CONTROLLED MOTION MODES OF A GYROSCOPIC PENDULUM IN A GIMBAL SUSPENSION

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Abstract

The issues of controlling the motion of a gyroscopic pendulum fixed in a gimbal suspension and having three degrees of freedom are discussed in the paper. It is proposed to use physical analogies and correlate their action with the dissipative forces of external viscous friction, friction in gimbal joints and internal friction in the pendulum rod, as well as inertial forces and gyroscopic forces to form the control torques in gimbal joints. Six modes of controlled movement of the system are considered, which allow achieving various goals. It is shown that it is possible to completely suppress the motions of a gyroscopic pendulum, switch it to the mode of a spherical or physical pendulum, enter the mode of rotation around a fixed vertical axis, bring it to the mode of regular precession, and also strengthen or weaken the gyroscopic structure of the motion equations. The discussion of these modes of controlled motion is accompanied by the construction of motion equations both within the framework of the original nonlinear model and for simplified linear or weakly nonlinear models. The obtained results are theoretically interesting and may be useful for specific practical applications in the field of gyroscopic technology.

Key words

gyroscopic pendulum, gimbal suspension, control torques.

1 Introduction

One of the most popular pendulum systems among solid structures is the Lagrange pendulum or top, which is a heavy axisymmetric rigid body with one fixed point. This pendulum got its name after the great French mathematician Lagrange, who first built its mathematical model and found its first integrals, bringing the solution of the problem to quadratures. Subsequently, this model was canonized by another famous French mathematician Poisson, who already performed a detailed analysis of the motion. Such a model turned out to be so elegant and interesting in its properties that it was included in most university textbooks on general mechanics and led to the appearance of numerous scientific publications in the field of rigid body dynamics. In addition, it also laid the foundation for a new field of instrument-making technology – the theory of gyroscopes [Greenhill, 1914; Nikolai, 1948; Magnus, 1971; Lunts, 1972]. Therefore, such a body is often called a gyroscopic pendulum. By the middle of the 19th century, the "Lagrange problem" has already firmly entered the circle of fundamental problems of analytical mechanics and rigid body dynamics, and has also served as a good example for many related disciplines – first of all, the theory of mechanical oscillations, the theory of motion stability, control theory, and many others [MacMillan, 1936; Wittenburg, 1977; Merkin, 1997; Merkin, Afagh, Bauer, and Smirnov, 2000; Borisov, and Mamaev, 2001; Chernousko, Akulenko, and Leshchenko, 2017].

The issues of controlling the motion of a rigid body are also of particular interest in mechanics, and many works are devoted to them, including those published recently [Kapitanyuk, Khvostov, and Chepinskiy, 2014; Balandin, and Malkin, 2017; Aleksandrov, and Tikhonov, 2018; Alekseev, Doroshin, Yeromenko, Krikunov, and Nedovesov, 2018; Materassi, and Morrison, 2018; Molodenkov, and Sapunkov, 2019; Akulenko, and Sirotin, 2020]. Along this path, a gyroscopic pendulum with three degrees of freedom is also a model object, on the example of which one can consider and analyze a number of controlled motion modes and discuss their common and distinctive features. The advantage of the results obtained in this case is that they will have not only theoretical significance, but will also be able to find practical application in various technical problems, and they can also be developed and generalized to other systems with many degrees of freedom [Sarvilov, and Smirnov, 2023].

It should be emphasized that when designing rational control actions, physical analogies play an extremely important role, which suggests how these controls should be formed to achieve certain goals [Merkin, and Smolnikov, 2003]. Such goals can be, for example, complete or partial damping of the system movements, as well as its bringing to any desired mode of motion. It is clear that the action of control factors can be correlated with dissipative forces of external or internal friction, inertial forces, or gyroscopic forces, which, as is well known, have their own features that deserve close attention [Smirnov, and Smolnikov, 2022]. In addition, of no small importance is the fact that the control actions built using the correspondence with natural force factors will have a very simple and clear structure with sufficient efficiency. This circumstance ensures convenience in their practical implementation and at the same time confirms the relevance of the state technique, which is subject to further development. Everything said makes it possible to consider rational control actions formed in such a way.

In view of the foregoing, the purpose of this study is to discuss the principles of the formation of control actions using the example of the problem of a gyroscopic pendulum in a gimbal suspension, as well as to construct and analyze several modes of controlled movement of this system, which allow achieving various final goals. It should be noted that the use of a three-degree gimbal suspension allows the practical implementation of control factors, since this makes it possible to create control torques in all gimbal joints. It remains to be emphasized that a detailed analysis of these modes of controlled motion of a gyroscopic pendulum is a rather complex mathematical problem due to the presence of three degrees of freedom. Therefore, in this article, all attention is focused precisely on the constructions and conclusions mostly of a qualitative character.

Looking ahead, we note that expressions for control torques that make it possible to provide six different motion modes of a gyroscopic pendulum are obtained in this paper. These control actions allow for complete suppression of system motion, transition to movement similar to motion of spherical or physical pendulum, and also to rotation around a vertical axis or regular precession, and, finally, they may lead to forced conservative motion. In addition, a detailed study of these modes is presented, which is accompanied by analytical expressions and conclusions of a general nature, as well as simplified equations of motion for small deviations. The novelty of the found results lies in the fact that their determination is based on key relations for the total energy and projections of the angular momentum vector onto the vertical axis and the angular velocity vector onto the axis of symmetry of a gyroscopic pendulum, which take place in the controlled motion. Moreover, the use of some physical analogies mentioned above also contributes to novelty, which often suggests how the indicated quantities will change over time, and also significantly facilitates the discussion of main features of controlled motion modes.

The remainder of the paper is structured as follows. Section 2 presents the calculation scheme and differential equations of controlled motion of a gyroscopic pendulum in a gimbal suspension. Section 3 is devoted to the derivation of basic relations in controlled motion which play a primary role in the formation of control actions. Section 4 is dedicated to the analysis of free motion of a gyroscopic pendulum which is important for subsequent actions. Section 5 carries out the construction and study of controlled motion modes of a gyroscopic pendulum which allow achieving various goals. Section 6 compares the controlled motion modes and summarizes the obtained results in tabular form. At last, the paper is concluded with final remarks in Section 7.

2 Calculation scheme and mathematical model of controlled motion of a gyroscopic pendulum in a gimbal suspension

We will assume that a gyroscopic pendulum is a heavy axisymmetric rigid body with one motionless point O fixed in a gimbal suspension. Let m be the mass of the rigid body; A, $B = A$ and C be the moments of inertia of the rigid body about the principal axes 123 passing through point O, and l be the distance from point O to the center of mass of the rigid body. The deflected position of the body is determined by three gimbal angles α, β and γ is shown together with a gimbal suspension device in Figure 1. Here, the outer frame can rotate around the fixed axis x, and α is the angle of rotation around this axis. The inner frame is suspended in the outer frame so that it can rotate around the inner axis, perpendicular to the axis x and in the initial position coinciding with the fixed axis y, and β is the angle of rotation around the axis of the inner frame. Finally, the body of a gyroscopic pendulum (rotor) can rotate around the axis that is perpendicular to the axis of the inner frame and in the initial position coincides with the axis z, and γ is the angle of rotation around axis 3 associated with the body. We note that in the initial position the system of axes 123 associated with a gyroscopic pendulum coincides with the fixed system of axes xyz.

Figure 2 shows a sequence of rotations that transforms the fixed axes xyz into axes 123 connected with the gyroscopic pendulum.

