1). The function h is used in the solution of the localization problem of compact invariant sets and is called a localizing function. Suppose that we are interested in the localization of all compact invariant sets located in some set $N \subset \mathbf{R}^n$ where N is an invariant set for the system (1) or a domain. By $S(h)$ we denote the set $\{x \in \mathbf{R}^n : L_F h(x) = 0\}$, where $L_F h(x)$ is a Lie derivative with respect to F. Further, we define $h_{\inf}(N) := \inf\{h(x) \mid x \in N \cap S(h)\}$, $h_{\sup}(N) := \sup\{h(x) \mid x \in N \cap S(h)\}$.

Proposition 1. *If $N \cap S(h) = \emptyset$ then the system (1) has no compact invariant sets located in N.*

Theorem 2. *For any $h(x) \in C^\infty(\mathbf{R}^n)$ all compact invariant sets of the system (1) located in N are contained in the set defined by the formula*

$$K(N) = \{x \in N : h_{\inf}(N) \leq h(x) \leq h_{\sup}(N)\}$$

as well.

Any of sets $K(N)$ is called a localization set. Now we remind another result called the iteration theorem, [Krishchenko, A.P. and Starkov, K.E., 2006].

Theorem 3. *Let $h_m(x), m = 1, 2, \ldots$ be a sequence of functions from $C^\infty(\mathbf{R}^n)$. Sets*

$$K_1 = K_{h_1}, \quad K_m = K_{m-1} \cap K_{m-1,m}, \quad m > 1,$$

with

$$\begin{aligned} &K_{m-1,m} = \{x : h_{m,\inf} \leq h_m(x) \leq h_{m,\sup}\}, \\ &h_{m,.\sup} = \sup_{S_{h_m} \cap K_{m-1}} h_m(x), \\ &h_{m,\inf} = \inf_{S_{h_m} \cap K_{m-1}} h_m(x), \end{aligned}$$

contain all compact invariant sets of of the system (1) and $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_m \supseteq \ldots$.

Below by F we denote the vector field which corresponds the system under investigation.

3 Localization bounds for systems with first integrals

Suppose that the system (1) possesses a polynomial first integral H of degree $k := \deg H$. Further, assume that there are a localizing function h and real parameter μ_* such that the set M defined by the equation $H(x) + \mu_* L_F h(x) = l$ is compact for some real l. Then we have that

$$K(H^{-1}(l)) = \{\inf_M h \leq h \mid_{H^{-1}(l)} \leq \sup_M h\} \tag{2}$$

is a localization set for all compact invariant sets contained inside the level set $H^{-1}(l)$. We notice that in this case each of sets $H^{-1}(l)$ and $S(h)$ may be noncompact.

4 Localization bounds for compact invariant sets for the cosmological system defined by the conformally coupled field

In this section we examine the system defined by the real Hamiltonian

$$H = \tfrac{1}{2}(-y_1^2 + y_2^2 - kx_1^2 + kx_2^2 + m^2x_1^2x_2^2 + \tfrac{\omega^2}{x_2^2}) + \tfrac{1}{4}(\Lambda x_1^4 + \lambda x_2^4).$$

Here we suppose that $H = 0$, parameters $\Lambda; \lambda; m^2$ are nonzero real; $k \in \{-1; 1\}$; $\omega^2 > 0$. This choice has a physical meaning, see in papers [Maciejewski, A., Radzki, W. and Rybicki, S., 2005],[Maciejewski, A., Radzki, W. , Stachowiak, T., and Szydlowski, M., 2008]. The corresponding dynamical system is defined by the following equations

$$\begin{aligned} \dot{x}_1 &= -y_1 \\ \dot{x}_2 &= y_2 \\ \dot{y}_1 &= kx_1 - m^2x_1x_2^2 - \Lambda x_1^3 \\ \dot{y}_2 &= -kx_2 - m^2x_1^2x_2 - \lambda x_2^3 + \frac{\omega^2}{x_2^3} \end{aligned} \tag{3}$$

Here we establish

Theorem 4.

