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CHARACTERIZING THE COMPLEXITY OF TIME SERIES NETWORK GRAPHS: A SIMPLICIAL APPROACH

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Abstract We use the techniques of time series network duality to map the time series of the logistic map, at several special parameter values, to equivalent networks utilizing the visibility algorithm. We discuss a set of measures which can analyse the simplicial structure of the resulting networks using the techniques of algebraic topology. We find that the complexity of the simplicial structure and the levels of hierarchy involved increase with the chaoticity of the system. We discuss the implications of our results.

1 Introduction

The analysis of the time series of the variables of evolving dynamical systems is an important tool for analyzing the dynamical behavior of nonlinear dynamical systems, as well as for making predictions for their behavior. A variety of well developed methods and tools are used to carry out this kind of analysis, which also define a set of precise metrics such as the Fourier transform, correlation dimensions and entropy, and Lyapunov exponents [Kantz and Schreiber, 2004]

It has recently been realized, that the network representation of time series offers additional ways of extracting the dynamical information of the system using a variety of metrics developed and tested for networks over the recent decades [Zhang and Small, 2006; Yang and Yang, 2008; Marwan, et. al, 2009]. Some of the recent methods used for mapping time series to graphs include the use of visibility graphs [Lacasa and Luque, 2008], and the quantile mapping [Campanharo, et. al, 2011]. The resulting networks have been analyzed using conventional network measures such as clustering coefficients and path lengths. In this paper we analyze the time series networks (TS networks) obtained from the time series obtained from the logistic map at different parameter values, using methods of algebraic topology [Atkin, 1972; Duckstein and Nobe, 1997; Kramer and Laubenbacher, 1998] and recently constructed measures which analyze the simplicial structure of graphs. We show that the methods are able to identify the crucial differences between the time series corresponding to distinct dynamical behaviors.

This paper is organized as follows. We use the time series data from the Logistic map at parameter values that show characteristic periodic as well as characteristic chaotic behaviour. (See Fig. 1). To map these time series data sets into their corresponding network representations we employ the visibility algorithm [Lacasa and Luque, 2008] in Section 2. We define the measures used to analyze the simplicial structures of the resulting graphs, and their connection with the topological structure and topological connectivity in Section 3. The results of the analysis are also tabulated in this section. We discuss the results, and the inferences drawn from these results in Section 4 and summarize and conclude in Section 5.

2 Visibility Graph

In this paper, we use the visibility algorithm developed by [Lacasa and Luque, 2008] to transform a time series (in this case, the time series obtained from the logistic map), into a network. Recent developments show that going from the time series representation to the network representation yields additional information of the underlying dynamics[Luque, et. al, 2011; Campanharo, et. al, 2011]. While there are a number of methods developed over recent years to convert a time series into a network [Zhang and Small, 2006; Yang and Yang, 2008; Campanharo, et. al, 2011], we use the visibility algorithm here, for the following reasons: i) the simplicity of the visibility approach, ii) the TS network resulting from the visibility algorithm conserves the structure of the time series, viz., a periodic time series gives rise to regular graphs, a random time series gives rise to random graphs, and a fractal time series gives rise to scale-free graphs [Lacasa and Luque, 2008].

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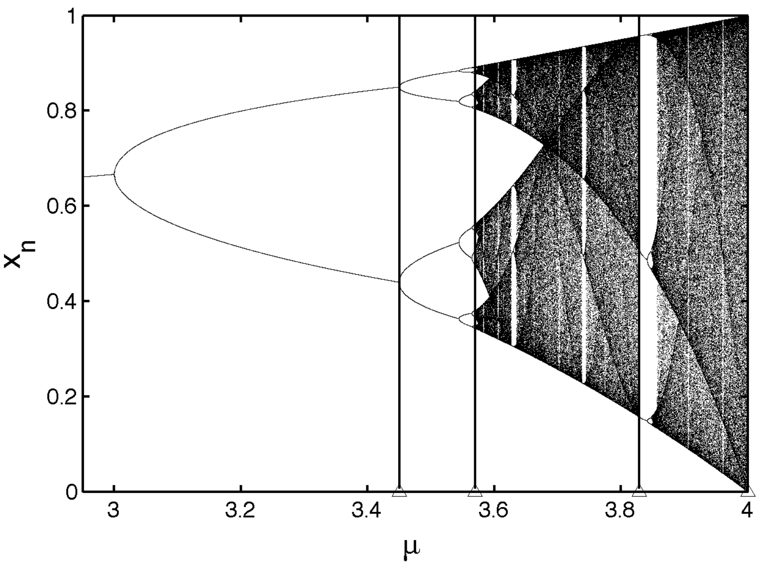

Figure 1. The bifurcation diagram of the logistic map used in our study with the four points identified with dark vertical lines at $\mu = 3.45, 3.56995, 3.82843$, and 4.0.

