STOCHASTIC BIFURCATIONS FOR RANDOM FORCED OSCILLATIONS

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Abstract: An analysis of the classical Hopf differential system perturbed by multiplicative and additive noises is carried out. The difference in the response of Hopf systems perturbed by additive and multiplicative random noises is investigated. A delaying shift of the Hopf bifurcation point induced by multiplicative noise is demonstrated. The phenomenon of inverse stochastic bifurcation in which auto-oscillations are suppressed by multiplicative noise is clearly observed.

Keywords: Limit cycle, bifurcation, stochastic disturbances

1. INTRODUCTION

Many phenomena of nonlinear dynamics are connected to the study of oscillations with limit cycles. As commonly known, it is important to take into account various effects of random perturbations on these limit cycles. The noise-induced transitions for systems with nonlinear stochastic auto-oscillations attract attention from both the theoretical and practical points of view (Horsthemke W. and Lefever R. (1984), Landa P.S. and McClintock P.V.E. (2000)).

Nonlinear stochastically forced systems with a transition zone from equilibrium point to limit cycle (Hopf bifurcation) have been studied by many authors. Stochastic Hopf bifurcation is investigated for different systems such as the Brusselator (Lefever R.and Turner J. (1984), Arnold *et al.* (1997)), Duffing-van der Pol oscillator (Schenk-Hoppe K.R. (1996)), Van-der Pol (Leung H.K. (1998)). During the study of these stochastically forced models by analytic approximation methods, various phenomena of noise-induced transitions are observed.

The important role of the Hopf system as a basic example for deterministic bifurcation theory is well-known. An explicit analytical representation of trajectories of deterministic Hopf model allows us to attain a clear understanding of the mechanism of Hopf bifurcation.

In this paper, we investigate stochastic variants of this dynamic system with respect to the effects of random perturbations. The stochastic Hopf model serves as a master example for which an exact analytic expression of the probability distribution of the random limit cycle can be found. This gives us a possibility to study especially the qualitative influence of multiplicative noises on the transition from stable equilibrium point to limit cycles, a shifted bifurcation point and inverse stochastic bifurcation. Consider Stratonovich-interpreted stochastic Hopf-type differential equations

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$$
\begin{aligned}
dx &= \Big(\mu x - y - (x^2+y^2)x\Big)dt + \\
&\quad + \sigma_1 x \circ dW_1(t) + \sigma_2 dW_2(t) \\
dy &= \Big(x + \mu y - (x^2+y^2)y\Big)dt + \\
&\quad + \sigma_1 y \circ dW_1(t) + \sigma_3 dW_3(t)
\end{aligned} \tag{1}
$$

driven by independent, standard Wiener processes W_i, where μ and σ_i are real-valued deterministic parameters. Here $\sigma_1 \geq 0$ is the intensity of parametric (multiplicative) noise perturbing the parameter μ, $\sigma_2 \geq 0$ and $\sigma_3 \geq 0$ are the intensities of external (additive) noises. The presence of a stable equilibrium or limit cycles depends on the choice of the parameters μ and σ_i.

This paper is organized as follows. We analyze a stationary probability distribution of stochastic Hopf systems in Section 2. Section 3 is devoted to a bifurcation analysis.

2. ON STATIONARY DISTRIBUTION

We study the equivalent Itô version of (1) given by

$$
\begin{aligned}
dx &= \Big(\mu x - y - (x^2+y^2)x + \tfrac{\sigma_1^2}{2}x\Big)dt + \\
&\quad + \sigma_1 x dW_1(t) + \sigma_2 dW_2(t) \\
dy &= \Big(x + \mu y - (x^2+y^2)y + \tfrac{\sigma_1^2}{2}y\Big)dt + \\
&\quad + \sigma_1 y dW_1(t) + \sigma_3 dW_3(t)
\end{aligned} \tag{2}
$$

Using the transformation into polar coordinates

$$
r = \sqrt{x^2+y^2}, \quad \varphi = \arctan\Big(\frac{y}{x}\Big),
$$

the system (2) can be rewritten in the form

$$
\begin{aligned}
dr &= (\mu r - r^3)dt + \frac{1}{2r}\left(\sigma_2^2\cos^2(\varphi) + \sigma_3^2\sin^2(\varphi)\right)dt + \\
&\quad + \sigma_1 r dW_1 + \sigma_2\cos(\varphi)dW_2 + \sigma_3\sin(\varphi)dW_3 \\
d\varphi &= dt + \frac{1}{r}(-\sigma_2\sin(\varphi)dW_2 + \sigma_3\cos(\varphi)dW_3)
\end{aligned} \tag{3}
$$

In order to have a separation of variables r from φ we suppose $\sigma_2 = \sigma_3$. It means that the intensities of external additive perturbations in the equations (2) are the same.

