ON SOME ASPECTS OF THE RESTRICTED THREE DIMENSIONAL THREE-BODY PROBLEM

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Abstract
In this paper we consider the three body problem in the variant, suggested by Kolmogorov, when two points of equal masses in the plane follow the elliptic orbits, symmetric with respect to the axis orthogonal to the plane; the third point of zero mass belongs to this axis during the motion. Considering the eccentricity as a small parameter of our problem, we prove the existence of infinitely many families (with respect to eccentricity) of regular long-periodic solutions. It means, the studied system is in a compound chaotic behavior. The elliptic solutions are stable not only in linear approximation, but they are orbitally stable in Lyapunov sense.

Key words
restricted three-body problem, stability

Let us consider the three body problem in the following setting (described in [1]). Two points of equal masses in the plane (x, y) follow the elliptic orbits, symmetric with respect to z-axis. The third point of zero mass belongs to the z-axis during the motion (Fig. 1) This variant of the restricted three body problem has been suggested by Kolmogorov to analyze the finite motion of the three gravitating bodies.

The force of gravitational interaction with the force function

$$U = \frac{\gamma}{\sqrt{z^2 + r^2(t)}},$$

acts on the point m; here $r(t) = \frac{1}{1+e\cos\varphi(t)}$, e being the elliptic orbit eccentricity. Further we set $\varphi \equiv t, \quad \gamma = 1$ and consider $\varepsilon = e \ll 1$ to be a small parameter. Then the Hamilton function

$$H = \frac{p^2}{2} - \frac{1}{\sqrt{z^2 + r^2}}$$

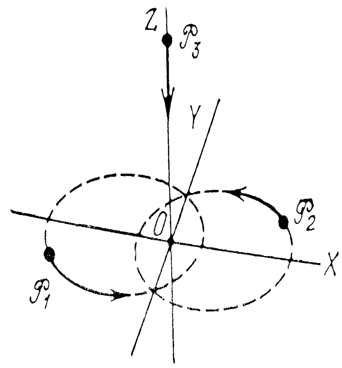

Figure 1.

can be cast into the form

$$H = H_0 + \varepsilon H_1 + o(\varepsilon)$$

$$H_0 = \frac{p^2}{2} - \frac{1}{(1+z^2)^{1/2}}, \qquad H_1 = -\frac{\cos t}{(1+z^2)^{3/2}}$$

To study periodic solutions we introduce the "action-angle" I, φ variables for the unperturbed system with the Hamiltonian H_0. From the energy integral $H_0 = h$ we obtain the condition

$$\frac{p^2}{2} = h + \frac{1}{\sqrt{1+z^2}} \geq 0$$

The phase portrait of the unperturbed system is shown in Fig.2. The oscillatory motion corresponds to the energy levels $h < 0$, i.e.

$$\frac{1}{\sqrt{1+z^2}} \geq -h = |h|$$

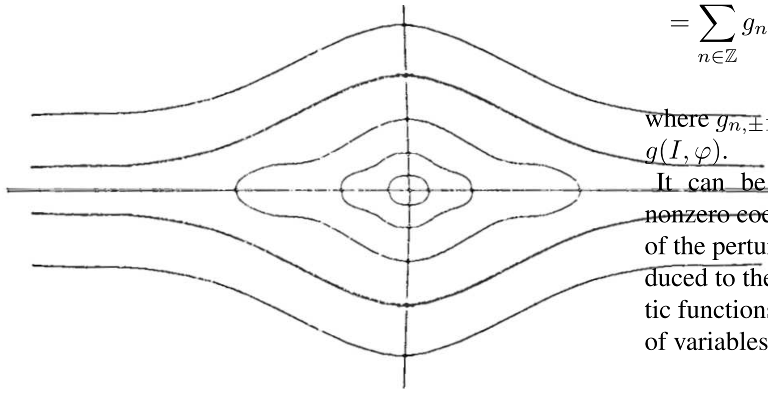

Figure 2.

