On experimental design for nonlinear guaranteed identification problems

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***Abstract*— The problem of choice of optimal inputs for control system parameters identification is studied. The uncertain items are assumed to be unknown but bounded, the problem is treated in the framework of guaranteed (set-membership) approach. The goal of input design is to get maximum information about system parameters from available observations. The integral of information function over the set of a priori constraints on parameters is considered as a criterion of optimality.**

I. Introduction

Generally differential equations used for modelling of physical and mechanical systems contain unknown parameters, which are estimated on the basis of information provided by indirect experimental observations. The goal of experimental design is to get maximal information about system parameters from available observations. The conventional approach to the experiments design is based on stochastic models for uncertain parameters and measurement errors. An alternative guaranteed approach states from deterministic model of uncertainty with set-membership description of the uncertain items [1-5]. These items are considered to be unknown but bounded with preassigned bounds. Such model of uncertainty arises in many applied problems of information processing in physics. Within the framework of guaranteed approach the set of parameters, consistent with the system equations, measurements, and a priori constraints called information (feasible) set is considered as the solution of estimation problem.

In this paper we consider the problem of optimal input choice [6] for guaranteed estimation of the parameters of dynamic system on the basis of indirect observation. The information sets in the problem may be described as the level sets for so-called information function (information state) [5,7,8]. An information function is defined as a value function for a certain auxiliary optimal control problem. The integral of information function over the set of a priori constraints on parameters is considered as a criterion of optimality. This allows to avoid, when designing an optimal input, the immediate construction of information sets. It is shown that considered problem may be reduced to an optimal control problem for the trajectory tubes of the system.

The paper consists of three parts. First, we consider the input design problem for a linear model in Hilbert space in order to clarify the details of the scheme used in the paper. Next, we describe an algorithm of solution of the problem in the case of the system described by nonlinear ODE with uncertain parameters in the right-hand side of the equation. In this case the systems with disturbances only in measurement equation are considered. In the last part the linear systems with disturbances in measurement and system equations are regarded.

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II. Linear models

Consider the linear model, connecting available for measurement output y with unknown vector of parameters $q = (q_1, ..., q_m)$ under the equation

$$y = \sum_{i=1}^{m} q_i a^i(u) + \xi.$$

Here $a_i : U \to H$ are given maps from the given set of control parameters U to the real Hilbert space H, $\xi \in H$ is treated as a measurement error, , $u \in U$ is a control.

Assume that all advance information on q and ξ is given by the conditions

$$q = (q_1, ..., q_m) \in Q \subset \mathbb{R}^m, \quad \xi \in \Xi = \{\xi : \langle \xi, \xi \rangle \le 1\}, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is an inner product in H, Q is a compact set in $\mathbb{R}^m$. An experimental design problem consists of two stages. The fist one is the identification problem, the second is the choice of optimal input, providing the best quality of identification. The identification problem is related to an estimation of unknown value of q on the basis of measurement of the output y. The solution of this problem is the information (feasible) set [2,5] consisting of all values of q consistent with the results of measurement and a priori data.

Let y, u are given. An information set is determined as follows

$$\hat{Q}(y, u) = \{q \in Q : \exists\, \xi,\ \langle \xi, \xi \rangle \le 1,\ y = \sum_{i=1}^{m} q_i a^i(u) + \xi\}.$$

The set $\hat{Q}$ contains an unknown true value of q.

The quality of identification usually is characterized by the value of some scalar functional $\Omega(\hat{Q}(y, u))$, which is defined on the class of sets. A radius, diameter or volume of the set may be considered as a such functional.

The problem of optimal input choice takes on the following form: to find the control (input) $u \in U$, solving the problem

$$\max_{y} \Omega(\hat{Q}(y, u)) \to \min_{u \in U}.$$

Here the maximum is taken in all possible values of y, or, equivalently, in all pairs q, ξ, satisfying constraints (1).

A disadvantage of such approach is a necessity of constructing of information sets for calculation of optimal inputs. The solution of last problem requires a laborious computing procedures (especially in the case of nonlinear identification problems, where an information set may be nonconvex or even nonconnected).