Let's take a look at these rotations. The first rotation is carried out around the axis x by the angle α, as a result of which the axis y passes into the intermediate axis c, and the axis z – into the intermediate axis a. The second rotation is made around the axis c by the angle β, as a result of which the axis x goes into the intermediate axis

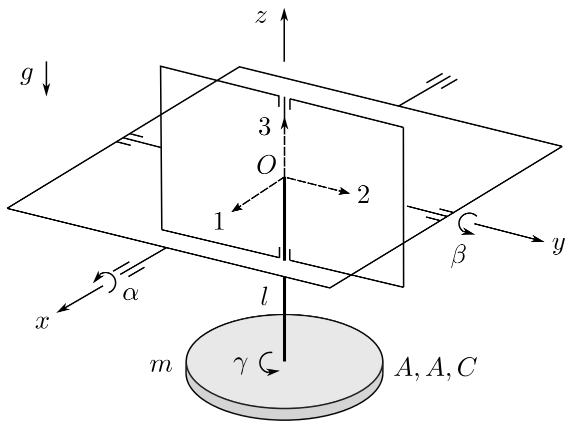

Figure 1. Gyroscopic pendulum and gimbal suspension device

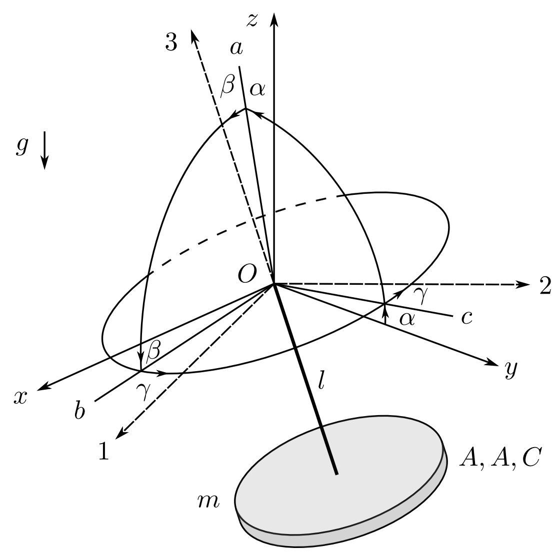

Figure 2. Gimbal angles α, β, γ and a sequence of rotations transforming the fixed axes xyz into axes 123 connected with the body

b, and the intermediate axis a goes into axis 3. Finally, the last rotation is made around axis 3 by the angle γ, and it leads to the fact that the intermediate axis b goes into axis 1, while the intermediate axis c goes to axis 2.

Starting to derive the equations of the controlled motion of a gyroscopic pendulum, we first find the kinematic relations binding the projections of the angular velocity vector onto the principal axes of inertia 123 with gimbal angles α, β and γ which can be determined using Figure 2. In fact, let us introduce into consideration the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$, directed along the fixed axes x, y and z, respectively, as well as the unit vectors $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$, directed along the moving axes 1, 2 and 3, respectively. In addition, we will need unit vectors $\underline{e}_a$, $\underline{e}_b$ and $\underline{e}_c$, which are directed along the intermediate axes a, b and c, respectively. It's not hard to understand that the angular velocity vector of a gyroscopic pendulum can be written in the form:

$$\underline{\omega} = \dot{\alpha}\underline{i} + \dot{\beta}\underline{e}_c + \dot{\gamma}\underline{e}_3. \tag{1}$$

Since we are interested in its projections onto the main axes of inertia 123, the vectors $\underline{i}$ and $\underline{e}_c$ should be expressed through the vectors $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$. To this end, we write out the relations following from Figure 2:

$$\begin{gathered}\underline{e}_b = \underline{e}_1\cos\gamma - \underline{e}_2\sin\gamma, \quad \underline{e}_c = \underline{e}_1\sin\gamma + \underline{e}_2\cos\gamma, \\ \underline{i} = \underline{e}_b\cos\beta + \underline{e}_3\sin\beta = \\ = \underline{e}_1\cos\gamma\cos\beta - \underline{e}_2\sin\gamma\cos\beta + \underline{e}_3\sin\beta.\end{gathered} \tag{2}$$

Substituting (2) into (1), we obtain the final expression for the angular velocity vector in the main axes:

$$\begin{gathered}\underline{\omega} = (\dot{\alpha}\cos\beta\cos\gamma + \dot{\beta}\sin\gamma)\underline{e}_1 + \\ + (-\dot{\alpha}\cos\beta\sin\gamma + \dot{\beta}\cos\gamma)\underline{e}_2 + (\dot{\gamma} + \dot{\alpha}\sin\beta)\underline{e}_3.\end{gathered} \tag{3}$$

As a result, the projections of the angular velocity vector onto the main axes of inertia 123 will be determined by the following kinematic relations [Magnus, 1971]:

$$\begin{cases} \omega_1 = \dot{\alpha}\cos\beta\cos\gamma + \dot{\beta}\sin\gamma, \\ \omega_2 = -\dot{\alpha}\cos\beta\sin\gamma + \dot{\beta}\cos\gamma, \\ \omega_3 = \dot{\gamma} + \dot{\alpha}\sin\beta. \end{cases} \tag{4}$$

We will also obtain at once the expressions connecting the Cartesian coordinates of the center of mass of the gyroscopic pendulum in a fixed coordinate system with gimbal angles. It is easy to understand that the radius vector $\underline{r}$ of the center of mass of a gyroscopic pendulum is determined by the expression:

$$\underline{r} = -l\underline{e}_3. \tag{5}$$

To obtain the required expressions here we need to connect the unit vector $\underline{e}_3$ with the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ of the fixed coordinate system, for which we will use the relations:

$$\begin{gathered}\underline{e}_a = -\underline{j}\sin\alpha + \underline{k}\cos\alpha, \\ \underline{e}_3 = \underline{e}_a\cos\beta + \underline{i}\sin\beta = \\ = \underline{i}\sin\beta - \underline{j}\cos\beta\sin\alpha + \underline{k}\cos\beta\cos\alpha.\end{gathered} \tag{6}$$

Taking into account formulas (5) and (6), we find the relations between the Cartesian coordinates of the center of mass of a gyroscopic pendulum and the gimbal angles in the following form [Smirnov, and Smolnikov, 2019]:

$$\begin{cases} x = -l\sin\beta, \\ y = l\cos\beta\sin\alpha, \\ z = -l\cos\beta\cos\alpha. \end{cases} \tag{7}$$

Next, we calculate the kinetic and potential energies of the system, using formulas (4) and last expression (7) [Magnus, 1971]:

$$\begin{gathered}T = \frac{1}{2}\left[A(\omega_1^2 + \omega_2^2) + C\omega_3^2\right] = \\ = \frac{1}{2}A\left(\dot{\alpha}^2\cos^2\beta + \dot{\beta}^2\right) + \frac{1}{2}C\left(\dot{\gamma} + \dot{\alpha}\sin\beta\right)^2, \\ \Pi = mg(z + l) = mgl\left(1 - \cos\alpha\cos\beta\right),\end{gathered} \tag{8}$$

where, for convenience, the potential energy is measured from the lower equilibrium position of a gyroscopic pendulum, when $\alpha = 0$ and $\beta = 0$. Therefore, in this position, the total mechanical energy of the system is zero.

