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REACHABLE SETS FOR A CLASS OF NONLINEAR IMPULSIVE CONTROL SYSTEMS

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Abstract
This paper deals with the state estimation problem for uncertain dynamical control systems with a special structure described by differential equations with impulsive control under bilinear uncertainty and with uncertainty in initial states. The matrix included in the differential equations of the system dynamics is uncertain but bounded. The bounds of admissible values of this matrix coefficients are known. The algorithms for constructing external ellipsoidal estimates of reachable sets for such bilinear uncertain systems are presented here.

Key words
Bilinear control systems, impulsive control, ellipsoidal calculus, trajectory tubes, estimation.

1 Introduction

In this paper we study impulsive control systems with unknown but bounded uncertainties related to the case of a set-membership description of uncertainty ([Kurzhanski and Valyi, 1997; Schweppe, 1973; Walter and Pronzato, 1997; Boyd, El Ghaoui, Feron and Balakrishnan, 1994]). Here we develop the set-membership approach based on ellipsoidal calculus for the special nonlinear system with uncertainty.

Bilinear dynamical systems are a special kind of nonlinear systems representing a variety of important physical processes [Boscain, Chambrion and Sigalotti, 2013; Boussaïd, Caponigro and Chambrion, 2013; Ceccarelli and etc., 2006; Gough, 2008; Nihtila, 2010]. A great number of results related to control problems for such systems has been developed earlier, among them we mention here [Chernousko, 1996; Filippova and Lisin, 2000; Kurzhanski and Filippova, 1993]. Reachable sets of bilinear systems in general are not convex, but have special properties (for example, they have star-shaped form). In the paper we consider the guaranteed state estimation problem and use ellipsoidal calculus for the construction of external estimates of reachable sets of such systems.

Also we consider a more complicated case and generalize earlier results [Filippova and Matviychuk, 2015; Matviychuk, 2016]. Now the constraints on the uncertain matrix of the dynamics systems has a more complex structure. The algorithms of constructing external ellipsoidal estimates for studied systems are given.

2 Basic Notations

Let $\mathbb{R}^n$ be the n–dimensional vector space, $\operatorname{comp}\mathbb{R}^n$ be the set of all compact subsets of $\mathbb{R}^n$, $\operatorname{conv}\mathbb{R}^n$ be the set of all convex and compact subsets of $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ stands for the set of all real $n\times n$–matrices and $x'y = (x,y) = \sum_{i=1}^n x_i y_i$ be the usual inner product of $x, y \in \mathbb{R}^n$ with prime as a transpose, $\|x\| = (x'x)^{1/2}$. Let $I \in \mathbb{R}^{n\times n}$ be the identity matrix, $\operatorname{Tr}(A)$ be the trace of $n\times n$-matrix A (the sum of its diagonal elements), $\operatorname{diag} b = \operatorname{diag}\{b_i\}$ be the diagonal matrix A with $a_{ii} = b_i$ where b_i are components of the vector b. For a set $A \subset \mathbb{R}^n$ we denote its closed convex hull as $\overline{\operatorname{co}}\, A$.

We denote by $B(a,r) = \{x \in \mathbb{R}^n : \|x-a\| \le r\}$ the ball in $\mathbb{R}^n$ with center $a \in \mathbb{R}^n$ and radius $r > 0$ and by

$$E(a,Q) = \{x \in \mathbb{R}^n : \ (Q^{-1}(x-a),(x-a)) \le 1\}$$

the ellipsoid in $\mathbb{R}^n$ with center $a \in \mathbb{R}^n$ and symmetric positive definite $n\times n$-matrix Q.

Denote by $h(A,B)$ the Hausdorff distance between sets $A, B \in \mathbb{R}^n$, $h(A,B) = \max\{h^+(A,B),\, h^-(A,B)\}$, where $h^+(A,B)$ and $h^-(A,B)$ are the Hausdorff semidistances between A and B, $h^+(A,B) = \sup\{d(x,B) : x \in A\}$, $h^-(A,B) = h^+(B,A)$, $d(x,A) = \inf\{\|x-y\| : y \in A\}$.