Further we modify the statement of the problem in order to avoid the direct constructing of information sets in the process of calculation of optimal input. The proposed approach is based on the notion of information function (information state) of the problem. In the case of linear model this function $V(y,u,q)$ is determined by the equality

$$V(y,u,q) = \langle y - \sum_{i=1}^{m} q_i a^i(u), y - \sum_{i=1}^{m} q_i a^i(u)\rangle.$$

Obviously, under given u, y

$$\hat{Q}(y,u) = \{q \in Q : V(y,u,q) \le 1\}.$$

If for the controls $\hat{u}$, $\bar{u}$ under given y

$$V(y,\hat{u},q) \le V(y,\bar{u},q) \tag{2}$$

for every $q \in Q$, then $\hat{Q}(y,\bar{u}) \subset \hat{Q}(y,\hat{u})$. Hence $\bar{u}$ is more preferable than $\hat{u}$ because it gives more precise estimate of unknown parameter.

Consider a scalar functional on the set of information states, which is monotone with respect relation (2) and is defined by the following way

$$I(y,u) = \int\limits_Q V(q,y,u)d\mu(q),$$

where μ is a nonnegative measure defined on the σ-algebra of Lebesgue measurable subsets of Q with the property $\mu(Q) = 1$. If Q has a nonzero Lebesgue measure, then for μ we can take the measure defined by the equality

$$\mu(E) = \int_E \alpha(q)dq,$$

where α is a given nonnegative function (weight function), the integral of which over the set Q equals 1. Another example is a measure μ concentrated at points of a given finite subset of Q.

Denote by q^* the true value of uncertain parameter and by ξ^*— the realization of disturbance in measurements. The output y^* is a function of q^*, ξ^* and input u: $y^* = y^*(q^*, \xi^*, u)$.

Depending on way of accounting the dependence of y from parameters the following statements of the problem are possible.

Problem 1: Find $u \in U$, maximizing the functional

$$I_1(y^*,u) = \int\limits_Q V(q, y^*(q^*,\xi^*,u),u)d\mu(q). \tag{3}$$

Problem 2: Find $u \in U$, maximizing the functional

$$I_2(q^*,u) = \inf_{\xi^* \in \Xi} \int\limits_Q V(q, y^*(q^*,\xi^*,u),u)d\mu(q). \tag{4}$$

Problem 3: Find $u \in U$, maximizing the functional

$$I_3(u) = \inf_{\xi^* \in \Xi, q^* \in Q} \int\limits_Q V(q, y^*(q^*,\xi^*,u),u)d\mu(q). \tag{5}$$

Introduce the following definitions. Let

$$K = \{k \in \mathbb{R}^m : k_i \in \{0,1\}, i = 1,...,m\},$$

$\bar{q} \in \mathbb{R}^m$. For $q \in \mathbb{R}^m$ denote as $p^k(q)$ a vector with coordinates $p_i = (-1)^{k_i} q_i$, $i = 1,...,m$. The Q $\bar{q}$ is said to be symmetrical, if from $q \in Q$ follows that $p^k(q-\bar{q})+\bar{q} \in Q$ for every $k \in K$. For the set $E \subset Q$ denote

$$E^k = p^k(E - \bar{q}) + \bar{q}.$$

It is obvious, that for measurable E the set E^k is also measurable; if Q $\bar{q}$ is symmetrical $E^k \subset Q$.

Assumption 1: There exists $\bar{q} \in Q$ such that the set Q and the measure μ are $\bar{q}$-symmetrical.

Let assumption 1 holds. Denote $\bar{Q} = Q - \bar{q}$, as $\bar{\mu}$ denote the measure $\bar{Q}$, defined by the equality $\bar{\mu}(E) = \mu(E + \bar{q})$.

Let

$$\tilde{y} = y - \sum_{i=1}^{m} \bar{q}_i a^i(u) = \sum_{i=1}^{m}(q_i - \bar{q}_i)a^i(u) + \xi.$$

Then

$$I(y,u) = \int\limits_Q V(q,y,u)d\mu(q) = \int\limits_Q \langle\tilde{y},\tilde{y}\rangle d\mu(q)$$

$$-2\sum_{i=1}^{m} \int\limits_Q (q_i - \bar{q}_i)d\mu(q)\langle\tilde{y}, a^i(u)\rangle +$$

$$+\sum_{i,j=1}^{m} p_{ij}(u) \int\limits_Q (q_i - \bar{q}_i)(q_j - \bar{q}_j)d\mu(q),$$

where $p_{ij}(u) = \langle a^i(u), a^j(u)\rangle$.