Substituting formulas (8) into the Lagrange equations of the second kind [Lurie, 2002], and assuming that the control torques M_α, M_β and M_γ act in gimbal joints for each of the degrees of freedom

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot\alpha} - \dfrac{\partial T}{\partial \alpha} = -\dfrac{\partial \Pi}{\partial \alpha} + M_\alpha, \\ \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot\beta} - \dfrac{\partial T}{\partial \beta} = -\dfrac{\partial \Pi}{\partial \beta} + M_\beta, \\ \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot\gamma} - \dfrac{\partial T}{\partial \gamma} = -\dfrac{\partial \Pi}{\partial \gamma} + M_\gamma, \end{cases} \tag{9}$$

we obtain after a series of transformations the following system of differential equations:

$$\begin{cases} \left(A\cos^2\beta + C\sin^2\beta\right)\ddot\alpha + C\left(\ddot\gamma\sin\beta + \dot\gamma\dot\beta\cos\beta\right) + \\ +2(C-A)\dot\alpha\dot\beta\cos\beta\sin\beta + mgl\sin\alpha\cos\beta = M_\alpha, \\ A\left(\ddot\beta + \dot\alpha^2\cos\beta\sin\beta\right) - C\left(\dot\gamma + \dot\alpha\sin\beta\right)\dot\alpha\cos\beta + \\ \qquad + mgl\sin\beta\cos\alpha = M_\beta, \\ C\left(\ddot\gamma + \ddot\alpha\sin\beta + \dot\alpha\dot\beta\cos\beta\right) = M_\gamma. \end{cases} \tag{10}$$

This system is a mathematical model of the controlled movement of a gyroscopic pendulum under the action of control torques in the suspension joints, which is the main subject of further research.

3 Derivation of basic relations in controlled motion

Based on equations (10), we obtain three basic relations that take place in the controlled motion of a gyroscopic pendulum. These expressions will play a significant role in the further formation of control actions, which should lead to one or another final goal.

The first relation is an energy one, and it can be obtained by compiling an expression for the total mechanical energy $E = T + \Pi$ according to formulas (8) and differentiating it with respect to time, taking into account the motion equations (10). As a result of these transformations, we can obtain the first key relation:

$$\dot E = M_\alpha\dot\alpha + M_\beta\dot\beta + M_\gamma\dot\gamma, \tag{11}$$

which was to be expected, since the rate of change of the total mechanical energy is equal to the total power of the acting control torques M_α, M_β and M_γ.

To obtain the second relation, we calculate the angular momentum of the gyroscopic pendulum relative to the fixed vertical axis z. For this purpose, we first write down the expression of the angular momentum vector in the main axes 123 [Merkin, and Smolnikov, 2003]:

$$\underline{K} = A\omega_1\underline{e}_1 + A\omega_2\underline{e}_2 + C\omega_3\underline{e}_3. \tag{12}$$

Next, it is necessary to recalculate the unit vectors $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ through the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$, for which we write the following relations:

$$\begin{gathered} \underline{e}_1 = \underline{e}_b\cos\gamma + \underline{e}_c\sin\gamma, \quad \underline{e}_2 = -\underline{e}_b\sin\gamma + \underline{e}_c\cos\gamma, \\ \underline{e}_3 = \underline{e}_a\cos\beta + \underline{i}\sin\beta, \quad \underline{e}_b = -\underline{e}_a\sin\beta + \underline{i}\cos\beta, \\ \underline{e}_a = -\underline{j}\sin\alpha + \underline{k}\cos\alpha, \quad \underline{e}_c = \underline{j}\cos\alpha + \underline{k}\sin\alpha, \end{gathered} \tag{13}$$

and then we find the required expressions:

$$\begin{gathered} \underline{e}_1 = \underline{i}\cos\beta\cos\gamma + \underline{j}\left(\sin\alpha\sin\beta\cos\gamma + \right. \\ \left. + \cos\alpha\sin\gamma\right) + \underline{k}\left(-\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma\right), \\ \underline{e}_2 = \underline{i}\cos\beta\sin\gamma + \underline{j}\left(-\sin\alpha\sin\beta\sin\gamma + \right. \\ \left. + \cos\alpha\cos\gamma\right) + \underline{k}\left(\cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma\right), \\ \underline{e}_3 = \underline{i}\sin\beta - \underline{j}\sin\alpha\cos\beta + \underline{k}\cos\alpha\cos\beta. \end{gathered} \tag{14}$$

Since we are interested only in the projection of the vector $\underline{K}$ onto the axis z, then we will pay attention only to the terms with $\underline{k}$ in the expressions for $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$. As a result of substituting (14) into (12) after simple transformations, we find the desired projection of the vector $\underline{K}$ onto the axis z, which we will denote as p:

$$\begin{aligned} p = A\left(\dot\beta\sin\alpha - \dot\alpha\cos\beta\sin\beta\cos\alpha\right) + \\ + C(\dot\gamma + \dot\alpha\sin\beta)\cos\alpha\cos\beta. \end{aligned} \tag{15}$$

Differentiating it with respect to time using the motion equations (10), we obtain the second key relation:

$$\begin{aligned} \dot p = (-M_\alpha\sin\beta\cos\alpha + M_\beta\sin\alpha\cos\beta + \\ + M_\gamma\cos\alpha)\,/\cos\beta. \end{aligned} \tag{16}$$

Finally, the third key relation is actually the last motion equation (10), taking into account the last expression (4) for ω_3, so that it can be represented as:

$$\dot\omega_3 = M_\gamma/C. \tag{17}$$

The key relations (11), (16) and (17) obtained above contain information about the character of the change in the quantities E (total mechanical energy), p (projection of the angular momentum vector onto the fixed vertical axis z) and ω_3 (projection of the angular velocity vector onto the principal axis of inertia 3) in time.

We will assume that sensors are installed in all gimbal joints that can read information about all state variables, i.e., about the values α, $\dot\alpha$, β, $\dot\beta$, γ, $\dot\gamma$ corresponding to the current configuration of a gyroscopic pendulum. It is clear that in order to achieve certain final control goals, the control torques M_α, M_β and M_γ should be formed according to the feedback principle, i.e., setting the law of their change depending on the specified state variables [Fradkov, 1999; Blekhman, and Fradkov (eds), 2001; Fradkov, 2007]. In this case, it is advisable to be guided by three key relationships (11), (16) and (17), as well as take into account the physical analogies mentioned above in the introduction. This will allow further consider several control options with a discussion of their main features. It remains to emphasize that the practical implementation of control torques formed according to the feedback principle is a separate technical problem that is beyond the scope of this work.

4 Free motion of a gyroscopic pendulum

Let us briefly consider here the free motion of a gyroscopic pendulum, the analysis of which is necessary for further comparisons. To do this, we put on the right sides of the equations (10)

$$M_\alpha = 0, \quad M_\beta = 0, \quad M_\gamma = 0. \tag{18}$$

Then from relation (17) we immediately find the integral

$$C\omega_3 = C\left(\dot\gamma + \dot\alpha \sin\beta\right) = H = \text{const}, \tag{19}$$

and, for definiteness, we can assume that $H > 0$. Then the first two equations (10) in this situation will be reduced to the form:

$$\begin{cases} A\left(\ddot\alpha\cos^2\beta - 2\dot\alpha\dot\beta\cos\beta\sin\beta\right) + H\dot\beta\cos\beta + \\ \qquad\qquad + mgl\sin\alpha\cos\beta = 0, \\ A\left(\ddot\beta + \dot\alpha^2\cos\beta\sin\beta\right) - H\dot\alpha\cos\beta + \\ \qquad\qquad + mgl\sin\beta\cos\alpha = 0. \end{cases} \tag{20}$$