The visibility algorithm introduced in [Lacasa and Luque, 2008] is implemented as follows. Let the pair of points (y_i, t_i) denote the data in the time series for $r = 1, 2, \ldots N$. For any two pairs (y_i, t_i) and (y_j, t_j) to be visible to each other (line of sight visibility), all other intermediate data pairs (y_r, t_r) should satisfy this condition:

$$y_j > y_r + \frac{y_j - y_i}{t_j - t_i}(t_j - t_r) \tag{1}$$

In this paper, our system of interest is the logistic map defined by $x_{n+1} = \mu x_n(1 - x_n)$, where the nonlinearity is introduced in the map by the parameter $\mu \in [0, 4]$, and $x_n \in [0, 1]$. The time series of the logistic map shows periodic behaviour for $\mu = 3.5$, and chaotic behaviour for $\mu = 4.0$, as shown in the top panels of Figs. 2 and 3, respectively. We employ the visibility algorithm to convert these time series into their corresponding network representations. The network representation of the periodic time series shows repetitive motifs (Fig. 2), and the corresponding network for the chaotic time series shows an irregular topology (Fig. 3). Similar network representations of the time series can be seen in [Luque, et. al, 2011], but have not been further analyzed by quantitative methods. In the next section we analyze the network representations obtained from the time series at various values of μ using the methods of algebraic topology. Since the dynamical system which contributes the time series is very well understood, the TS networks studied here constitute good test beds for analyzing the effectiveness of the algebraic topology methods.

3 The simplicial analysis of the TS networks

In this section, we study the topological structural properties of the networks generated by the visibility algorithm. The connectivity and topological properties of the network graphs reflect the connections between the dynamical states of the system in time [Maletić and Rajković, 2012; Kramer and Laubenbacher, 1998]. The networks so obtained are further classified using the concepts of *cliques* and *simplices* [Bron and Kerbosch, 1973; Andjelković, et. al, 2014].

A graph or a network represents interacting nodes interconnected by the links/edges. We consider here the simplicial complexes of graphs. A simplex with $q + 1$ nodes or vertices is a q-dimensional simplex. For instance, a 0-simplex is an isolated point, a 1-simplex is two vertices connected by a line segment, a 2-simplex is a triangle of three connected nodes, 3-simplex is a tetrahedron with 4 connected nodes, and so on. Further, if two simplices have $q+1$ nodes in common, they share a q-face. A collection of simplices – not just the nodes, but their shared faces as well – forms a simplicial complex. The dimension of the simplicial complex is defined as the dimension of the largest simplex in the structure. If we can find a sequence of simplices such that each successive pair share a q-face, then all the simplices in this sequence are said to be q-connected. Simplices which are q-connected are also connected at all lower levels.

In this study, we consider a simplicial complex where the simplices are cliques. A clique is a maximal complete subgraph – the nodes of a clique are not part of a larger complete subgraph. Using the adjacency matrix of the network, we employ the Bron-Kerbosch algorithm [Bron and Kerbosch, 1973] to find the cliques. In our study, we carry out the structural and connec-

