PERIODIC, QUASI–PERIODIC & CHAOTIC MOTION IN MASS–SPRING MODEL

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Abstract
The dynamics of nonlinear mass–spring model under periodic force is considered. Through the application of Banach's fixed point theorem, some conditions for the existence of periodic solution are obtained. At last by simulations, quasi–periodic and chaotic behaviors of model are shown.

Key words
Mass-Spring model, periodic, quasi–periodic, chaos.

1 Introduction

Many known planar models such as Duffing [Jing (2005)], Van der Pol [Guckenheimer (2003)], Pendulum [Hubbard (1999)] models have periodic solutions which are implication of the Hopf bifurcation. To design a model whose periodic solutions are not obtained by Hopf bifurcation is the second order nonlinear forced differential equation that is given in this paper. This model also shows the quasi–periodic and chaotic behaviors for a suitable rang of parameters.

There have been extensive researches on the existence of periodic solution for the nonlinear non-autonomous system of ordinary differential equations. One of the standard methods employed for this, is to transform the system into a Fredholms type integral operator by making use of a Green or generalized Green function with a suitable periodic or semi-periodic boundary conditions, and then to apply the topological fixed point theorems like one of Banach or LeraySchauder to prove the existence of a solution for the integral operator, see for example [Cronin (1964)], [Lazer (1968)], [Mehri (1997)].

In this paper first we introduce a new model of well known mechanical model mass-spring by a new nonlinearity. Then we analyze the model in some simple cases. These include section 2. In section 3, using Banach's fixed point theorem we shall prove the existence of the periodic solution and the conditions that system has a periodic solution. Then in section 4 quasi-periodic and chaotic behaviors of model are shown by simulations.

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2 Dynamic of Model

Consider the mass-spring friction system represented in Fig. 1. The friction forces include viscous friction is assumed to behave nonlinearly in accordance with the square of the velocity $b\dot{x}^2$, the spring force is a nonlinear function of x which can be modeled as $kx|x|$ and is similar to the classical nonlinear spring which is described by kx^3 and external force is $A\cos(\omega t)$. So the

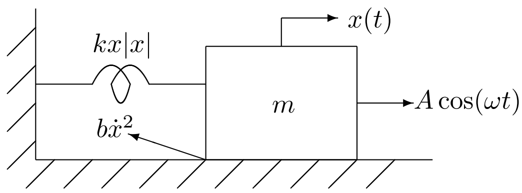

Figure 1. Scheme of nonlinear mass-spring model

differential equation of system is:

$$m\ddot{x}(t) + b\dot{x}^2(t) + kx(t)|x(t)| = A\cos(\omega t),$$

which can be written as

$$\ddot{x}(t) + \beta\dot{x}^2(t) + \alpha x(t)|x(t)| = \gamma\cos(\omega t), \qquad (1)$$

where α, β and γ are nonnegative parameters. For $\alpha = 0$, one can easily obtain the exact solution. Now we analyze (1) with respect to parameters.

Let $\gamma = 0$. The origin is the only equilibrium point of system (1) and linearization about the origin shows that it is a nonhyperbolic equilibrium point. To study the system (1) we first consider two cases:

Case 1.$\alpha \neq 0$, $\beta = 0$
In this case we choose Lyapunov function $V(x, \dot{x}) = 2\alpha x^2|x| + 3\dot{x}^2$. The derivative of V with respect to time along the trajectory of system is $\dot{V}(x, \dot{x}) = 6\alpha x|x|\dot{x} + 6\dot{x}\ddot{x} = 0$. Since the origin dose not lie on the close curve $2\alpha x^2|x| + 3\dot{x}^2 = c^2$, it is a periodic solution of system for the initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ and $c^2 = 2\alpha x_0^2|x_0| + 3\dot{x}_0^2$. Figure 2 shows the phase portrait of model (1) for $\alpha = 1$ with different initial conditions.