Lemma 1: Under assumption 1 the following equalities hold

$$\int\limits_Q (q_i - \bar{q}_i)d\mu(q) = 0, \; i = 1,...,m,$$

$$\int\limits_Q (q_i - \bar{q}_i)(q_j - \bar{q}_j)d\mu(q) = 0, i,j = 1,...,m, i \ne j.$$

This lemma follows from the elementary properties of Lebesgue integral. From lemma 1 it follows that

$$\int\limits_Q V(q,y,u)d\mu(q) = \langle\tilde{y},\tilde{y}\rangle + \sum_{i=1}^{m} A_i p_{ii}(u),$$

where $A_i = \int_Q (q_i - \bar{q}_i)^2 d\mu$.

Calculating the infimum in ξ^*, we get

$$I_2(q^*,u) = \inf_{\xi^* \in \Xi} \int\limits_Q V(q, y^*(q^*,\xi^*,u),u)d\mu(q)$$

$$= \phi((q^* - \bar{q})^\top P(u)(q^* - \bar{q})) + \sum_{i=1}^{m} A_i p_{ii}(u),$$

where $P(u)$ is a matrix with the elements $p_{ij}(u)$, $i, j = 1, ..., m$ and the function $\phi(x)$ is defined by the equality

$$\phi(x) = \begin{cases} 0 & 0 \le x \le 1, \\ (\sqrt{x} - 1)^2 & x \ge 1. \end{cases}$$

Calculating the infimum I_2 in q^*, we have

$$I_3(u) = \inf_{q^*} I_2(q^*, u) = \sum_{i=1}^{m} A_i p_{ii}(u).$$

Statement 1: Let for every $u \in U$, $q^* \in Q$

$$(q^* - \bar{q})^\top P(u)(q^* - \bar{q}) \le 1, \qquad (6)$$

and assumption 1 holds. Then the solutions of problems 2, 3 coincide.

If inequality (6) holds, the first term in formula (4) for functional I_2 equals to zero. This implies the validity of the statement.

III. Nonlinear systems with noise in measurements

Consider the control system

$$\dot{x} = f(t, q, x, u(t)), \;\; t \in [t_0, t_1], \; x(t_0) = x^0, \qquad (7)$$

($x \in R^n, \;\; u \in R^r$) with the right-hand side f depending on unknown parameter $q \in R^m$. We assume that all a priori information on q is given by the inclusion $q \in Q$ where Q is a compact set in R^m. As an admissible control (input) we will consider a Lebesque-measurable function $u : [t_0, t_1] \to U$, where $U \subset R^r$. We assume that $f(t, q, x, u)$ is continuously differentiable in x on $[t_0, t_1] \times Q \times \mathbb{R}^n \times U$. The solution of system (7) is denoted as $x(t, q)$ (or $x(t, q, u(\cdot))$.

Consider the measurement equation on $[t_0, t_1]$

$$y(t) = g(t, x(t)) + \xi(t), \qquad t \in [t_0, t_1], \qquad (8)$$

corrupted by unknown but bounded noise $\xi(t)$. An advance information on $\xi(t)$ is assumed to be given by the inclusion

$$\xi(\cdot) \in \Xi, \qquad (9)$$

where Ξ is a bounded set in the space $L_2^k[t_0, t_1]$. Suppose that

$$\Xi = \{\xi(\cdot) : W(\xi(\cdot)) \le 1\},$$

where

$$W(\xi(\cdot)) = \int_{t_0}^{t_1} \xi^\top(t) R \xi(t) dt.$$

Here R is a given positively defined matrix. Let $y(t)$ be the result of measurements, generated by unknown ”true” value of $q^* \in Q$, input $u(t)$, and measurement error $\xi(t)$. The function $q \to V(q, y(\cdot), u(\cdot))$, defined by the equality

$$V(q, y(\cdot), u(\cdot)) = W(y(\cdot) - g(\cdot, x(\cdot, q)))$$

is said to be an information function(information state) of the problem (7),(8). The set $Q(y(\cdot), u(\cdot))$ of all parameters $q \in Q$ that are consistent with (7), (8) and a priori constraints is referred to as the information set relative to measurement $y(t)$ []. It follows directly from definitions that

$$Q(y(\cdot), u(\cdot)) = \{q \in Q : V(q, y(\cdot), u(\cdot)) \le 1\}.$$

Unknown q^* belongs to the information set.

We shall consider an integral of information function as a functional of the problem

$$I(y(\cdot), u(\cdot)) = \int_Q V(q, y(\cdot), u(\cdot)) d\mu(q). \qquad (10)$$

Here μ is a nonnegative measure defined on Lebesque subsets of Q such that $\mu(Q) = 1$. The functional I is nonnegative, the most value of I corresponds to a more accurate estimate of unknown quantity of parameter q.