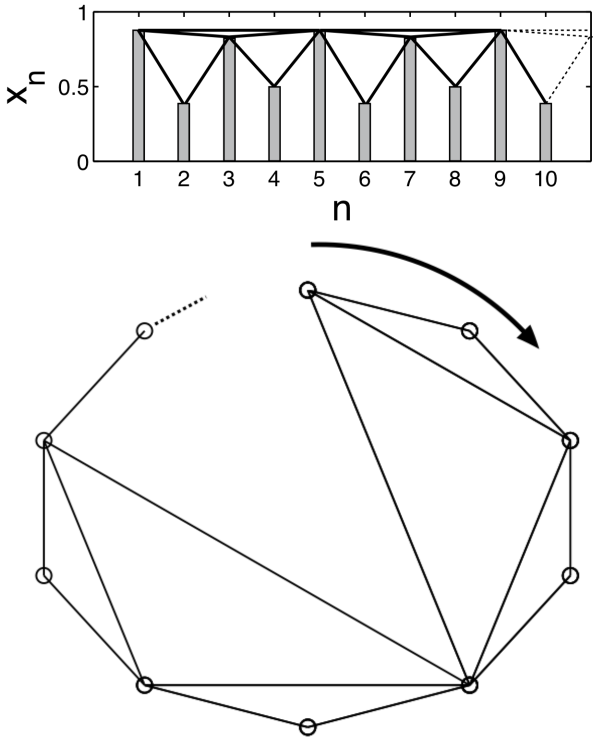

Figure 2. A periodic time series (top panel) obtained from the logistic map at $\mu = 3.5$ is mapped to a network using visibility algorithm that shows repetition of motifs periodically (bottom panel).

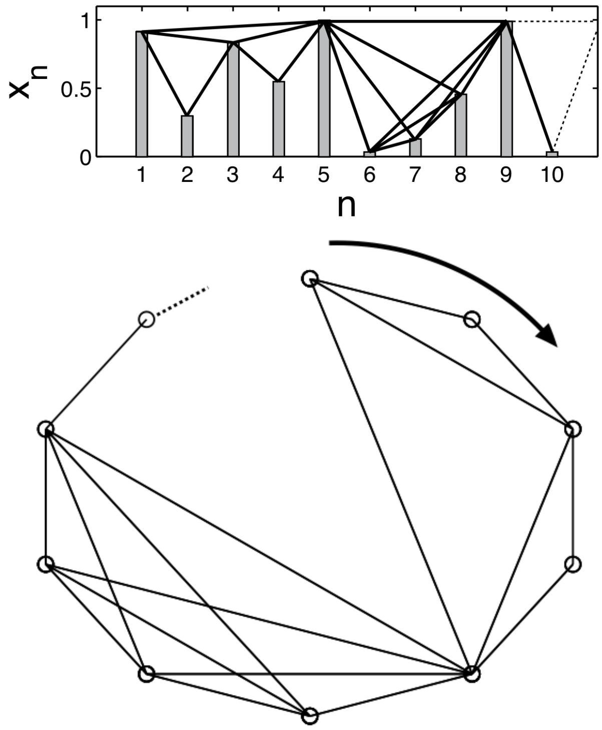

Figure 3. A chaotic time series (top panel) obtained from the logistic map at $\mu = 4.0$ and its network realization (bottom panel). This network show distinct features that are different from the network generated using periodic time series in Fig.2.

tivity analysis of the TS networks obtained from the logistic map time series, using six topological characterizers defined as follows [Kramer and Laubenbacher, 1998; Atkin, 1972; Duckstein and Nobe, 1997; Maletić and Rajković, 2012]. The first characterizer is the vector $\mathbf{Q}$, known as the first structure vector, which is a measure of the connectivity of the clique complex at various levels. The qth component of the $\mathbf{Q} = \{Q_0, Q_1, \ldots Q_{qmax}\}$ is the number of $q-$ connected components at the $q-th$ level. The next vector quantity, which we denote by $\tilde{\mathbf{f}}$, is defined to have the number of q-dimensional simplices as its q-th component. The third quantity $\mathbf{N_s} = \{n_0, n_1, \ldots, n_{qmax}\}$, known as the second structure vector, has the number of simplices of dimension q and higher as its qth component. The third structure vector, $\widehat{\mathbf{Q}}$, is defined in terms of the previously defined structure vectors $\mathbf{Q}$ and $\mathbf{N_s}$. Its q-th component, $\widehat{Q}_q$ is given by $\left(1 - \frac{Q_q}{n_q}\right)$. A fifth quantity $\dim Q^i$, is a local quantity, which defines the topological dimension of node i of the simplicial complex, given by

$$\dim Q^i = \sum_{q=0}^{q_{\max}} Q_q^i, \tag{2}$$

where $q_{\max}$ is the dimension of the simplicial complex, and Q_k^i is the number of different simplices of dimension k in which the node i participates.