It can be seen that the terms with H in these nonlinear equations are gyroscopic terms. In this case, relations (11) and (16) also imply the energy and momentum integrals

$$E = \text{const}, \quad p = \text{const}, \tag{21}$$

so there are three integrals in total in free motion. If small oscillations of a gyroscopic pendulum near the lower equilibrium position are considered, then the angles α and β can be assumed to be small, and system (20) will take on a simpler form:

$$\begin{cases} A\ddot\alpha + H\dot\beta + mgl\alpha = 0, \\ A\ddot\beta - H\dot\alpha + mgl\beta = 0. \end{cases} \tag{22}$$

It can be seen that the equations of the system motion are connected in linear model (22) precisely due to gyroscopic terms [Aleksandrov, Semenov, and Zhan, 2019]. It should be noted that by introducing the complex variable $\delta = \alpha + i\beta$ equations (22) can be written in a compact form as a single equation. To do this, we multiply the second equation of system (22) by the imaginary unit i and add it to the first equation, after which we get:

$$A\ddot\delta - iH\dot\delta + mgl\delta = 0. \tag{23}$$

This equation can be easily solved analytically [Merkin, and Smolnikov, 2003], but in this paper we will be primarily interested in the structure of such equations for a complex variable. Indeed, in the study of controlled motion modes, the solution of these equations turns out to be much more complicated, and in most cases it can be carried out only by approximate methods under certain assumptions. The construction of such solutions is a rather voluminous problem and, therefore, it is not considered in this paper.

5 Construction and study of controlled motion modes of a gyroscopic pendulum

Let us now turn to the analysis of six modes of controlled motion of a gyroscopic pendulum, which lead to different goals. As these goals, we take the complete and partial damping of the system movements (with the transfer to the modes of oscillations of a spherical or physical pendulum, as well as rotation around a fixed axis), its introduction to the regular precession mode, and also the creation of a forced conservative movement.

5.1 Complete suppression of movements of a gyroscopic pendulum

As the first goal of control, we set the complete suppression of the movements of a gyroscopic pendulum. To form control torques that meet this goal, one should turn to physical principles. They suggest that it is advisable to compose the control torques in such a way that they act against the forces of inertia that arise during the system braking. It is known that a collinear control has a similar feature, the effective braking properties of which were previously demonstrated using the example of many applied problems of analytical mechanics [Smolnikov, 1991; Merkin, and Smolnikov, 2003; Leontev, Smirnov, and Smolnikov, 2020; Smirnov, and Smolnikov, 2022]. This implies that this control is kinetic, i.e., it takes into account the dynamic properties of the system. The formation of control torques according to the principle of collinear control in a mechanical system with many degrees of freedom means that these torques are assumed to be proportional to the corresponding generalized impulses, i.e., $\partial T/\partial\dot\alpha$, $\partial T/\partial\dot\beta$ and $\partial T/\partial\dot\gamma$. Taking into account that in the problem under consideration the expression for the kinetic energy has the form (8), we obtain the following control law:

$$\begin{gathered} M_\alpha = -b\frac{\partial T}{\partial\dot\alpha} = \\ = -b\left[\left(A\cos^2\beta + C\sin^2\beta\right)\dot\alpha + C\dot\gamma\sin\beta\right], \\ M_\beta = -b\frac{\partial T}{\partial\dot\beta} = -bA\dot\beta, \\ M_\gamma = -b\frac{\partial T}{\partial\dot\gamma} = -bC\left(\dot\gamma + \dot\alpha\sin\beta\right), \end{gathered} \tag{24}$$

where b here and below is a positive coefficient, which for simplicity can be considered constant, and the "–" sign is placed so that this control has a brake character, although collinear control can also be used to excite the accelerating motions of the system [Smirnov, and Smolnikov, 2021]. It is also interesting to emphasize that here an analogy with external viscous friction in the environment can also be drawn [Routh, 1955; Krivtsov, 2000; Ivanova, 2001]. Indeed, the dissipative Rayleigh function for this case has the form:

$$\begin{gathered} R = \frac{1}{2}b\left[A(\omega_1^2 + \omega_2^2) + C\omega_3^2\right] = \\ = \frac{1}{2}b\left[A\left(\dot\alpha^2\cos^2\beta + \dot\beta^2\right) + C\left(\dot\gamma + \dot\alpha\sin\beta\right)^2\right] = bT, \end{gathered} \tag{25}$$

where expressions (4) and (8) are used. Thus, the dissipative function (25) is proportional to the kinetic energy, and this once again confirms that the use of collinear control leads to effective suppression of motions.

Referring further to relation (17) and taking into account the last of expressions (24), we obtain:

$$C\frac{d}{dt}\left(\dot{\gamma}+\dot{\alpha}\sin\beta\right)=-bC\left(\dot{\gamma}+\dot{\alpha}\sin\beta\right), \tag{26}$$

whence after integration we will have:

$$C\omega_3=C\left(\dot{\gamma}+\dot{\alpha}\sin\beta\right)=H_0e^{-bt}, \tag{27}$$

where $H_0 = \text{const}$ (we further assume for definiteness that $H_0 > 0$). Instead of the integrals of energy and momentum, which took place in the free motion of a gyroscopic pendulum, in this case, according to (11) and (16) and taking into account (23), we will have the following relations:

$$\dot{E}=-2bT, \quad \dot{p}=-bp, \tag{28}$$

which demonstrate a decrease in both the total mechanical energy and the angular momentum about the vertical axis, and it follows from the second equation (28) that

$$p=p_0e^{-bt}, \tag{29}$$

where p_0 is the initial value of momentum p at $t = 0$ (we assume that $p_0 > 0$). Relations (27) – (29) demonstrate the complete attenuation of the movement of the gyroscopic pendulum, at which the values of E, p and ω_3 will decrease to zero. In this case, instead of equations (20), we now have from (10), taking into account (24):

$$\begin{cases} A\left(\ddot{\alpha}\cos^2\beta-2\dot{\alpha}\dot{\beta}\cos\beta\sin\beta\right)+bA\dot{\alpha}\cos^2\beta+ \\ \qquad +H_0e^{-bt}\dot{\beta}\cos\beta+mgl\sin\alpha\cos\beta=0, \\ A\left(\ddot{\beta}+\dot{\alpha}^2\cos\beta\sin\beta\right)+bA\dot{\beta}- \\ \qquad -H_0e^{-bt}\dot{\alpha}\cos\beta+mgl\sin\beta\cos\alpha=0, \end{cases} \tag{30}$$

and for the case of small deviations, these equations are rewritten in the following form:

$$\begin{cases} A\ddot{\alpha}+bA\dot{\alpha}+H_0e^{-bt}\dot{\beta}+mgl\alpha=0, \\ A\ddot{\beta}+bA\dot{\beta}-H_0e^{-bt}\dot{\alpha}+mgl\beta=0. \end{cases} \tag{31}$$

It can be seen that equations (30) and (31) differ from (20) and (22) by the presence of dissipative terms [Aleksandrov, Semenov, and Zhan, 2019], as well as the factor e^{-bt} in gyroscopic terms. Writing equations (31) as a single equation for the complex variable $\delta=\alpha+i\beta$

$$A\ddot{\delta}+(bA-iH_0e^{-bt})\dot{\delta}+mgl\delta=0, \tag{32}$$

we see that it is linear, but unlike (23) it has a variable coefficient, which causes the complexity of its analytical solution.