1. Let $\lambda < 0; m^2 < 0; k = 1$ and $\omega \neq 0$. Then all compact invariant sets are contained in the set

$$K_1 = \{\mid x_2y_2 \mid \leq \frac{1}{2} \mid \lambda \mid^{-1}\}.$$

2. Let $\lambda < 0; m^2 < 0; k = -1$ and $\omega \neq 0$. Then the system (3) has no compact invariant sets.

3. Let $\lambda < 0; \Lambda > 0; k = -1$ and $\omega \neq 0$. Then the system (3) has no compact invariant sets.

4. Let $\Lambda > 0; m^2 > 0; k = -1$ and $\omega \neq 0$. Then the system (3) has no compact invariant sets.

5 Localization bounds for compact invariant sets for the cosmological system defined by the minimally coupled field

Here we take the system

$$\begin{aligned}\dot{x}_1 &= -y_1 \\ \dot{x}_2 &= \frac{1}{x_1^2}y_2 \\ \dot{y}_1 &= 2kx_1 - 4x_1^3(\Lambda + m^2x_2^2) + \frac{1}{x_1^3}y_2^2 + \frac{2\omega^2}{x_1^3x_2^2} \\ \dot{y}_2 &= -2m^2x_1^4x_2 + \frac{2\omega^2}{x_1^2x_2^3}\end{aligned} \tag{4}$$

for the real Hamiltonian

$$H = \frac{1}{2}(-y_1^2 + \frac{1}{x_1^2}y_2^2) - kx_1^2 + \Lambda x_1^4 + m^2x_1^4x_2^2 + \frac{\omega^2}{x_1^2x_2^2}.$$

In equations (4) $H = 0$, parameters Λ; m^2 are nonzero real; $k \in \{-1; 1\}$; $\omega^2 > 0$. This choice has a physical meaning, see in papers [Maciejewski, A., Radzki, W. and Rybicki, S., 2005],[Krishchenko, A.P. and Starkov, K.E., 2007]. We mention that it is established in [Maciejewski, A., Radzki, W. and Rybicki, S., 2005] that if we put $\omega = 0$ in equations (4) then the corresponding generic system is non- integrable. In the extended version of this text we present localization results in case $\omega = 0$ as well.

Here we give

Theorem 5.
1. Let $\omega \neq 0; m^2 < 0$. Then the system (4) has no compact invariant sets.

2. Let $\omega \neq 0$; $\Lambda > 0; k = -1$ and $m^2 > 0$. Then the system (4) has no compact invariant sets.

3. Let $\omega \neq 0$; $\Lambda > 0; k = 1$ and $m^2 > 0$. Then all compact invariant sets are contained in the frustum K_2 defined by

$$| x_1 | \leq \min\{\sqrt{\frac{1}{\Lambda}}; \frac{1}{4 \mid m\omega \mid}\}.$$

4. Let $\omega \neq 0$; $\Lambda < 0$ and $m^2 > 0$. Then all compact invariant sets are contained in the set K_3 defined by

$$| y_1 | \geq \sqrt{-\frac{2}{3\Lambda}}\}.$$

5. Let $\omega \neq 0$; $\Lambda < 0; k = -1$ and $m^2 > 0$. Then all compact invariant sets are contained in the set K_4 defined by

$$| y_1 | \geq \sqrt{2} \mid x_1 \mid\}.$$

6 Localization bounds for compact invariant sets of the generalized Hénon-Heiles Hamiltonian system

Below we obtain the polytopic localization bound for all compact invariant sets of a class of Hamiltonian systems with a cubic potential and two degrees of freedom provided one quadratic inequality imposed on parameters of these systems is imposed. Namely, we consider the following generalization of the Hénon-Heiles Hamiltonian system:

$$\begin{aligned}\dot{p}_1 &= -aq_1 - 2cq_1q_2 - eq_2^2 \\ \dot{p}_2 &= -bq_2 - cq_1^2 - dq_2^2 - 2eq_1q_2 \\ \dot{q}_1 &= p_1 \\ \dot{q}_2 &= p_2\end{aligned} \tag{5}$$

which is defined by the Hamiltonian $H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2 + aq_1^2 + bq_2^2) + cq_1^2q_2 + \frac{1}{3}dq_2^3 + eq_1q_2^2$. Parameters a and b are supposed to be positive. We show in a few technical steps that if

$$cd > e^2 \tag{6}$$

then this system possesses a polytope containing all compact invariant sets of (5) and provide explicit formulae for it.

Below we assume everywhere that the condition (6) holds.

Step 1. Localization bounds for $p_2 + \lambda q_2$.

By using the localizing function $h_1 = p_2 + \lambda q_2$, with λ be a real nonzero parameter we derive

Proposition 6. *All compact invariant sets are contained in the one-parameter family of localization sets*

$$K(h_1; \lambda) = \left\{ \begin{array}{c} p_2 + \lambda q_2 \geq r_1(\lambda) := -\frac{(\lambda^2+b)^2c}{4\lambda(cd-e^2)}, \\ \lambda > 0 \end{array} ; \right\}$$

$$K(h_1; \lambda) = \{p_2 + \lambda q_2 \leq r_1(\lambda), \lambda < 0\}.$$

Step 2. Localization bounds for p_2.