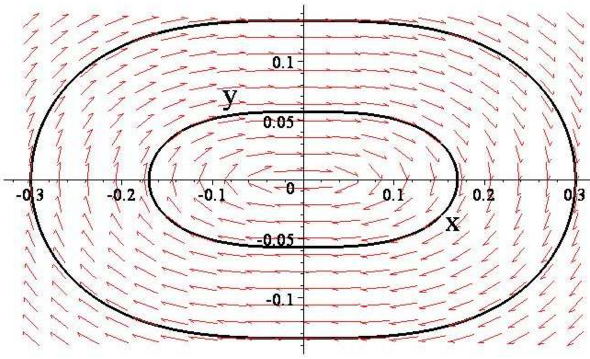

Figure 2. Phase portrait of system in case 1, $\alpha = 1$.

Case 2. $\alpha \neq 0$, $\beta \neq 0$
In this case we have the following differential equation

$$\ddot{x} + \beta\dot{x}^2 + \alpha x|x| = 0. \tag{2}$$

In [Sabatini (2004)] the model $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$ has been studied. By Choosing $f(x) = \beta$, $g(x) = \alpha x|x|$, the assumptions of corollary 1. in [Sabatini (2004)] are satisfied, therefore the origin is a center for (2) and all solutions started at some neighborhood of the origin are periodic. Figure 3 shows the phase portrait in this case for $\alpha = \beta = 1$ with different initial conditions.

3 Existence and Uniqueness of Periodic solution

Here we find by Green's function the time dependent solution of the forced model (1) as a nonlinear boundary value problem and then extend it to the real line. Consider (1) with the initial conditions

$$x(T) = 0, \qquad \dot{x}(0) = 0, \tag{3}$$

where $T = \frac{\pi}{2\omega}$. We establish the following theorem.

Theorem 1. If there exits $M > 0$ such that $2MT\max(\alpha, \beta)(1 + T) < 1$, and $\big((\alpha + \beta)M^2 + \gamma\big)T\max(1, T) \leq M$, then the equation (1) has a

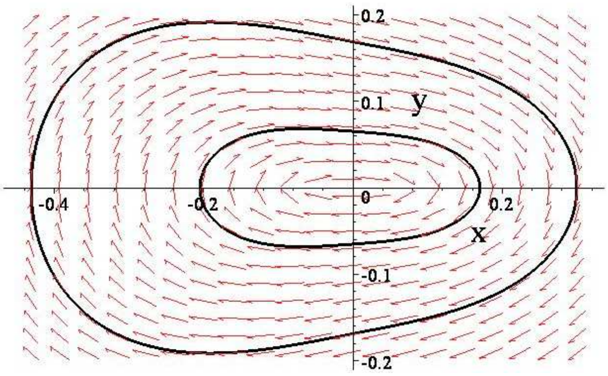

Figure 3. Phase portrait of system in case 2, $\alpha = \beta = 1$.

unique solution satisfying boundary conditions (3).
Proof: Consider the following function

$$G(t, s) = \begin{cases} s - T & 0 \leq t \leq s \leq T \\ t - T & 0 \leq s \leq t \leq T \end{cases} \tag{4}$$

$G(t, s)$ satisfies continuity, jump and boundary conditions (3) [Kondo (1991)] and we have

$$\max_{0 \leq t, s \leq T} |G(t, s)| = T.$$

Then

$$G_t(t, s) = \begin{cases} 0 & 0 < t < s < T \\ 1 & 0 < s < t < T \end{cases}. \tag{5}$$

and consequently

$$\int_0^T |G(t, s)ds| \leq T^2, \qquad \int_0^T |G_t(t, s)ds| \leq T.$$

Now, let $M > 0$ be given and $Q = (\alpha + \beta)M^2 + \gamma$. Let $B = \{x \in C^1[0, T] : |x(t)| \leq M, |\dot{x}(t)| \leq M\}$, and define the map L on B by

$$(Lx)(t) = \int_0^T G(t, s)f(s, x(s), \dot{x}(s))ds,$$

where

$$f(\cdot, \cdot, \cdot) = \beta\dot{x}^2(s) + \alpha x(s)|x(s)| - \gamma\cos(\omega s). \tag{6}$$

Then

$$|(Lx)(t)| \leq QT^2, \quad \text{and} \quad \Big|\frac{d}{dt}(Lx)(t)\Big| \leq QT. \tag{7}$$