5.2 Entering the motion mode of a spherical pendulum

If it is required to bring the motions of a gyroscopic pendulum to the mode similar to the oscillations of a spherical pendulum, it is sufficient to somewhat modify the control law (24). Let us show further that it is possible for this to exclude the terms with the moment of inertia A from relations (24), as a result of which we obtain the following expressions for the control torques:

$$\begin{gathered} M_\alpha=-bC\left(\dot{\gamma}+\dot{\alpha}\sin\beta\right)\sin\beta, \\ M_\beta=0, \quad M_\gamma=-bC\left(\dot{\gamma}+\dot{\alpha}\sin\beta\right). \end{gathered} \tag{33}$$

Since the expression for M_γ has not changed from (24), here ω_3 will still decrease to zero in accordance with expression (27). As for the quantity E, then from relation (11), taking into account (33), we will have an equation for it:

$$\dot{E}=-bC\left(\dot{\gamma}+\dot{\alpha}\sin\beta\right)^2=-\frac{bH_0^2}{C}e^{-2bt}<0, \tag{34}$$

which shows that the total energy decreases over time. In addition, equation (34) is easy to integrate, after which it is possible to establish a specific law of energy change in time:

$$E=E_0-\frac{H_0^2}{2C}(1-e^{-2bt})\to E_0-\frac{H_0^2}{2C} \text{ at } t\to\infty, \tag{35}$$

where E_0 is the initial value of total energy E (at $t = 0$). So that, in contrast to the case of the previously considered collinear control, now the total energy will decrease not to zero, but to some constant non-zero value. It is easy to understand that the value p will also tend to a non-zero level in accordance with the equation

$$\begin{aligned} \dot{p}=-bC\left(\dot{\gamma}+\dot{\alpha}\sin\beta\right)\cos\alpha\cos\beta= \\ =-bH_0e^{-bt}\cos\alpha\cos\beta, \end{aligned} \tag{36}$$

following from relation (16), while for deviations of a gyroscopic pendulum in angles α and β not exceeding $\pi/2$ in absolute value, the right side of relation (36) is negative, i.e., in this case, p will be monotonically decreasing. It remains to write the first two equations (10) for the considered control mode (33):

$$\begin{cases} A\left(\ddot{\alpha}\cos^2\beta-2\dot{\alpha}\dot{\beta}\cos\beta\sin\beta\right)+H_0e^{-bt}\dot{\beta}\cos\beta+ \\ \qquad +mgl\sin\alpha\cos\beta=0, \\ A\left(\ddot{\beta}+\dot{\alpha}^2\cos\beta\sin\beta\right)-H_0e^{-bt}\dot{\alpha}\cos\beta+ \\ \qquad +mgl\sin\beta\cos\alpha=0. \end{cases} \tag{37}$$

It is already easy to understand from this that the role of terms with H_0 decreases over time, and all other terms in these equations correspond to conservative motion, in contrast to equations (30), where dissipative terms are clearly observed. This means that for the system of equations (37), in contrast to (30), there will be a finite conservative mode of motion, to which a gyroscopic pendulum will go over time, and it is described by the following equations:

$$\begin{cases} A\left(\ddot{\alpha}\cos^2\beta - 2\dot{\alpha}\dot{\beta}\cos\beta\sin\beta\right) + \\ \qquad\qquad + mgl\sin\alpha\cos\beta = 0, \\ A\left(\ddot{\beta} + \dot{\alpha}^2\cos\beta\sin\beta\right) + mgl\sin\beta\cos\alpha = 0. \end{cases} \tag{38}$$

These equations are similar to the equations of motion of a spherical pendulum written in gimbal angles, which was required to be shown [Smirnov, and Smolnikov, 2019]. We note that if it is necessary to trace in more detail the process of the system entering the mode of motion of a spherical pendulum, then we can consider small deviations, for which equations (38) take the form:

$$\begin{cases} A\ddot{\alpha} + H_0 e^{-bt}\dot{\beta} + mgl\alpha = 0, \\ A\ddot{\beta} - H_0 e^{-bt}\dot{\alpha} + mgl\beta = 0, \end{cases} \tag{39}$$

and, of course, they can be written as a single equation for the complex variable $\delta = \alpha + i\beta$:

$$A\ddot{\delta} - iH_0 e^{-bt}\dot{\delta} + mgl\delta = 0, \tag{40}$$

which differs from (32) by the absence of only one term.

5.3 Entering the motion mode of plane oscillations of a physical pendulum

Now let's set as the goal of control the entering of the system movements to the mode of plane oscillations like a physical pendulum. Let us show that in this case the control torques should again be constructed similarly to (24), only with the exception of the torque M_β and without changes in the rest:

$$\begin{gathered} M_\alpha = -b\left[\left(A\cos^2\beta + C\sin^2\beta\right)\dot{\alpha} + C\dot{\gamma}\sin\beta\right], \\ M_\beta = 0, \quad M_\gamma = -bC\left(\dot{\gamma} + \dot{\alpha}\sin\beta\right). \end{gathered} \tag{41}$$

Indeed, here ω_3 again decreases to zero in accordance with formula (27). Substituting expressions (41) into (11), we obtain the following relation for the total mechanical energy:

$$\begin{aligned} \dot{E} &= -b\left[A\cos^2\beta\dot{\alpha}^2 + C(\dot{\gamma} + \dot{\alpha}\sin\beta)^2\right] = \\ &= -b\left(A\cos^2\beta\dot{\alpha}^2 + \frac{H_0^2}{C}e^{-2bt}\right) < 0, \end{aligned} \tag{42}$$

which shows the decrease in energy over time. As for the value p, according to (16), taking into account (27) and (41), we will have:

$$\begin{aligned} \dot{p} &= b\cos\alpha\cos\beta\left[A\dot{\alpha}\sin\beta - C(\dot{\gamma} + \dot{\alpha}\sin\beta)\right] = \\ &= b\cos\alpha\cos\beta\left(A\dot{\alpha}\sin\beta - H_0 e^{-bt}\right). \end{aligned} \tag{43}$$

It can be seen that, based on this expression, the behavior of value p in time has an unobvious character, which requires a separate study. To deal with this issue, we write the first two equations (10), taking into account the expressions for the control torques (41):

$$\begin{cases} A\left(\ddot{\alpha}\cos^2\beta - 2\dot{\alpha}\dot{\beta}\cos\beta\sin\beta\right) + bA\dot{\alpha}\cos^2\beta + \\ \qquad + H_0 e^{-bt}\dot{\beta}\cos\beta + mgl\sin\alpha\cos\beta = 0, \\ A\left(\ddot{\beta} + \dot{\alpha}^2\cos\beta\sin\beta\right) - H_0 e^{-bt}\dot{\alpha}\cos\beta + \\ \qquad + mgl\sin\beta\cos\alpha = 0. \end{cases} \tag{44}$$

In contrast to (30), where dissipative terms were presented in both equations, in (44) the second equation does not contain such a term. Analyzing equations (44), it is easy to understand that the angle α will eventually tend to zero, while the angle β in the final conservative motion mode will be determined from the equation:

$$A\ddot{\beta} + mgl\sin\beta = 0. \tag{45}$$

It represents the equation of oscillations of a physical pendulum [Markeev, 2007], which, as is known, is similar to the equation of oscillations of a mathematical pendulum [Merkin, and Smolnikov, 2003]. From the foregoing, we can conclude that p, as well as ω_3, will tend to zero, while the total energy E will go to a constant non-zero level. As before, here, if necessary, it is also possible to trace the process of a gyroscopic pendulum reaching the specified mode of plane oscillations by writing equations (44) for small deviations:

$$\begin{cases} A\ddot{\alpha} + bA\dot{\alpha} + H_0 e^{-bt}\dot{\beta} + mgl\alpha = 0, \\ A\ddot{\beta} - H_0 e^{-bt}\dot{\alpha} + mgl\beta = 0, \end{cases} \tag{46}$$

which for the complex variable $\delta = \alpha + i\beta$ can be rewritten in the following form:

$$A\ddot{\delta} + bA\,\mathrm{Re}\,\dot{\delta} - iH_0 e^{-bt}\dot{\delta} + mgl\delta = 0. \tag{47}$$