Finally, the topological entropy $\mathbf{S}$ is defined as

$$S_Q(q) = -\frac{\sum_i p_q^i \log p_q^i}{\log N_q}. \tag{3}$$

Here, $p_q^i = Q_q^i / \sum_i Q_q^i$ is the probability of a particular node i participating in a q-simplex, and $N_q = \Sigma_i \left(1 - \delta_{Q_q^i, 0}\right)$ denotes the number of nodes that participate in at least one q-simplex. We illustrate the calculation of these quantities for a simple example here.

3.1 Example:

Let us take the simplicial complex in Fig. 4 (ii) where two triangular simplices have two vertices in common, to illustrate how the six characterizers are calculated.

The simplicial complex consists of two simplices $A = \{1, 2, 4\}$ and $B = \{2, 3, 4\}$, with the vertices labelled as shown. The incidence matrix is given by

$$\mathbf{\Lambda} = \begin{pmatrix} 1\ 1\ 0\ 1 \\ 0\ 1\ 1\ 1 \end{pmatrix}$$

	$\mu = 3.45$			$\mu = 3.56995$			$\mu = 3.82843$			$\mu = 4.0$		
q-level	$\mathbf{Q}$	$\mathbf{N_s}$	$\widehat{\mathbf{Q}}$	$\mathbf{Q}$	$\mathbf{N_s}$	$\widehat{\mathbf{Q}}$	$\mathbf{Q}$	$\mathbf{N_s}$	$\widehat{\mathbf{Q}}$	$\mathbf{Q}$	$\mathbf{N_s}$	$\widehat{\mathbf{Q}}$
0	1	1499	0.999	1	1988	0.999	1	1333	0.999	1	1984	0.999
1	501	1499	0.665	12	1988	0.993	667	1333	0.499	18	1984	0.990
2	1498	1498	0	1987	1987	0	1332	1332	0	1981	1981	0

Table 1. Three structure vectors evaluated for the TS networks of the Logistic map at four parameter values. The time series considered is of length 2000 therefore, the number of nodes/vertices of the TS networks is also $N = 2000$.

	$\mu = 3.45$		$\mu = 3.56995$		$\mu = 3.82843$		$\mu = 4.0$	
q-level	$\mathbf{S}$	$\tilde{\mathbf{f}}$	$\mathbf{S}$	$\tilde{\mathbf{f}}$	$\mathbf{S}$	$\tilde{\mathbf{f}}$	$\mathbf{S}$	$\tilde{\mathbf{f}}$
0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	0.95915	3
2	0.97745	1498	0.95455	1987	0.98852	1332	0.95506	1981

Table 2. The structure vector and the entropy evaluated for the TS networks ($N = 2000$) for four parameter values of the logistic map.

Both the simplices A and B are of dimension 2. Therefore, the first structure vector $\mathbf{Q}$ has three components. Recall that if two simplices are to be q-connected, they should have at least $q + 1$ nodes in common, and also that if two simplices are q-connected, they are also connected at all lower

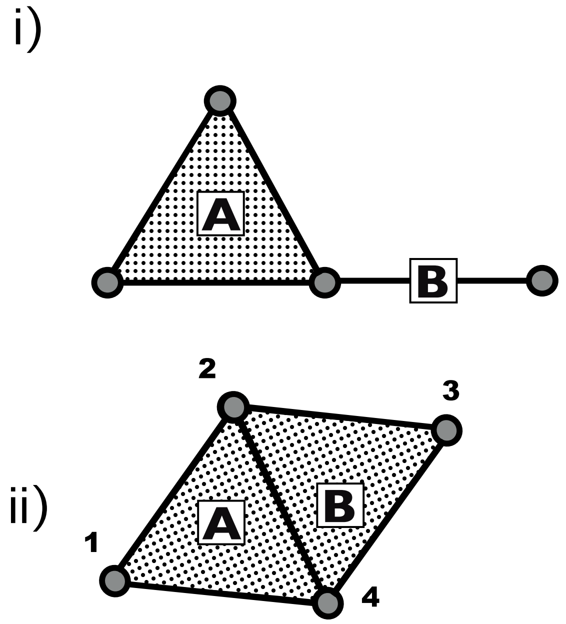

Figure 4. Illustrations to demonstrate connectivity between simplices in a simplicial complex: i) Two simplices A of dimension $q = 2$ and B of dimension $q = 1$ are 0–connected, which means that they have a single vertex in common. ii) In contrast, here two simplices A and B both of dimension $q = 2$ are 1–connected, meaning that have two vertices in common. We use this example to illustrate the calculation of all the six characterizers in the main text.