Hence by (7), L maps B into itself provided

$$QT\max(1,T) \leq M. \tag{8}$$

For $x_1, x_2 \in B$ let

$$d(x_1,x_2) = \max_{0\leq t\leq T}\{|x_1 - x_2| + |\dot{x_1} - \dot{x_2}|\},$$

Then (B, d) is a complete metric space. Whe show that L is a contraction map.
For $x_1, x_2 \in B$, we have

$$\begin{aligned} |Lx_1 - Lx_2| &= \Big|\int_0^T G(t,s)\{\beta(\dot{x}_1^2(s) - \dot{x}_2^2(s)) \\ + &\quad \alpha(x_1(s)|x_1(s)| - x_2(s)|x_2(s)|)\}ds\Big| \\ \leq &\quad \int_0^T |G(t,s)|\{\beta|\dot{x}_1 - \dot{x}_2||\dot{x}_1 + \dot{x}_2| \\ &\quad +\alpha(|x_1| + |x_2|)|x_1 - x_2|\}ds \\ \leq &\quad 2MT^2\max(\alpha,\beta)d(x_1,x_2). \end{aligned} \tag{9}$$

Similarly

$$\begin{aligned} &\Big|\frac{d}{dt}(Lx_1)(t) - \frac{d}{dt}(Lx_2)(t)\Big| \leq \\ &\quad 2MT\max(\alpha,\beta)d(x_1,x_2) \end{aligned} \tag{10}$$

So

$$d(Lx_1, Lx_2) \leq 2MT\max(\alpha,\beta)(1+T)d(x_1,x_2)$$

and L is a contraction with respect to the metric define on B provided that

$$2MT\max(\alpha,\beta)(1+T) < 1. \tag{11}$$

Hence, if M satisfies both (8) and (11), then by the Banach's fixed point theorem, equation (1) with boundary conditions (3) has a unique solution $x(t)$ with $|x(t)| \leq M$ and $|\dot{x}(t)| \leq M$.■

Theorem 2. The unique solution established in Theorem 1 can be extended to $4T$-periodic solution.

Proof: Let $x(t)$ be the unique solution defined on $[0, T]$. Consider the following function on $[0, 4T]$:

$$z(t) = \begin{cases} x(t) & 0 \leq t \leq T \\ -x(2T - t) & T \leq t \leq 2T \\ -x(t - 2T) & 2T \leq t \leq 3T \\ x(4T - t) & 3T \leq t \leq 4T \end{cases}$$

It is easy to verify that $z(t)$ is a solution of model (1) and it satisfies the following periodic boundary conditions:

$$z(0) = z(4T), \qquad \dot{z}(0) = \dot{z}(4T)$$

■

For convenience, we show the periodic solution obtained in theorems 1 and 2 by $x(t)$. The periodic solution of the model (1) is shown in Fig. 4 in phase portrait and time history on the top. Here, $M = .22$

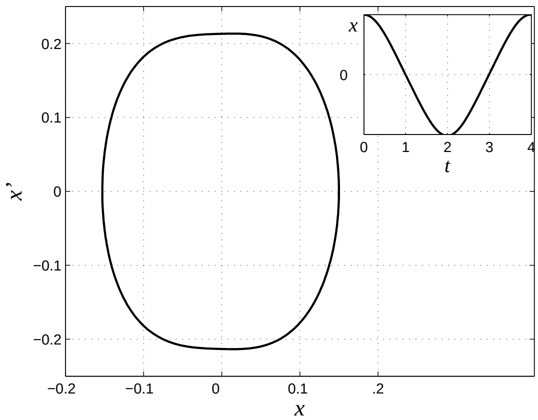

Figure 4. Periodic solution of system for $\alpha = 19.6, \beta = \gamma = 0.01, \omega = 2\pi, x(0) = x(4), \dot{x}(0) = \dot{x}(4)$.

4 Quasi–Periodic and Chaotic Solution of the Model

Periodic solution of the model was obtained in the preceding section is not an asymptotically stable solution. This is because by chancing the initial conditions other solutions are observed. Also this shows that the model (1) has quasi–periodic solutions. It is not so unexpected: quasi–periodic solutions occur in the forced linear systems where the free system has a periodic solution with different (not rational multiple of each other) period from the corresponding forced one.