5.4 Entering the rotation mode around the vertical axis

Let us now assume that the goal of control is to bring a gyroscopic pendulum to the rotation mode around the vertical axis with a constant angular velocity. It is easy to understand that in this case it is appropriate to form control actions similar to the torques of viscous friction in two joints of a gimbal suspension corresponding to rotations of its outer and inner frames by angles α and β, namely [Lunts, 1972]:

$$M_\alpha = -b_0\dot{\alpha}, \quad M_\beta = -b_0\dot{\beta}, \quad M_\gamma = 0, \tag{48}$$

where b_0 here and below is a positive and constant coefficient [Voronkov, and Denisov, 1999]. We note that the introduction of another dissipative coefficient b_0 instead of the previously used coefficient b here is only due to the fact that its dimension turns out to be different. It is easy to understand that in accordance with (17) in this situation ω_3 is constant. Turning to relation (11), we obtain, taking into account (48), the following formula:

$$\dot{E} = -b_0 \left(\dot{\alpha}^2 + \dot{\beta}^2\right) = -2R < 0, \tag{49}$$

demonstrating a gradual decrease in total mechanical energy over time, where the value R is determined by the formula

$$R = \frac{1}{2} b_0 \left(\dot{\alpha}^2 + \dot{\beta}^2\right) \tag{50}$$

and plays the role of the dissipative function of viscous friction in two gimbal joints. In this case, of course, from the traditional relations

$$M_\alpha = -\frac{\partial R}{\partial \dot{\alpha}}, \quad M_\beta = -\frac{\partial R}{\partial \dot{\beta}}, \quad M_\gamma = -\frac{\partial R}{\partial \dot{\gamma}} \tag{51}$$

expressions for the control torques (48) will follow. As for the value p, then according to (16) and (48) we have the relation:

$$\dot{p} = b_0 \left(\dot{\alpha} \sin\beta \cos\alpha - \dot{\beta} \sin\alpha \cos\beta\right) / \cos\beta. \tag{52}$$

This formula does not give any clear representation about the process of changing the value p over time. To deal with this issue in more detail and investigate the final mode of motion, let us turn to the first two equations of controlled motion (10) taking into account (48) and also using the integral (19) that takes place in the case under consideration:

$$\begin{cases} A\left(\ddot{\alpha}\cos^2\beta - 2\dot{\alpha}\dot{\beta}\cos\beta\sin\beta\right) + b_0\dot{\alpha} + H\dot{\beta}\cos\beta + \\ \qquad + mgl\sin\alpha\cos\beta = 0, \\ A\left(\ddot{\beta} + \dot{\alpha}^2\cos\beta\sin\beta\right) + b_0\dot{\beta} - H\dot{\alpha}\cos\beta + \\ \qquad + mgl\sin\beta\cos\alpha = 0, \end{cases} \tag{53}$$

where one can clearly see again both gyroscopic and dissipative terms in both equations. It is quite clear that in the final mode of motion the angles α and β will tend to zero. This process can be traced in more detail by considering, as usual, the case of sufficiently small deviations, i.e., assuming the angles α and β to be small, as a result of which we obtain from equations (53) the following simplified equations [Lamb, 1929]:

$$\begin{cases} A\ddot{\alpha} + b_0\dot{\alpha} + H\dot{\beta} + mgl\alpha = 0, \\ A\ddot{\beta} + b_0\dot{\beta} - H\dot{\alpha} + mgl\beta = 0. \end{cases} \tag{54}$$

Using the complex variable $\delta = \alpha + i\beta$, these equations can be rewritten as:

$$A\ddot{\delta} + (b_0 - iH)\dot{\delta} + mgl\delta = 0. \tag{55}$$

It is important to emphasize that this equation, like equation (23) for the free motion, contains constant coefficients. Therefore, it can be solved analytically by exact methods, in contrast to the equations (32), (40) and (47) obtained earlier in the three previous subsections and containing a variable coefficient.

Despite the fact that the angles α and β tend to zero over time, this will not mean complete attenuation of the movements of a gyroscopic pendulum due to the presence of the integral (19). It follows from it that in the final mode of motion the angle γ will change linearly with time, and the system will enter the mode of rotation around the vertical axis, as it was originally stated, and this rotation will be carried out at a constant angular velocity equal to H/C. Hence it follows that the total mechanical energy E and angular momentum p relative to the vertical axis in accordance with relations (8) and (15) will reach constant non-zero values, namely:

$$E \to \frac{H^2}{2C}, \quad p \to H \quad \text{at} \quad t \to \infty. \tag{56}$$

5.5 Entering the regular precession mode

As the next control goal, we will choose the transition of a gyroscopic pendulum to the regular precession mode. To do this, we can use the analogy with internal friction, which requires a detailed discussion. It is clear that such a mode of controlled motion will be of particular interest, since the issue of taking into account internal friction during oscillations of various mechanical systems is extremely relevant [Smirnov, and Smolnikov, 2019; Smirnov, and Smolnikov, 2022]. To this end, we construct the control in such a way that the values p and ω_3 are constant, and the total mechanical energy E decreases over time, and its dissipation would stop on the circular motion of the center of mass of the system. Let us form the control torques as:

$$\begin{gathered} M_\alpha = -b_0\left(\dot{\alpha}\sin\alpha\cos\beta + \dot{\beta}\sin\beta\cos\alpha\right)\sin\alpha\cos\beta, \\ M_\beta = -b_0\left(\dot{\alpha}\sin\alpha\cos\beta + \dot{\beta}\sin\beta\cos\alpha\right)\sin\beta\cos\alpha, \\ M_\gamma = 0, \end{gathered} \tag{57}$$

and we will explain further from what considerations these expressions are obtained. Of course, we immediately have from relation (17) that $\omega_3 = \mathrm{const}$, and, in addition, according to relation (16), expressions (57) also provide the required momentum integral $p = \mathrm{const}$. In addition, there is also an energy relation based on (11):

$$\dot{E} = -b_0\left(\dot{\alpha}\sin\alpha\cos\beta + \dot{\beta}\sin\beta\cos\alpha\right)^2 = -2S < 0, \tag{58}$$

where the quantity S, determined by the formula

$$S = \frac{1}{2} b_0 \left(\dot{\alpha}\sin\alpha\cos\beta + \dot{\beta}\sin\beta\cos\alpha\right)^2, \tag{59}$$

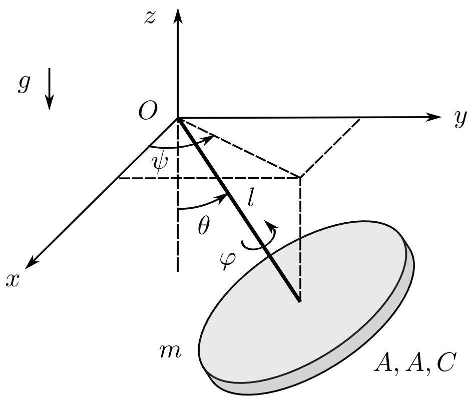

Figure 3. Gyroscopic pendulum and Euler angles ψ, θ, φ

can be interpreted as some dissipative function, and then the control torques (57) will be determined by the standard expressions:

$$M_\alpha = -\frac{\partial S}{\partial \dot\alpha}, \quad M_\beta = -\frac{\partial S}{\partial \dot\beta}, \quad M_\gamma = -\frac{\partial S}{\partial \dot\gamma}. \qquad (60)$$