topological levels. Here, simplices A and B have two nodes in common, namely, node-2 and node-4. So they are 1-connected and are also 0-connected. Because the two simplices are connected at both the levels $q = 0$ and $q = 1$, the simplicial complex at these two levels therefore form a single entity. As a result, the corresponding components of the first structure vector $\mathbf{Q}$ are $Q_0 = 1$ and $Q_1 = 1$. The simplices A and B are, however, not connected at the $q = 2$ level, because they do not have three nodes in common. So the simplicial complex has 2 seperate entities at this level, and the corresponding component of $\mathbf{Q}$ is $Q_2 = 2$. Thereby, we can write the full vector as $\mathbf{Q} = (1, 1, 2)$.

The simplicial complex does not have a 0-dimensional or a 1-dimensional simplex. Both simplices A and B are of dimension 2. Thus, the vector $\tilde{\mathbf{f}}$, whose qth component is the number of simplices of dimension q, is $\tilde{\mathbf{f}} = (0, 0, 2)$.

The qth component of the second structure vector $\mathbf{N_s}$ is the number of simplices of dimension q and higher. This means that it is an inverse cumulative counting giving, in this case, the first and the second components the same value as that of the last component, thereby the vector is $\mathbf{N_s} = (2, 2, 2)$.

The qth component of the third structure vector $\widehat{\mathbf{Q}}$, is given by $1 - Q_q/n_q$, where Q_q and n_q are the qth components of the first and second structure vectors, respectively. So we can get $\widehat{\mathbf{Q}} = (1/2, 1/2, 0)$.

To find the topological dimension of node i, we need to find its $\mathbf{Q}^i$ vector, whose qth component is the number of q-dimensional simplices the node i participates in. Nodes 1 and 3 are part of only one simplex, while nodes 2 and 4 participate in both simplices. So we get $\mathbf{Q}^1 = (0,0,1)$, $\mathbf{Q}^3 = (0,0,1)$, $\mathbf{Q}^2 = (0,0,2)$, and $\mathbf{Q}^4 = (0,0,2)$. The topological dimension of a node is just the sum of the components of its $\mathbf{Q}^i$ vector, that is, the total number of simplices it participates in. So $\dim Q^1 = 1$, $\dim Q^3 = 1$, $\dim Q^2 = 2$, and $\dim Q^4 = 2$.

Thus, the maximum topological dimension of the simplicial complex is $\max(\dim) Q^i = 2$.

To find the entropy of each topology level, we need to first find the occupation probability of each level, which is given by

$$p_q^i = \frac{Q_q^i}{\sum_i Q_q^i}$$

where the subscript denotes the topology level and the superscript is the node index. Computing this for each topology level, we get $p_0^i = 0$ and $p_1^i = 0$ for all the four nodes. Now, $p_2^1 = 1/6$, $p_2^3 = 1/6$, $p_2^2 = 1/3$, and $p_2^4 = 1/3$.

Using these, we can compute the entropy for each topology level q, given by

$$S_Q(q) = -\frac{\sum_i p_q^i \log p_q^i}{\log N_q}$$

where $N_q = \sum_i \left(1 - \delta_{Q_q^i,0}\right)$ gives the number of nodes which participate in at least one q-simplex. Clearly, $S_Q(0) = 0$ and $S_Q(1) = 0$. Now,

$$S_Q(2) = -\frac{\sum_{i=1}^{4} p_2^i \log p_2^i}{\log 4} = 0.9591$$

Therefore, the entropy vector is $\mathbf{S} = (0, 0, 0.9591)$. The quantities defined here can now be computed for the actual TS networks in a similar way.