Now, define the new random processes W_r and W_φ by

$$
\begin{aligned}
dW_r &= \cos(\varphi)\, dW_2 + \sin(\varphi)\, dW_3 \\
dW_\varphi &= -\sin(\varphi)\, dW_2 + \cos(\varphi)\, dW_3
\end{aligned} \tag{4}
$$

Formulas (4) present an orthogonal transformation of the standard independent Wiener processes W_2 and W_3. Hence, thanks to the well-known Theorem of R.A. Fisher on the invariance of Gaussian distributions with respect to orthogonal transformations, the processes W_r and W_φ are standard independent Wiener processes too (cf. Gardiner C.W. (1996)).

It follows from (3), (4) and assumption $\sigma_2 = \sigma_3$ that

$$
\begin{aligned}
dr &= (\mu r - r^3 + \frac{\sigma_2^2}{2r})dt + \sigma_1 r dW_1 + \sigma_2 dW_r \\
d\varphi &= dt + \frac{\sigma_2}{r}\, dW_\varphi
\end{aligned} \tag{5}
$$

Clearly, there is no feedback in the stochastic dynamics of variable r (i.e. the amplitude) of the Hopf system (5) resulting from the dynamics of φ (i.e. the phase). Therefore, we can consider the equation for r as separated from the dynamics of variable φ.

The Fokker-Planck equation for the probability density function $\rho(r)$ of the stationary distribution of scalar random variable r is given by

$$
\frac{1}{2}\left[(\sigma_1^2 r^2 + \sigma_2^2)\rho\right]'' = \left[(\mu r - r^3 + \frac{\sigma_1^2}{2}r + \frac{\sigma_2^2}{2r})\rho\right]' \tag{6}
$$

One can find the analytical solution of this equation.

Case $\sigma_1 = 0,\ \sigma_2 \neq 0$ (no parametric noise).

The stationary probability density of amplitude r can be written in explicit form

$$
\rho(r) = N r \exp\left(\frac{2\mu r^2 - r^4}{2\sigma_2^2}\right), \tag{7}
$$

where $N > 0$ is the standardizing integration constant. For the probability density $p(x,y)$ of the stationary distribution in terms of Cartesian coordinates x and y for the Hopf system (2), we obtain

$$
p(x,y) = K \exp\left(\frac{2\mu(x^2+y^2) - (x^2+y^2)^2}{2\sigma_2^2}\right) \tag{8}
$$

with standardizing integration constant $K > 0$.

Case $\sigma_1 \neq 0$ (with parametric noise).

Here we have

$$
\rho(r) = N r \exp\left(-\frac{r^2}{\sigma_1^2}\right)(\sigma_1^2 r^2 + \sigma_2^2)^\alpha \tag{9}
$$

in polar coordinates and

$$
\begin{aligned}
p(x,y) &= K \exp\left(-\frac{x^2+y^2}{\sigma_1^2}\right) \times \\
&\quad \times (\sigma_1^2(x^2+y^2) + \sigma_2^2)^\alpha,
\end{aligned} \tag{10}
$$

in Cartesian coordinates, where

$$
\alpha = \frac{\mu}{\sigma_1^2} + \frac{\sigma_2^2}{\sigma_1^4} - 1
$$

and standardizing integration constants $N > 0$ and $K > 0$.

3. STOCHASTIC BIFURCATIONS

3.1 Influence of additive noises

Let $\sigma_1 = 0$ and $\sigma_2 = \sigma_3 \neq 0$ in the system (2). We especially discuss main features of the probability density function $p(x, y)$ under changes of parameters μ and σ_2. Let ∇ denote the gradient in R^2. Then $\nabla p(x, y) = 0$ provides the extremal points of p. This necessary requirement to locate the extremes of p is obviously equivalent to either

$$x = y = 0 \text{ or } \mu = x^2 + y^2$$

as the only possible sets of real-valued solutions. The analysis of this set of conditions leads to the following conclusions on the shape of p.