By applying the function $h_2 = p_2$ and results of the first step we come to

Theorem 7. *All compact invariant sets are located in the frustum*

$$K(h_2) = \{-r_2 \leq p_2 \leq r_2\},$$

where

$$r_2 := \frac{6bc + c + b^2c}{cd - e^2}.$$

Step 3. Localization bounds for p_1

Let us apply a localizing function $h_3 = \lambda p_1 + p_2$. Here $\lambda \in (\lambda_-, \lambda_+)$, with

$$\lambda_\pm = \frac{-e \pm \sqrt{4cd - 3e^2}}{2c}.$$

Next, we have introduce notations

$$\eta_1(\lambda) = \frac{(bc - \lambda^2 ac - \lambda ae)^2}{4c(cd - \lambda^2 c^2 - \lambda ce - e^2)^2} + \frac{\lambda^2 a^2}{4c}$$

and

$$\eta_2 = \left\{ \begin{array}{c} \sqrt{\frac{c\eta_1}{cd - \lambda^2 c^2 - \lambda ce - e^2}} + \\ \frac{bc - \lambda^2 ac - \lambda ae}{2(cd - \lambda^2 c^2 - \lambda ce - e^2)}; \end{array} \right\}$$

$$\eta_3 = \sqrt{\frac{\eta_1}{c}} + \frac{\lambda a}{c} + \frac{\lambda c + e}{c}\eta_2.$$

Using these notations we show that the polytope P_3 :

$$\begin{aligned} -\eta_2 &\le q_2 \le \eta_2; \\ -\eta_3 &\le q_1 \le \eta_3 \end{aligned}$$

in the $(q_1, q_2)-$ linear space contains $S(h_3)$. Now we derive from the formula $H(p, q) = l$ that the inequality

$$p_1^2 \le 2l + 2c\eta_2\eta_3^2 + \frac{2d}{3}\eta_2^3 + 2e\eta_2^2\eta_3 + r_2 + a\eta_3^2 + b\eta_2^2 \quad (7)$$

holds on $K(h_2) \cap H^{-1}(l) \cap P_3$. By r_3^2 we denote the right side of (7), $r_3 > 0$. Then we get the localization set

$$K(h_3) := K(H^{-1}(l); h_3) = \{-r_3 \le p_1 \le r_3\}.$$

Step 4. Localization bounds for q_1 and q_2. We have

from the main theorem in [Starkov, K.E., 2008]:

Proposition 8. *Let $h_u = p_1 q_1 + p_2 q_2$. Take any real $l > 0$. Then*

$$\begin{aligned} K(H^{-1}(l); h_u) = \{-\frac{6\sqrt{5}l}{5}(\frac{\sqrt{a}}{a} + \frac{\sqrt{b}}{b}) \le \\ p_1 q_1 + p_2 q_2 \le \frac{6\sqrt{5}l}{5}(\frac{\sqrt{a}}{a} + \frac{\sqrt{b}}{b})\}. \end{aligned}$$

Now we introduce notations

$$\begin{aligned} r_4 &= \frac{6\sqrt{5}l}{5}(\frac{\sqrt{a}}{a} + \frac{\sqrt{b}}{b}) + \frac{b^2 c}{8(cd - e^2)}, \\ r_5 &= \sqrt{\frac{2cr_4}{cd - e^2}} + \frac{bc}{2(cd - e^2)}, \\ r_6 &= \sqrt{\frac{2r_4}{c}} + \frac{r_4}{c}. \end{aligned}$$

In these notations by applying the iteration theorem to localizing functions h_u and $h = p_2 + q_1^2 + q_2^2$ and the last proposition one can derive

Theorem 9. *All compact invariant sets contained in the level set $H^{-1}(l)$, with $l > 0$, are contained in the polytope*

$$\begin{aligned} -r_3 \le p_1 &\le r_3; \qquad (8) \\ -r_2 \le p_2 &\le r_2; \\ -\sqrt{r_5^2 + r_6^2 + 2r_2} \le q_i &\le \sqrt{r_5^2 + r_6^2 + 2r_2}, i = 1, 2, \end{aligned}$$

Similarly, in case $l = 0$ we introduce notations

$$\begin{aligned} r_7 &:= \frac{bc}{cd - e^2}, \\ r_8 &:= \frac{b}{2\sqrt{cd - e^2}} + \frac{be}{cd - e^2}. \end{aligned}$$

Then we establish

Theorem 10. *All compact invariant sets contained in the level set $H^{-1}(0)$ are contained in the polytope defined by the first two inequalities in (8) and inequality* $-\sqrt{r_8^2 + r_7^2 + 2r_2} \le q_i \le \sqrt{r_8^2 + r_7^2 + 2r_2}, i = 1, 2.$

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