4 Results and Discussion for the logistic map time series

We calculated the above six quantities for the networks obtained using the time series data ($N = 2000$) of the logistic map at four parameter values: $\mu = 3.45$ (period-4 behaviour), $\mu = 3.56995$ (Feigenbaum point, edge of chaos), $\mu = 3.82843$ (period-3 window), and $\mu = 4.0$ (fully chaotic state). As mentioned above the time series data is transformed to the network using the visibility algorithm, and the clique structure of the resulting network is extracted using the Bron-Kerbosch algorithm. The results of the analysis are presented in Tables 1 and 2.

There are two central aspects of the networks quantified by the above six topological quantities: topological structure and topological connectivity. Of these, the topological structure of the simplicial complex is clearly identified by the $\tilde{\mathbf{f}}$ vector. The $\tilde{\mathbf{f}}$ vector counts the number of simplices at each topological level $q = 0, 1$ and 2. We see that almost all the simplices are at the highest topological level, $q = 2$. At the parameter values where the logistic map is chaotic ($\mu = 3.56995$ and $\mu = 4.0$, respectively), we see that $\tilde{\mathbf{f}}$ has a larger number of simplices (1987, and 1981) in comparison to the number of simplices (1498, and 1332) seen for the periodic states($\mu = 3.45$, and $\mu = 3.82843$, respectively) (cf. Table 2).

Next, from $\mathbf{Q}$ we can find the topological connectivity between the simplices at each topology level. The vector $\mathbf{Q}$ measures the number of connected components of the network at each topology level – for $q = 0$ (at least one vertex in common), $q = 1$ (at least two vertices in common) and $q = 2$ (at least three vertices in common). We observe that at the lowest topology level ($q = 0$) the components of $\mathbf{Q}$ have a value of 1 confirming that there is no isolated node, for any of the parameter values, either periodic or chaotic. However, at $q = 1$ we see that the chaotic states have fewer simplicial components($Q = 12, 18$), in other words, are more connected, than the corresponding periodic states that show a higher number ($Q = 501, 667$) of disconnected simplicial components. From the number of simplicial components of $\mathbf{Q}$ seen at the higher topological level of $q = 2$, that is Q_2, we can infer that the chaotic parameter values yield higher-dimensional simplices which are more connected in comparison to the periodic parameter values.

The quantity $\max(\dim) Q^i$ gives the maximum value of the topological dimension of all the nodes in the network. We see a clear distinction between the periodic states and the chaotic states in this case. For the periodic states, the node in the network which participates in the most number of simplices, participates in very few simplices, the values being 4 and 3. However, for the chaotic states there is at least one node which participates in as many as 19 and 25 simplices (cf. Table 3). It is also clear that the entropies of the chaotic states are lower than those of the periodic states, yielding the information that the networks representing the chaotic states are more connected than the networks corresponding to the periodic states. The other vectors N_s, $\widehat{Q}$ that are derived from the vectors $\tilde{\mathbf{f}}$ and $\mathbf{Q}$, give much the same information, and are consistent with the information yielded by the other quantities.

5 Conclusion

To summarise, we examine the TS networks obtained from the time series of the logistic map using algebraic topology methods. Our characterisers are clearly able to distinguish between chaotic and periodic regimes. Both regimes contain graphs whose simplicial struc-

μ	$\max(\dim)\, Q^i$
3.45000	4
3.56995	19
3.82843	3
4.00000	25

Table 3. The table shows the maximum value of the topological dimension of all the nodes in the TS networks ($N = 2000$) at the four parameter values of the logistic map.

ture contain nodes, links and triangular faces, and also contain fully connected clique complexes. No higher structures are found, indicating that the connections between the dynamical states probe short scales. The periodic regimes are characterised by regular graphs and fewer simplicial structures of dimensions one and two. In contrast, the simplicial structures in the chaotic regimes, contain many more connections at levels one and two. The entropies are higher for the periodic cases, and are significantly lower for the chaotic cases, indicating that the chaotic regimes have higher complexity.

We note that the local quantities pick up the differences in the two cases most sharply, especially the maximum dimension which counts the number of simplices in which the most highly connected node participates. The utility of the algebraic topological quantifiers is thus demonstrated in a simple context where the dynamical behaviour is well understood. Hence, they look like promising candidates for revealing the hidden geometry of networks which represent time series with nontrivial correlations between dynamical states. We expect them to be particularly useful in situations which exhibit phase transitions or other radical changes, such as crisis, intermittency and unstable dimension variability. We hope our study will motivate future work in these directions.

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