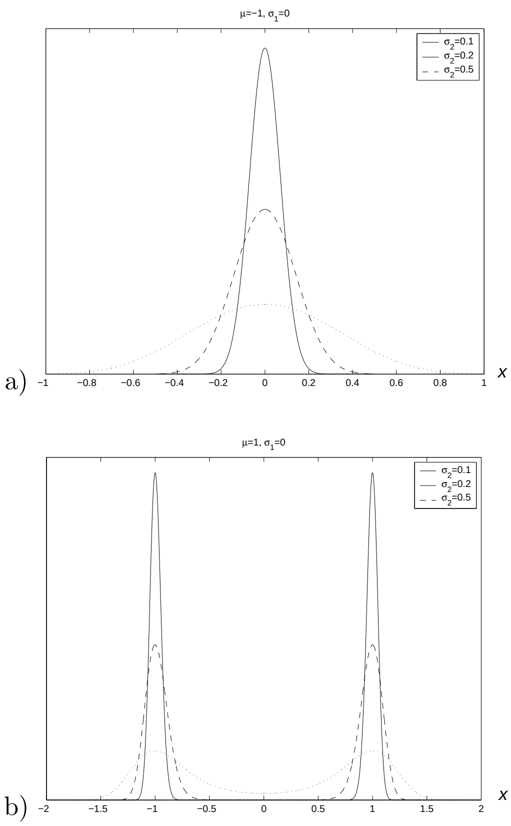


Fig.1. Stationary probability distribution $p(x, 0)$ of Hopf system for $\sigma_1 = 0$, $\sigma_2 = 0.1$ (solid), $\sigma_2 = 0.2$ (dashed), $\sigma_2 = 0.5$ (dotted) a) $\mu = -1$, b) $\mu = 1$.

For $\mu \leq 0$, the function p can only have a single maximum at the point $(0, 0)$. The graph of p has a single-peak bell-shaped form. The random trajectories of Hopf system under additive noise perturbations are concentrated in the vicinity of point $(0, 0)$. It is natural to name this type of stochastic attractor by *stochastic equilibrium point.*

For $\mu > 0$, the function p possesses a maximum at the points of deterministic cycle characterized by $x^2 + y^2 = \mu$ and a single minimum at the point $(0, 0)$. The graph of p has a crater-like form. The random trajectories of Hopf system are concentrated in the vicinity of deterministic cycle given by $x^2+y^2 = \mu$. So the randomly forced Hopf model is said to have a *stochastic limit cycle.*

In Fig.1, the graphs of $p(x, 0)$ for two values of parameter $\mu = -1$ (Fig.1a), $\mu = 1$ (Fig.1b) and various intensities of additive noise $\sigma_2 = 0.1$ (solid), $\sigma_2 = 0.2$ (dashed), $\sigma_2 = 0.5$ (dotted) are presented in details.

An variation of the parameter σ_2 of additive Gaussian noise does not change the location of extremal points of the related probability density function. An increase of σ_2 just results into a growth of the dispersion of the random Hopf system around the deterministic attractors (i.e. around the stable point $(0, 0)$ for $\mu \leq 0$, and around the stable limit cycle $x^2 + y^2 = \mu$ for $\mu > 0$). Note that the deterministic bifurcation value $\mu_* = 0$ under additive Gaussian noise perturbations is not changed.

3.2 Influence of multiplicative noise

Let $\sigma_1 \neq 0$. Then $\nabla p(x, y) = 0$ provides the extremal points of p and requires that

$$x = y = 0 \text{ or } \mu - \sigma_1^2 = x^2 + y^2$$

as the only possible sets of real-valued solutions. The analysis of this set of conditions leads to the following conclusions on the shape of p.

For $\mu \leq \sigma_1^2$, the function p can only have a single maximum at the point $(0, 0)$. In this case the Hopf system (2) is characterized by the stochastic equilibrium point $(0, 0)$.