3 Problem Formulation

Consider the following bilinear impulsive control system

$$
\begin{aligned}
dx(t) &= A(t)x(t)dt + B(t)du(t), \\
x(t_0 - 0) &= x_0, \quad t \in [t_0, T],
\end{aligned} \tag{1}
$$

here $x \in \mathbb{R}^n$, vector-function $B(\cdot) \in \mathbb{R}^n$ is continuous on $[t_0, T]$. The $n \times n$–matrix function $A(t)$ in (1) has the special form

$$
A(t) = A^0 + A^1(t) + A^2(t), \quad t \in [t_0, T] \tag{2}
$$

where $A^0 \in \mathbb{R}^{n\times n}$ is given and the measurable $A^1(t), A^2(t) \in \mathbb{R}^{n\times n}$ are unknown but bounded

$$
A(t) \in \mathcal{A} = A^0 + \mathcal{A}^1 + \mathcal{A}^2, \quad t \in [t_0, T], \tag{3}
$$

$$
\begin{aligned}
A^1(t) \in \mathcal{A}^1 = \big\{A = \{a_{ij}\} \in \mathbb{R}^{n\times n} : \\
|a_{ij}| \le c_{ij}, i, j{=}1, \dots, n\big\},
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
A^2(t){\in}\mathcal{A}^2 = \{A \in \mathbb{R}^{n\times n} \colon\ A = \operatorname{diag} a, \\
a = (a_1, \dots, a_n) \in \mathbf{A}_0\},
\end{aligned} \tag{5}
$$

$$
\mathbf{A}_0 = \{a \in \mathbb{R}^n :\ \sum_{i=1}^{n} |a_i|^2 \le 1\}
$$

where $c_{ij} \ge 0$ $(i, j = 1, \dots n)$ are given.

The impulsive control function $u \in \mathbb{R}^n$ is the scalar function of bounded variation on $[t_0, T]$, monotonically increasing and right-continuous in $t \in [t_0, T]$. We assume that for some $\mu > 0$ we have

$$
\underset{t\in[t_0,T]}{\operatorname{Var}}\ u(t) = \sup_{\{t_i\}} \sum_{i=1}^{k} |u(t_i) - u(t_{i-1})| \le \mu,
$$

supremum is taken over all $\{t_i\}$ such that $t_i\ :\ t_0 \le t_1 \le \ldots \le t_k = T$.

Denote by $\mathcal{U}$ the class of all admissible controls $u(\cdot)$.

The initial condition $x(t_0 - 0) = x_0$ for the system (1) is assumed to be unknown but bounded

$$
x_0 \in \mathcal{X}_0 = E(a_0, Q_0). \tag{6}
$$

Let the function $x(t) = x\big(t; t_0, x_0, A(\cdot), u(\cdot)\big)$ be a *solution* to dynamical system (1)–(6) with impulsive control $u(\cdot) \in \mathcal{U}$, with initial state $x_0 \in \mathcal{X}_0$ and with a matrix $A(\cdot) \in \mathcal{A}$.

The *trajectory tube* $\mathcal{X}(\cdot) = \mathcal{X}(\cdot; \mathcal{X}_0, \mathcal{A}, \mathcal{U})$ of the system (1) is defined as the following set (see also [Filippova and Matviychuk, 2011])

$$
\begin{aligned}
\mathcal{X}(\cdot) = \bigcup \big\{x(\cdot) = x\big(\cdot;\, t_0, x_0, A(\cdot), u\big) : \\
x_0 \in \mathcal{X}_0,\ A(\cdot) \in \mathcal{A},\ u \in \mathcal{U}\big\}
\end{aligned} \tag{7}
$$

and the *reachable set* of the system (1) at the time t is the cross-sections $\mathcal{X}(t)$ of the tube $\mathcal{X}(\cdot)$ (7) at the instant t ($t \in [t_0, T]$).

The main problem considered in this paper is to find the external ellipsoidal estimates for reachable sets $\mathcal{X}(t)$ of the dynamic control systems (1)–(6) with uncertain matrix of the system and uncertain initial state basing on the special structure of the data $\mathcal{A}, \mathcal{U}$ and $\mathcal{X}_0$.

4 Main Results

In this section we apply the techniques of the ellipsoidal calculus [Kurzhanski and Valyi, 1997; Chernousko, 1994] to find the estimates of the reachable sets $\mathcal{X}(t)$, $t \in [t_0, T]$. Consider first some auxiliary results.