Hence $|z| \leq \sqrt{1+z^2}/|h| = a$. By definition:

$$I = \frac{1}{2\pi}\oint pdz$$

or in the explicit form

$$I = \frac{1}{\pi}\int_{-a}^{a} pdz = \frac{\sqrt(2)}{\pi}\int_{-a}^{a}\sqrt{h + \frac{1}{\sqrt{1+z^2}}}dz \quad (1)$$

We introduce the frequency of the periodic motion $\omega = \partial H_0/\partial I \quad (H \equiv h)$. Denoting the derivative with respect to I by prime, from (1) we obtain

$$\frac{1}{h'} = \frac{1}{\omega} = \frac{1}{\pi\sqrt(2)}\int_{-a}^{a}\frac{dz}{\sqrt{h + 1/\sqrt{1+z^2}}}$$

Let now $h \to 0$. Then

$$a \to \infty, \quad \frac{1}{\omega} \to \frac{1}{\pi\sqrt(2)}\int_{-a}^{a}\sqrt[4]{1+z^2}dz = \infty$$

Thus, as the separatrix is approached, ω tends to zero and the action variable I tends to infinity:

$$I \to \frac{\sqrt{2}}{\pi}\int\limits_{-\infty}^{\infty}\frac{dz}{\sqrt{1+\sqrt[4]{1+z^2}}} = \infty$$

Therefore, we can always choose a sequence $\{I_n\}_{n=1}^{\infty}$ of the action-variable values in a way that $\omega(I_n) = 1/n, \quad I_n \to \infty$.

The perturbation function H_1 can be represented in the form

$$H_1(I,\varphi,t) = -\frac{1}{(1+z^2)^{3/2}}\cos t = g(I,\varphi)\frac{e^{it}+e^{-it}}{2} =$$

$$= \sum_{n\in\mathbb{Z}} g_{n,1}(I)e^{i(n\varphi+t)} + \sum_{n\in\mathbb{Z}} g_{n,-1}(I)e^{i(n\varphi-t)} \quad (2)$$

where $g_{n,\pm1}$ are the Fourier coefficients of the function $g(I,\varphi)$.

It can be shown that there exist infinitely many nonzero coefficients $g_{n,\pm1}$ in the Fourier expansion (2) of the perturbing function H_1. This problem can be reduced to the analysis of Fourier expansion of the elliptic functions. Indeed, performing the following change of variables $\{z,p\} \to \{x,y\}$:

$$\sqrt{1+z^2} = \frac{1}{x}, \quad p = -yx^2\sqrt{1-x^2}$$

we obtain

$$H_0 = \frac{y^2}{2}x^4(1-x^2) - x, \quad H_1 = -x^2\cos t$$

The canonical variables φ (conjugate to I) are determined by the equation:

$$\varphi = \frac{\partial S}{\partial I}$$

where, S is the generating function of the canonical transformation from the variables x, y to the "action-angle" variables:

$$S = \int_{x_0}^{x} ydx$$

Then

$$\varphi = \frac{\partial}{\partial I}\left(\sqrt{2}\int\limits_{x_0}^{x}\frac{\sqrt{x+h}}{x^2\sqrt{1-x^2}}dx\right) =$$

$$= h'\frac{1}{\sqrt{2}}\int\limits_{x_0}^{x}\frac{dx}{x^2\sqrt{(x+h)(1-x^2)}} = \omega\int\limits_{x_0}^{x}\frac{dx}{x^2\sqrt{2(x+h)(1-x^2)}}$$

Hence, x is an elliptic function of φ/ω (see, for example, [2]).

Now we apply the nonautonomous variant of the Poincaré theorem on the existence of periodic solutions [3] Suppose the following conditions are fulfilled:

1. the frequency ω is rational for $I = I_0$;
2. $\partial^2 H_0/\partial I^2 \neq 0$ for $I = I_0$
3. for some $\varphi = \varphi_0$ we have $\partial\bar{H}_1/\partial\varphi = 0$ and $\partial^2\bar{H}_1/\partial\varphi^2 \neq 0$ for $I = I_0$, where

$$\bar{H}_1(I_0,\varphi_0) = \frac{1}{T}\int\limits_{0}^{T} H_1(I_0, \omega t + \varphi_0)dt,$$

here T is the period of the function $H_1(I, \varphi, t)$ on an invariant resonance torus.

Then for small $\varepsilon \neq 0$ there exists a periodic solution of the perturbed problem. This solution is of oeriod T and depends analytically on the parameter ε, moreover for $\varepsilon = 0$ it coincides with a periodic solution of the unperturbed problem

$$I = I_0, \quad \varphi = \omega t + \varphi_0$$

The characteristic exponents $\pm\alpha$ of this solution can be expanded in the convergent series in $\sqrt{\varepsilon}$:

$$\alpha = \alpha_1\sqrt{\varepsilon} + \alpha_2\varepsilon + \alpha_3\varepsilon\sqrt{\varepsilon} + \dots,$$

$$\alpha^2 = -\frac{\partial^2 \bar{H}_1}{\partial\varphi^2}\frac{\partial^2 \bar{H}_0}{\partial I_0^2} \neq 0$$

Let us note that for $\alpha^2 < 0$, there is the condition of orbital stability of elliptic solutions (in Lyapunov sense) [5]:

$$5\left(\frac{\partial^3 \bar{H}_1}{\partial\varphi^3_)}\right)^2 - 3\left(\frac{\partial^2 \bar{H}_1}{\partial\varphi^2_)}\right)\cdot 5\left(\frac{\partial^4 \bar{H}_1}{\partial\varphi^4_)}\right)$$

In our case we set $I_0 = I_n$, $\omega(I_n) = 1/n$. As it was noted above, there are infinitely many values I_n of this kind.