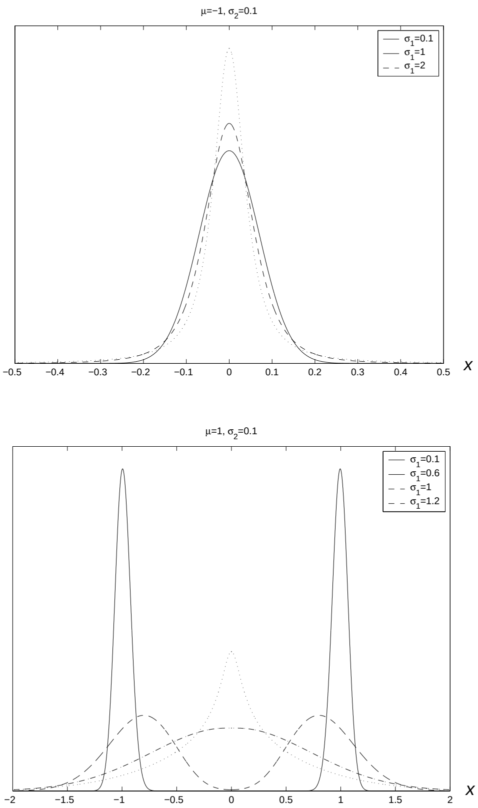
For $\mu > \sigma_1^2$, the function p possesses a maximum at the points of circle $x^2 + y^2 = \mu - \sigma_1^2$ and a minimum at the point $(0, 0)$. The random trajectories of Hopf system (2) are concentrated at the neighborhood of this circle. The value

$$r_s = \sqrt{\mu - \sigma_1^2}$$

can be interpreted as the radius of this stochastic cycle. Note that additive Gaussian noises have no influence on the location of the extremes of density function p.

Fig.2 demonstrates a response of the stationary probability density function p subject to changes of intensity σ_1 of multiplicative noise. In Fig.2a, the graphs of $p(x, 0)$ for $\mu = -1$, $\sigma_2 = 0.1$ and three distinct values of intensities $\sigma_1 = 0.1$ (solid), $\sigma_1 = 1$ (dashed), $\sigma_1 = 2$ (dotted) of multiplicative noise are shown. An increase of the intensity of multiplicative noise results into a decrease of the dispersion of random trajectories around equilibrium point $(0, 0)$.

In Fig.2b, the graphs of $p(x, 0)$ for $\mu = 1$, $\sigma_2 = 0.1$ and four values of multiplicative noise intensity $\sigma_1 = 0.1$ (solid), $\sigma_1 = 0.6$ (dashed), $\sigma_1 = 1$ (dash-dotted), $\sigma_1 = 1.2$ (dotted) are shown. As we can see in Fig.2b, the increase of multiplicative noise is accompanied by suppressing auto-oscillations and re-localization of random trajectories in vicinity of zero point.



a)

b)

Fig.2. Stationary probability distribution $p(x, 0)$ of Hopf system for $\sigma_2 = 0.1$,
a) $\mu = -1$, $\sigma_1 = 0.1$ (solid), $\sigma_1 = 1$ (dashed), $\sigma_1 = 2$ (dotted),
b) $\mu = 1$, $\sigma_1 = 0.1$ (solid), $\sigma_1 = 0.6$ (dashed), $\sigma_1 = 1$ (dash-dotted), $\sigma_1 = 1.2$ (dotted).

An variation of σ_1 shifts the extremal points of related probability density function. The magnitude of this shift is equal to σ_1^2.

Thus a presence of multiplicative noise changes the bifurcation point from $\mu_* = 0$ to $\mu_* = \sigma_1^2$. In Fig.3, a bifurcation diagram of the stochastic Hopf system is shown. The curve $\mu = \sigma_1^2$ characterizes the border line between the zones of stochastic equilibrium ($\mu \leq \sigma_1^2$) and stochastic limit cycle ($\mu > \sigma_1^2$).