This analogy is about the dissipative function of internal friction, and it is indicated by the fact that for the mode under consideration there is a continuous dissipation of energy while maintaining the momentum integral. This is precisely the characteristic feature of internal dissipation [Smirnov, and Smolnikov, 2022]. We note that expressions similar in structure were obtained earlier in the study of the motion of a spherical pendulum with internal friction in its rod [Smirnov, and Smolnikov, 2019], as well as in a similar problem for a gyroscopic pendulum [Smolnikov, 2008]. In addition, the problem of the motion of a rigid body with a cavity filled with a viscous fluid deserves attention, where the phenomenon of continuous energy dissipation also took place while maintaining the momentum integral [Chernousko, Akulenko, and Leshchenko, 2017].

It remains to show in more detail that the final motion mode in this case will be the mode of regular precession of a gyroscopic pendulum. Indeed, referring to formulas (57), one can see that the action of the control torques stops when

$$\dot\alpha \sin\alpha \cos\beta + \dot\beta \sin\beta \cos\alpha = 0, \qquad (61)$$

and in this case, according to (58), the dissipation of energy also stops, i.e., the total mechanical energy will tend to some non-zero value. It is easy to understand that the indicated equality (61) is fulfilled exactly in the case when $\cos\alpha\cos\beta = \mathrm{const}$. Remembering that according to (7) $z = -l\cos\alpha\cos\beta$, we establish that $z = \mathrm{const}$, and, therefore, the center of mass of a gyroscopic pendulum will eventually move in a circle like a conical pendulum. The general motion of the system will indeed represent a regular precession, as can be illustrated by the following simple reasoning, if we take into account that usually a regular precession is described using the Euler angles ψ (precession angle), θ (nutation angle) and φ (self-rotation angle), which together with a gyroscopic pendulum are shown in Figure 3.

Using the Euler angles, we write the projections of the angular velocity vector of the system onto the principal axes of inertia of a gyroscopic pendulum in the form of well-known kinematic relations [Magnus, 1971]:

$$\begin{cases} \omega_1 = \dot\psi \sin\theta \sin\varphi + \dot\theta\cos\varphi, \\ \omega_2 = \dot\psi \sin\theta\cos\varphi - \dot\theta \sin\varphi, \\ \omega_3 = \dot\varphi + \dot\psi\cos\theta. \end{cases} \qquad (62)$$

Since $z = -l\cos\theta$, we have $\theta = \mathrm{const}$. It follows from the condition of conservation of total mechanical energy in the final motion mode that at $\omega_3 = \mathrm{const}$ we will have

$$\omega_1^2 + \omega_2^2 = \dot\psi^2 \sin^2\theta + \dot\theta^2 = \mathrm{const}, \qquad (63)$$

where the expression for the kinetic energy (8) and the first two formulas (62) are taken into account, and since $\theta = \mathrm{const}$, then $\dot\psi = \mathrm{const}$. Finally, from the equality $\omega_3 = \mathrm{const}$ and the last formula (62) we establish that $\dot\varphi = \mathrm{const}$. The resulting equalities

$$\theta = \mathrm{const}, \quad \dot\psi = \mathrm{const}, \quad \dot\varphi = \mathrm{const} \qquad (64)$$

determine the regular precession of a gyroscopic pendulum, which is the final mode of motion for the considered control option (57).

We note that the first two equations of controlled motion (10), taking into account expressions (57) and integral (19), take the form:

$$\begin{cases} A\left(\ddot\alpha\cos^2\beta - 2\dot\alpha\dot\beta\cos\beta\sin\beta\right) + H\dot\beta\cos\beta + \\ \quad +b_0\left(\dot\alpha\sin\alpha\cos\beta + \dot\beta\sin\beta\cos\alpha\right)\sin\alpha\cos\beta + \\ \quad\quad + mgl\sin\alpha\cos\beta = 0, \\ A\left(\ddot\beta + \dot\alpha^2\cos\beta\sin\beta\right) + b_0\dot\beta - H\dot\alpha\cos\beta + \\ \quad +b_0\left(\dot\alpha\sin\alpha\cos\beta + \dot\beta\sin\beta\cos\alpha\right)\sin\beta\cos\alpha + \\ \quad\quad + mgl\sin\beta\cos\alpha = 0. \end{cases} \qquad (65)$$

For the case of sufficiently small deviations of the gyroscopic pendulum, we will have from (65) simplified equations in the framework of a weakly nonlinear model:

$$\begin{cases} A\ddot\alpha + H\dot\beta + b_0(\alpha\dot\alpha + \beta\dot\beta)\alpha + mgl\alpha = 0, \\ A\ddot\beta - H\dot\alpha + b_0(\alpha\dot\alpha + \beta\dot\beta)\beta + mgl\beta = 0, \end{cases} \qquad (66)$$

in which dissipative effects are of the third order of smallness in generalized coordinates and velocities. As a result, here the complexity of constructing an analytical solution arises again, which, of course, can be produced only by approximate methods. If we introduce the former complex variable $\delta = \alpha + i\beta$, then system (66) will take the form of one equation:

$$A\ddot\delta - iH\dot\delta + mgl\delta = -\frac{b_0}{2}\left(\delta\dot{\overline{\delta}} + \overline{\delta}\dot\delta\right)\delta. \qquad (67)$$

5.6 Forced conservative movement of a gyroscopic pendulum

Finally, as the last goal of control, we require the creation of the mode of forced conservative motion of a gyroscopic pendulum. It is clear that for this it is necessary to ensure the constancy of the value E in the presence of control torques [Leontev, Smirnov, and Smolnikov, 2020]. In this situation, it is advisable to build control torques like gyroscopic forces that do not change the total mechanical energy. This property is possessed by the orthogonal control mode, which was previously successfully applied in the problem of reorientation of a rigid body in space and in robotics problems [Smolnikov, 1979; Smolnikov, 1991]. Based on what has been said, we now form the control torques, for example, in the following form:

$$M_\alpha = \kappa\dot{\beta}\cos\beta, \quad M_\beta = -\kappa\dot{\alpha}\cos\beta, \quad M_\gamma = 0, \quad (68)$$

where κ is a constant coefficient that can have any sign, and the multiplication of the control torques by $\cos\beta$ is not fundamental in qualitative terms, but represents a certain convenience and contributes to a more visual further study of the controlled motions of the system. It is easy to see that the control torques (68) have a gyroscopic structure, while it follows from relation (11) that $E = \mathrm{const}$, so we really have a conservative motion, which, however, here is no longer free [Leontev, Smirnov, and Smolnikov, 2020]. Orthogonality of the column of control torques $[M_\alpha, M_\beta, M_\gamma]^{\mathrm{T}}$ to the column of generalized velocities $[\dot{\alpha}, \dot{\beta}, \dot{\gamma}]^{\mathrm{T}}$ explains the name of the control. In addition, for the value p from (16) we now have the relation:

$$\dot{p} = -\kappa\left(\dot{\beta}\sin\beta\cos\alpha + \dot{\alpha}\sin\alpha\cos\beta\right), \quad (69)$$