The integral (13) depends on $u(\cdot)$ and the result of measurements $y(\cdot)$. In turn, $y(t) = y^*(t) + \xi(t)$, where $y^*(t) = g(t, x(t, q^*, u(\cdot)))$ and $\xi(t)$ is the measurement error. In the worst case, the value of I is equal to

$$J(u(\cdot)) = \inf_{W(\xi(\cdot)) \le 1} \int_Q V(q, y(\cdot) + \xi(\cdot), u(\cdot)) d\mu(q).$$

Direct calculations lead to the following formula for J

$$J(u(\cdot)) = I_1(u(\cdot)) + \phi(I_2(u(\cdot))), \qquad (11)$$

where

$$I_1(u(\cdot))) = \int_{t_0}^{t_1} \int_Q r(t, q)^\top R r(t, q) d\mu(q) dt,$$

$$I_2(u(\cdot))) = \int_{t_0}^{t_1} \int_Q r(t, q)^\top d\mu(q) R \int_Q r(t, q) d\mu(q) dt,$$

$$r(t, q) = g(t, x(t, q^*)) - g(t, x(t, q)),$$

$$\phi(x) = \begin{cases} -x & \text{if} \quad 0 \le x \le 1 \\ 1 - 2\sqrt{x} & \text{if} \quad x \ge 1. \end{cases}$$

Thus the problem of optimal input design is equivalent to the maximization of the functional J on the tubes of trajectories of uncertain system (7). It is similar in a certain way to the problems of of beam optimization [9].

The necessary conditions of optimality for this problem constitute the basis for constructing the numerical algorithms [10,11]. In the next picture the results of numerical simulation for the system describing oscillations of nonlinear pendulum are presented. The orange and blues lines denotes the boundaries of information sets corresponding to optimal and some non optimal inputs. These information sets are constructed for the case of hard (magnitude) constraints on measurement noise: $|\xi(t)| \le 1, \; t \in [t_0, t_1]$.

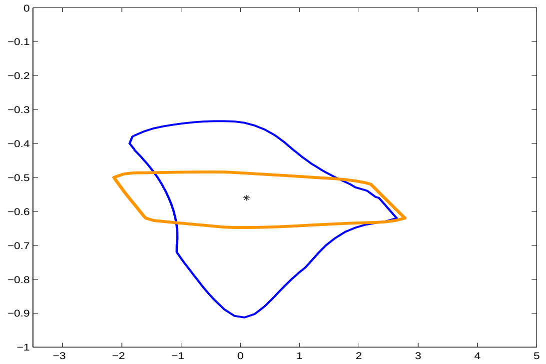

Fig. 1. Information sets for nonlinear pendulum

IV. Linear systems with uncertain disturbances

In this section we consider the choice of optimal inputs for identifying the parameters of a control system whose dynamics is corrupted by unknown but bounded noise.

Consider the following control system

$$\dot{z} = A(q)z + Bu(t) + w(t),\ z(t_0) = z^0, \quad (12)$$

with matrix A depending on unknown parameter $q \in R^m$. We assume that all the "a priori" information on q is given by the inclusion $q \in Q$, where Q is a compact set in R^m. As an admissible control (input) we will consider a Lebesgue-measurable function $u : [t_0, t_1] \to U$, where $U \subset R^r$.

Consider the measurement equation to be specified by the equality

$$y(t) = Cz(t) + \xi(t),\ t \in [t_0, t_1].$$

where $\xi(t)$ is the measurement error. The advance information on $\xi(t)$ and $w(t)$ is assumed to be given by inequalities

$$W_1(\xi(\cdot)) = \int_{t_0}^{t_1} \xi^\top(t) R_1 \xi(t) dt \le 1,$$

$$W_2(w(\cdot)) = \int_{t_0}^{t_1} w^\top(t) R_2 2(t) dt \le 1.$$

Here $R_1,\ R_2$ are given positive matrices.

An information set relative to measurement $y^*(t)$ may be expressed as a level set

$$\hat{Q}(y^*(\cdot)) = \{q \in Q : V(q, y^*(\cdot), u(\cdot)) \le 1\},$$

where "information state" $V(q, y^*(\cdot), u(\cdot))$ is the value function for the following extremal problem

$$V(q, y^*(\cdot), u(\cdot)) = \min_{W_2(w(\cdot)) \le 1} W_1(y^*(\cdot) - Cz(\cdot)).$$

We shall consider as a criterion of optimality

$$I(y^*(\cdot), u(\cdot)) = \int\limits_Q V(q, y^*(\cdot), u(\cdot)) d\mu(q), \quad (13)$$

the problem will therefore consist in maximizing I over $u(t) \in U$, $t \in [t_0, t_1]$.