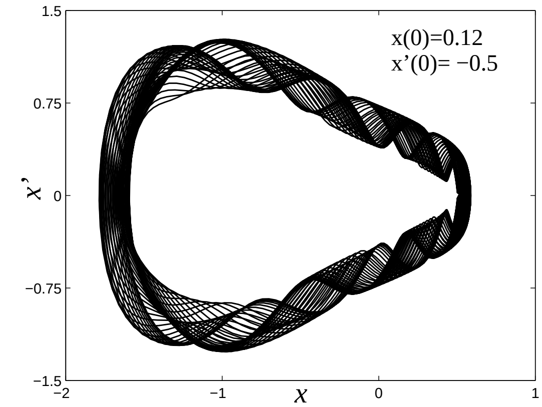

Figure 5. Quasi–Periodic solution of the model.

A quasi–periodic solution with $\alpha = \beta = \gamma = 1$, and $\omega = 1.9\pi$ is shown in figure 5.

Also for some values of parameters α, β, γ and ω in the nonlinear forced model (1) another complex dynamic, chaos, is observed. In Fig. 6 Poincare section of the model is plotted for $\alpha = \beta = \gamma = 1$ and $\omega = 0.8\pi$ which is seen to be chaotic. For another sets of values

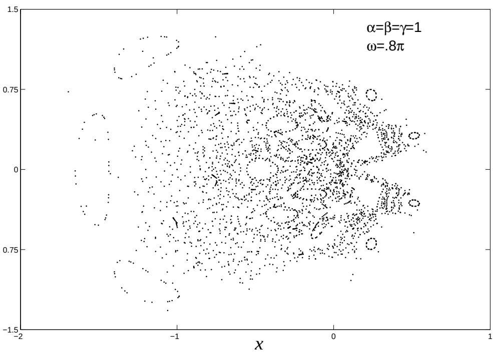

Figure 6. Poincare section of the model corresponding to the chaotic behavior.

of parameters, chaos is shown in the $(x, \dot{x})$ phase plane. These are in figures 7,8.

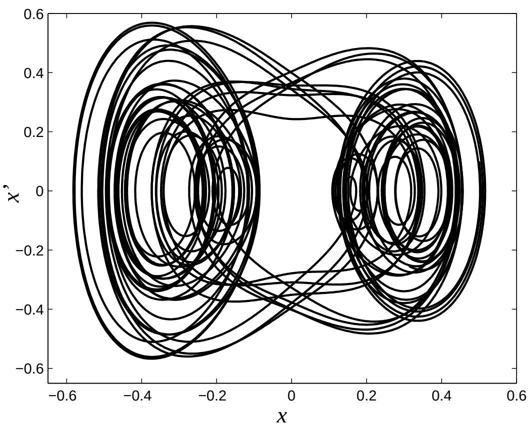

Figure 7. Phase portrait of the model corresponding to the chaotic behavior for $\alpha = 8, \beta = 0.1, \gamma = 1$ and $\omega = 0.1\pi$.

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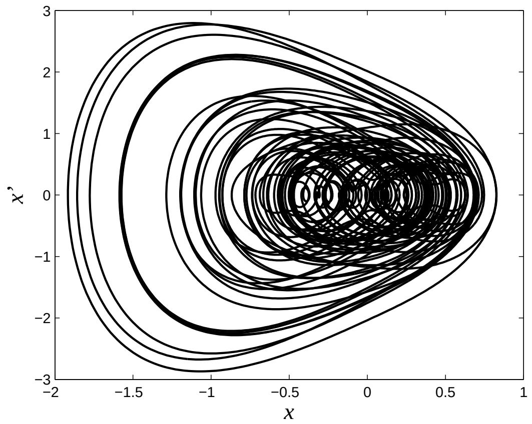

Figure 8. As same as fig. 7 $\alpha = 3, \beta = 0.5, \gamma = 1$ and $\omega = 0.8\pi$.

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