and after integration we find specific law for value p:

$$p = p_0 + \kappa\cos\alpha\cos\beta = p_0 - \kappa\frac{z}{l}, \quad (70)$$

where it is taken into account that $z = -l\cos\alpha\cos\beta$, and $p_0 = \mathrm{const}$ is the value p that it would have in the case $\kappa = 0$ (i.e., in the absence of orthogonal control), so this is not the initial value of momentum p at $t = 0$. Thus, in contrast to free motion, p is not constant, and it is a linear function of the vertical coordinate z, so that the character of the change p in time repeats the behavior of this coordinate. It follows from the formula (70) that p can take values in the range $[p_0 - \kappa, p_0 + \kappa]$. Finally, using (17), here we again obtain the integral (19). As a result, the equations of controlled motion (10), taking into account (68), will have the following form:

$$\begin{cases} A\left(\ddot{\alpha}\cos^2\beta - 2\dot{\alpha}\dot{\beta}\cos\beta\sin\beta\right) + (H-\kappa)\dot{\beta}\cos\beta + \\ \qquad\qquad + mgl\sin\alpha\cos\beta = 0, \\ A\left(\ddot{\beta} + \dot{\alpha}^2\cos\beta\sin\beta\right) - (H-\kappa)\dot{\alpha}\cos\beta + \\ \qquad\qquad + mgl\sin\beta\cos\alpha = 0, \end{cases} \quad (71)$$

and they differ from the equations (20) for the free motion mode only by gyroscopic terms, or rather, only by the coefficient in them – in (71) we have the factor $H-\kappa$ instead of the factor H in (20). Considering that it was previously assumed that $H > 0$, it is easy to understand that by proper choice of the control parameter κ, one can both weaken (for $|H-\kappa| < H$, i.e., for $0 < \kappa < 2H$) and strengthen (for $|H-\kappa| > H$, i.e. for $\kappa < 0$ or $\kappa > 2H$) gyroscopic terms in equations (71). It is clear that the case $\kappa = 0$ corresponds to the free motion of a gyroscopic pendulum, while for $\kappa = 2H$ we will have that the factor $H-\kappa$ differs from H only in sign. It is also interesting to note that, assuming $\kappa = H$, one can completely eliminate gyroscopic terms in equations (71) and arrive at system (38), which describes a motion similar to that of a spherical pendulum. It is important to note, however, that in the controlled motion mode based on the modification of the collinear control law and ultimately leading to equations (38), E decreases not to zero, p tends to the non-zero value, and ω_3 decreases to zero over time. In the current control option E and ω_3 are constant, and p depends linearly on the coordinate z, herewith, there is no transient process, and for $\kappa = H$ we will immediately have a trajectory in projection onto a horizontal plane, similar to the trajectory of a spherical pendulum. These differences must be kept in mind for the most distinct comparison of the two modes of controlled motion. Finally, if, for example, we put $\kappa = -H$ or $\kappa = 3H$, then we will amplify (here again by amplification we mean the increase in the value of factor $H-\kappa$ in absolute value compared to H) gyroscopic terms in equations (71) by a factor of two, and if put $\kappa = -2H$ or $\kappa = 4H$, then we obviously triple them.

6 Comparison of controlled motion modes

At the end of the conversation about the controlled movements of a gyroscopic pendulum, it is convenient to tabulate the main qualitative features of the six considered control modes, indicating physical analogies (Table 1) and noting how the values E, p and ω_3 behave in each of these modes (Table 2).

Despite the fact that the conclusions presented above give an important representation of the processes of controlled motion of a gyroscopic pendulum, they are only of a qualitative character. Of course, in the general case, the problem of the controlled motion of a gyroscopic pendulum should be solved by a harmonious combination of analytical research methods and numerical integration of the motion equations (10) in any software package with various options for the formation of control torques M_α, M_β and M_γ. In this case, not only obtaining dependencies $\alpha = \alpha(t)$, $\beta = \beta(t)$ and $\gamma = \gamma(t)$ through numerical integration is of great interest, but also constructing the trajectory of the center of mass of a gyroscopic pendulum in projection onto the horizontal

Table 1. Physical analogies in controlled motion modes

No.	Controlled motion mode	Physical analogy
1	Complete suppression of movements (collinear control)	Inertial forces or external viscous friction
2	Entering the motion mode of a spherical pendulum	Inertial forces (with some modification)
3	Entering the motion mode of plane oscillations of a physical pendulum	Inertial forces (with some modification)
4	Entering the rotation mode around the vertical axis	Viscous friction in two gimbal joints
5	Entering the regular precession mode	Internal friction
6	Forced conservative movement (orthogonal control)	Gyroscopic forces

Table 2. Behavior of values E, p and ω_3 in controlled motion modes

No.	E	p	ω_3
1	Decreases to zero	Decreases to zero	Decreases to zero
2	Decreases not to zero	Tends to a non-zero value	Decreases to zero
3	Decreases not to zero	Tends to zero	Decreases to zero
4	Decreases not to zero	Tends to a non-zero value	Is constant
5	Decreases not to zero	Is constant	Is constant
6	Is constant	Is a linear function of the vertical coordinate	Is constant

plane xy according to first two formulas (7), which gives a visual representation of its movement [Webster, 1959; Magnus, 1971]. It is clear that the specific character of this trajectory when considering the free motion of a gyroscopic pendulum essentially depends on the geometric (length l) and inertial (mass m, moments of inertia A and C) parameters of the system, as well as on the initial conditions of motion (they should be set as $\alpha = \alpha_0$, $\dot{\alpha} = \dot{\alpha}_0$, $\beta = \beta_0$, $\dot{\beta} = \dot{\beta}_0$, $\gamma = \gamma_0$, $\dot{\gamma} = \dot{\gamma}_0$ at $t = 0$). In this case, as is known [Merkin, and Smolnikov, 2003], the trajectory can have both a sinusoidal and a loop-like character, and a boundary variant can also be realized when the trajectory has a cycloidal character and has cusps. When considering controlled modes of motion, the character of the trajectory also depends on the control parameters (b, b_0 or κ), which increases the number of parameters. Thus, the issues of numerical integration of the equations of controlled motion of a gyroscopic pendulum represent a serious problem that requires separate research. At the same time, the qualitative conclusions made in the work and regarding the behavior of the quantities E, p and ω_3 over time will be able to confirm the correctness of the numerical integration procedure for all the above control modes. As for the analytical part of the study, it can also be continued by constructing solutions of simplified equations of controlled motion obtained in the present paper for the case of small deviations from the lower equilibrium position of a gyroscopic pendulum with respect to a complex variable $\delta = \alpha + i\beta$, using approximate methods, for example, the averaging method [Bogolyubov, and Mitropolskiy, 1958; Nayfeh, 1981].

7 Conclusion

Summing up the results of the presented study, we can conclude that the physical principles of control formation outlined in the paper really lead to very simple expressions for control torques depending on the state variables and make it possible to implement a number of interesting modes of controlled motion of a gyroscopic pendulum, each of which has its own characteristics and ultimate goal of control. It was shown that it is possible to completely or partially suppress the movements of a gyroscopic pendulum by bringing it to the modes of motion of a spherical or physical pendulum, as well as to the mode of regular precession, and, in addition, to carry out its forced conservative motion. It is interesting to note that in all variants of the formation of control torques, they are linear in generalized velocities. The conclusions drawn are of theoretical interest and can be applied to other mechanical systems with several degrees of freedom, and they are of no small importance for practical applications in various technical problems.

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