For calculating of $V(q, y^*(\cdot), u(\cdot))$ we pass to a solution of the dual convex programming problem

$$V(q, y^*(\cdot), u(\cdot)) = \sup_{\alpha > 0} \phi(\alpha),$$

where $\alpha \in R$ and $\phi(\alpha) = \phi(\alpha, q, u(\cdot))$

$$\phi(\alpha) = \min_{w(\cdot)} \{W_1(y^*(\cdot) - Cz(\cdot)) + \alpha W_2(w(\cdot)) - \alpha\}. \quad (14)$$

Here (14) is a linear-quadratic tracking problem whose solution may be obtained in explicit form. Let $\bar{x}(t, q)$ be the solution of system

$$\dot{\bar{x}} = A(q)\bar{x}(t) + Bu(t),\ x(t_0) = z^0, \quad (15)$$

and $\bar{y}(t) = y^*(t) - C\bar{x}(t, q)$. Then the value of $\phi(\alpha)$ may be expressed as follows

$$\phi(\alpha) = \min_{w(\cdot)} J_\alpha(w(\cdot)),$$

where

$$J_\alpha(w(\cdot)) = \int_{t_0}^{t_1} (\bar{y} - Cx)^\top R_1 (\bar{y} - Cx) dt$$

$$+\alpha \int_{t_0}^{t_1} w^\top R_2 w dt - \alpha,$$

the minimum is sought for among the trajectories of system

$$\dot{x} = A(q)x + w(t),\ x(t_0) = 0.$$

Solving the last problem, we come to the expression

$$\varphi(\alpha) = (x(t_0) - g(t_0), K(t_0)(x(t_0) - g(t_0))),$$

where $K(t), g(t)$ arrive due to the Riccati system

$$\dot{K} = -KA - A^\top K$$

$$+1/\alpha K R_2^{-1} K - C^\top R_1 C, \quad (16)$$

with

$$\dot{g} = -[A - 1/\alpha R_2^{-1} K]^\top g$$

$$-C^\top R_1 (y^*(t) - C\bar{x}), \quad (17)$$

and boundary conditions $K(t_1) = 0,\ g(t_1) = 0$.

Since $x(t_0) = 0$ we have $\varphi(\alpha) = (g(t_0), K(t_0)g(t_0))$. Thus,

$$\int_Q V d\mu = \int_Q \sup_{\alpha > 0} \phi(\alpha, q, u(\cdot)) d\mu.$$

Substituting the supremum over scalar α by one over continuous functions $\alpha(q)$, we will have

$$I(y^*(\cdot), u(\cdot)) = \int_Q \sup_{\alpha(q) > 0} \phi(\alpha(q), q, u(\cdot))) d\mu$$

$$= \sup_{\alpha(q) > 0} \int_Q \phi(\alpha(q), q, u(\cdot)) d\mu.$$

Thus, the resulting problem is as follows

$$\int_Q \phi(\alpha(q), q, u(\cdot)) d\mu \to \max_{\alpha(\cdot), u(\cdot)} \min_{\xi^*(\cdot), w^*(\cdot)} \quad (18)$$

over the solutions of system (15), (16), (17). Here (18) is a nonstandard control problem, since the control function actually consists of two parts – a conventional function of time $u(t)$ and a distributed control $\alpha(q)$.

V. Conclusion

In the present work, a problem of constructing an optimal input for identification of control system parameters is considered. The integral of an information function of the system over the set of a priori constraints on parameters is suggested for the optimality criterion. This allows us to avoid, when designing an optimal input, the immediate construction of information sets. The problems of constructing the information sets and of design of optimal inputs thus are separated. The proposed scheme may be applied to design of optimal inputs for nonlinear systems with measurements corrupted by unknown but bounded noise. Under the assumption that measurement errors satisfy integral quadratic constraints, the considered problem is transformed into an optimal control problem in which equations of the system depend on a parameter and the functional contains the integral over the parameter. Optimality conditions in the form of the Pontryagin maximum principle and a formula of the functional gradient for the last problems constitute a basis for developing the numerical algorithms. The integration measure may be used to control the parameters of algorithms.

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