4.1 Bilinear System

Bilinear dynamic systems [Kurzhanski and Filippova, 1993] are a special class of nonlinear systems representing a variety of important physical processes [Boscain, Chambrion and Sigalotti, 2013; Boussaïd, Caponigro and Chambrion, 2013; Gough, 2008; Nihtila, 2010]. Reachable sets of bilinear systems in general are not convex, but have special properties (for example, they are star-shaped). We use ellipsoidal calculus and consider here the external estimates of reachable sets of such uncertain control systems.

Consider first the following bilinear system

$$
\begin{aligned}
\dot{x} = A(t)\,x, \quad t_0 \le t \le T, \\
x_0 \in \mathcal{X}_0 = E(a_0, Q_0), \quad A(t) \in \mathcal{A},
\end{aligned} \tag{8}
$$

where $x \in \mathbb{R}^n$, the set $\mathcal{A}$ is defined in (3).

The *reachable set* $\mathcal{X}(t) = \mathcal{X}(t; \mathcal{X}_0, \mathcal{A})$ at time t ($t_0 < t \le T$) of system (8) is defined as the following set

$$
\begin{aligned}
\mathcal{X}(t) = \bigcup \big\{x(t) = x(t;\, t_0, x_0, A(t)) : \\
x_0{\in}\mathcal{X}_0,\ A(t){\in}\mathcal{A}\big\}.
\end{aligned} \tag{9}
$$

Note that the reachable sets $\mathcal{X}(t)$ need not be convex for considering bilinear system. However, these sets have other geometrical properties.

A set $Z \subseteq \mathbb{R}^n$ is called *star-shaped* (with center c) if $c + \lambda(Z - c) \subseteq Z$ for all $\lambda \in [0, 1]$.

The set of all star-shaped compact subsets $Z \subseteq \mathbb{R}^n$ with center c will be denoted as $\mathrm{St}(c, \mathbb{R}^n)$, $\mathrm{St}\,\mathbb{R}^n = \mathrm{St}(0, \mathbb{R}^n)$.

Assumption 1. *For every $t \in [t_0, T]$ the inclusion $0 \in \mathcal{U}$ is true. The inclusion $0 \in \mathcal{X}_0$ is true.*

We will assume further that Assumption 1 is satisfied.

Theorem 1. [Kurzhanski and Filippova, 1993] *Under Assumption 1 the reachable sets $\mathcal{X}(t)$ are star-shaped and compact sets for all $t \in [t_0, T]$ ($\mathcal{X}(t) \in \mathrm{St}\,\mathbb{R}^n$).*

Let $\rho(l|C)$ be the *support function* of a convex compact set $C \in \operatorname{conv}\mathbb{R}^n$, i.e.,

$$\rho(l|C) = \max\{l'c : \ c \in C\}, \quad l \in \mathbb{R}^n .$$

We will denote the *Minkowski function* of a set $M \in \operatorname{St}\mathbb{R}^n$ by

$$h_M(z) = \inf\{t > 0 : \ z \in tM, \ z \in \mathbb{R}^n\}.$$

We need the following notation

$$\mathcal{M} * X = \{z \in \mathbb{R}^n : \ z = Mx, \ M \in \mathcal{M}, \ x \in X\},$$

where $\mathcal{M} \in \operatorname{conv}\mathbb{R}^{n\times n}$, $X \in \operatorname{conv}\mathbb{R}^n$.

Then the evolution equation known as the *integral funnel equation* [Kurzhanski and Filippova, 1993; Kurzhanski and Valyi, 1997] that describes the dynamics of star-shaped trajectory tubes is given in the following theorem.

Theorem 2. [Filippova and Lisin, 2000] *The trajectory tube $\mathcal{X}(t)$ of the bilinear differential system (8) with constraints (3)–(6) is the unique solution to the evolution equation*

$$\lim_{\sigma\to+0} \sigma^{-1} h\big(\mathcal{X}(t+\sigma), (I+\sigma\mathcal{A}) * \mathcal{X}(t)\big) = 0, \quad (10)$$

with initial condition $\mathcal{X}(t_0) = \mathcal{X}_0, \ t \in [t_0, T]$.