We shall check the fulfillment of the condition $\partial^2 H_0/\partial I^2 \neq 0$ in a neighborhood of the singular point $h = 0$. We have

$$\frac{\partial H_0}{\partial I} = h' = \frac{\pi\sqrt{2}}{\int\limits_{-a}^{a}\frac{dz}{\sqrt{h+\frac{1}{\sqrt{1+z^2}}}}}$$

$$\frac{\partial^2 H_0}{\partial I^2} = h'' = \frac{\pi\sqrt{2}}{\left(\int\limits_{-a}^{a}\frac{dz}{\sqrt{h+\frac{1}{\sqrt{1+z^2}}}}\right)^2}\cdot$$
$$\cdot\frac{d}{dI}\left(\int\limits_{-a}^{a}\frac{dz}{\sqrt{h+\frac{1}{\sqrt{1+z^2}}}}\right)$$

Denoting

$$f(h) = \int\limits_{-a}^{a}\frac{dz}{\sqrt{h+\frac{1}{\sqrt{1+z^2}}}}$$

we obtain

$$\frac{df}{dI} = \frac{df}{dh}\frac{dh}{dI} = \omega\frac{df}{dh}$$

Since $dh/dI = \omega \neq 0$, it is necessary to prove that

$$\frac{df}{dh} \neq 0$$

in a neighborhood of the singular point. One can easily check that

$$\frac{df}{dh} = \frac{1}{2}\int_{-a}^{a}\frac{dz}{(h+1/\sqrt{1+z^2})^{3/2}} +$$

$$+\left.\frac{2}{(\sqrt{h+1/\sqrt{1+z^2}})^{3/2}}\right|_{z^2=a^2}\cdot\frac{1-2h^2}{h^2\sqrt{1-h^2}} \to \infty$$

when $h \to 0$, i.e. the condition 2. of the Poincaré theorem is fulfilled.

Let us consider the condition 3. The perturbation function has the following form:

$$H_1 = \sum_{n\in\mathbb{Z}} g_{n,1}e^{i(n\varphi+t)} + \sum_{n\in\mathbb{Z}} g_{n,-1}e^{i(n\varphi-t)}$$

Suppose

$$\frac{\partial\bar{H}_1}{\partial\varphi_0} = \frac{1}{T}\int_0^T\left(\sum_{n\in\mathbb{Z}} g_{n,1}ie^{i(n\omega+1)t+i\varphi_0} -\right.$$
$$\left.-\sum_{n\in\mathbb{Z}} g_{n,-1}ie^{i(n\omega-1)t-i\varphi_0}\right)dt =$$
$$= i(g_{n,1}e^{i\varphi_0} - g_{n,-1}e^{-i\varphi_0}) = 0$$

for some value of φ_0. Then $\partial^2\bar{H}_1/\partial\varphi_0^2 \neq 0$, as if $\varphi^* = \varphi_0$ is a root of the equation

$$a\sin\varphi_0 + \beta\cos\varphi_0 = 0$$

where α, β are some coefficients that are not linearly independent of $g_{n,\pm1}$, then $\varphi^* = \varphi_0$ is a root of the equation

$$\frac{\partial}{\partial\varphi_0}(a\sin\varphi_0 + \beta\cos\varphi_0) = a\cos\varphi_0 - \beta\sin\varphi_0 = 0$$

only in case when

$$|\alpha| + |\beta| = 0$$

or, equivalently, when

$$|g_{n,1}| + |g_{n,-1}| = 0$$

That results in contradiction.

Thus, since the conditions 1-3 of the Poincaré theorem are fulfilled, the perturbed problem admits infinitely many families (with respect to ε) of regular long-periodic solutions. These solutions are accumulated in a neighborhood of separatrices od the unperturbed problem. It means that the considered system is in compound chaotic motion within the indicated domain. The elliptic solutions are stable not only in linear approximation, but also orbitally stable in Lyapunov sense. The existence of an infinite number of regular long-periodic solutions implements the qualitative picture of stochastic behavior of this system considered in [4].

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