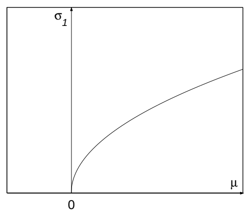
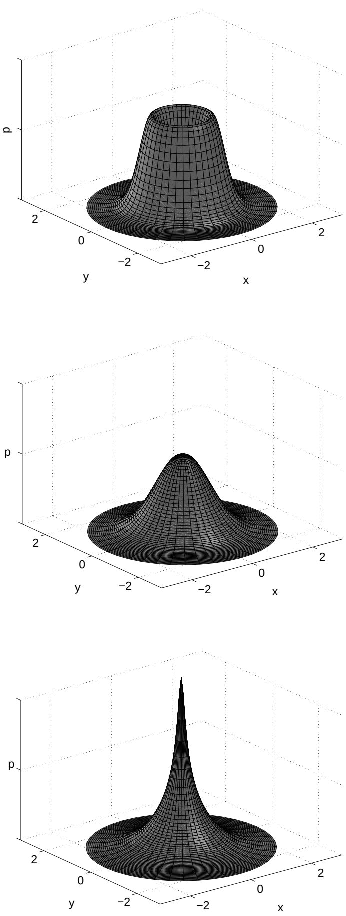


Fig.3. Bifurcation diagram of Hopf system
for $\mu > \sigma_1^2$ stochastic limit cycle zone,
for $\mu \leq \sigma_1^2$ stochastic equilibrium zone.

As we can clearly see, for any fixed $\mu > 0$, an increase of σ_1 results into a transition from the stochastic limit cycle ($0 < \sigma_1 < \sqrt{\mu}$) to the stochastic equilibrium ($\sigma_1 \geq \sqrt{\mu}$) at the point $\sigma_{1*} = \sqrt{\mu}$. Here we observe a so-called *inverse stochastic bifurcation*. The underlying reason of this phenomenon rests on the nonlinear response of the Hopf system to the random parametric perturbations.



a)

b)

c)

Fig.4. The stationary probability density of Hopf system for $\mu = 1$, $\sigma_2 = 0.1$,
a) $\sigma_1 = 0.6$, b) $\sigma_1 = 1$, c) $\sigma_1 = 1.2$

In Fig.4, a demonstration of inverse stochastic bifurcation as a qualitative change of the form of the related stationary probability density is shown. Here, for fixed value $\mu = 1$, the stochastic bifurcation point is $\sigma_{1*} = 1$. An increase of the intensity of multiplicative noise from $\sigma_1 = 0.1$ (Fig.4a) to $\sigma_1 = 1$ (Fig.4b) results into a transition from a crater-like to a single-peak bell-shaped form of the graph of function $p = p(x, y)$. A further growth of σ_1 changes the single-peak form of function $p(x, y)$ to higher levels and sharper ones (Fig.4c).

4. SUMMARY AND CONCLUSIONS

We have considered the classical Hopf differential system perturbed by parametric multiplicative and external additive noises. From the related Fokker-Planck equation, an analytical representation of the stationary probability density function is found. The zone of transition from the trivial equilibrium point to limit cycle (called Hopf bifurcation) is investigated. The details of an observed delaying shift of the Hopf bifurcation point and the mechanism of inverse stochastic bifurcation induced by multiplicative noise are presented and discussed. Additive (Gaussian) noise does not effect any change in the location of the extremal points of stationary probability density. This type of state-independent noise only changes the dispersion of trajectories around the deterministic attractors.

From our exposition, it becomes clear that the stochastic Hopf system can be considered as a master example for the demonstration of phenomena of the theory of noise-induced transitions.

REFERENCES

Horsthemke W., Lefever R. (1984) *Noise-Induced Transitions.* Springer, Berlin.

Landa P.S. and McClintock P.V.E. (2000) Changes in the dynamical behavior of nonlinear systems induced by noise. *Physics Reports,* 323, 1–80.

Lefever R.and Turner J. (1984) Sensitivity of a Hopf bifurcation to external multiplicative noise. *Fluctuations and Sensitivity in Equilibrium Systems, ed. by W. Horsthemke and D.K. Kondepudi, Berlin: Springer,* 143–149.

Arnold, L., G. Bleckert and K. Schenk-Hoppe (1997). The stochastic Brusselator: parametric noise destroys Hopf bifurcation. In *Stochastic dynamics*, 71–92. Springer, New-York.

Schenk-Hoppe K.R. (1996) Bifurcation scenarios of the noisy Duffing-van der Pol oscillator. *Nonlinear dynamics,* 11, 255–274.

Leung H.K. (1998) Stochastic Hopf bifurcation in a biased van der Pol model.*Physica A,* 254, 146–155.

Gardiner C.W. (1996) *Handbook of Stochastic Methods. 2nd ed.* Springer, New York.