From Theorem 2 we have

$$\mathcal{X}(t_0+\sigma) \subseteq (I+\sigma\mathcal{A}) * \mathcal{X}_0 + o(\sigma)B(0,1),$$

where $\sigma^{-1}o(\sigma) \to 0$ for $\sigma \to +0$. Taking into account (3), we note that

$$\begin{gathered}(I+\sigma\mathcal{A}) * \mathcal{X}_0 = \\ = (I+\sigma(A^0+\mathcal{A}^1)) * \mathcal{X}_0 + \sigma\mathcal{A}^2 * \mathcal{X}_0,\end{gathered} \quad (11)$$

where sets $\mathcal{A}^1$ and $\mathcal{A}^2$ are defined in (4) and (5) respectively.

Consider the auxiliary bilinear system

$$\begin{gathered}\dot{x} = A(t)\,x, \quad t \in [t_0, T], \\ x_0 \in \mathcal{X}_0 = E(a_0, Q_0), \quad A(t) \in A^0 + \mathcal{A}^1.\end{gathered} \quad (12)$$

The external ellipsoidal estimate of set $(I+\sigma(A^0+\mathcal{A}^1)) * \mathcal{X}_0$ may be found by applying the following theorem.

Theorem 3. [Chernousko, 1996] *Let $a^*(t)$ and $Q^*(t)$ be the solutions of the following system of nonlinear differential equations*

$$\dot{a}^* = A^0 a^*, \quad a_1^+(t_0) = a_0, \quad (13)$$

$$\begin{gathered}\dot{Q}^* = A^0 Q^* + Q^* A^{0'} + qQ^* + q^{-1}G, \quad (14) \\ Q^*(t_0) = Q_0, \quad t_0 \le t \le T, \\ q = \big(n^{-1}\operatorname{Tr}((Q^+)^{-1}G)\big)^{1/2}, \\ G = \operatorname{diag}\Big\{(n-v)\Big[\sum_{i=1}^n c_{ji}|a_i^+| + \\ +\big(\max_{\sigma=\{\sigma_{ij}\}} \sum_{p,q=1}^n Q^+_{pq} c_{jp} c_{jq} \sigma_{jp}\sigma_{jq}\big)^{1/2}\Big]^2\Big\}.\end{gathered}$$

Here the maximum is taken over all $\sigma_{ij} = \pm 1$, $i,j = 1,\ldots,n$, such that $c_{ij} \neq 0$ and v is a number of such indices i for which we have: $c_{ij} = 0$ for all $j = 1,\ldots,n$. Then the following external estimate for the reachable set $\mathcal{X}(t)$ of the system (12) *is true*

$$\mathcal{X}(t) \subseteq E(a^*(t), Q^*(t)), \quad t_0 \le t \le T. \quad (15)$$

Corollary 1. *Under conditions of the Theorem 3 the following inclusion holds*

$$\begin{gathered}(I+\sigma(A^0+\mathcal{A}^1)) * \mathcal{X}_0 \subseteq \\ \subseteq E(a^*(t_0+\sigma), Q^*(t_0+\sigma)) + o(\sigma)B(0,1),\end{gathered} \quad (16)$$

where $\sigma^{-1}o(\sigma) \to 0$ for $\sigma \to +0$.

The following theorem is hold.

Theorem 4. [Filippova and Lisin, 2000] *For every $z \in \mathbb{R}^n$ such that $z_i \neq 0$ $(i = 1,\ldots,n)$ the following formula is true:*

$$\begin{gathered}h_{\mathcal{A}^2 * \mathcal{X}_0}(z) = \min\Big\{\max_{l\neq 0} \frac{1}{\rho(l|\mathcal{X}_0)} \sum_{i=1}^n l_i z_i a_i^{-1} : \\ a \in \mathbf{A}_0, \ a_i \neq 0, \ i = 1,\ldots,n\Big\}.\end{gathered}$$

Remark 1. [Filippova and Lisin, 2000] Let now the set $\mathcal{A}^2$ be defined in (5) and $\mathcal{X}_0 = E(0, Q_0)$, then the following formula is true

$$h_{\mathcal{A}^2 * E(0,Q_0)}(z) = \|Q_0^{-\frac{1}{2}} z\|_{l_1}.$$

The external ellipsoidal estimate of set $\sigma\mathcal{A}^2 * \mathcal{X}_0$ may be found by applying the following theorem.

Theorem 5. [Matviychuk, 2016] *For $\mathcal{X}_0 = E(a_0, Q_0)$ and all $\sigma > 0$ the following external estimate is true*

$$\sigma \mathcal{A}^2 * \mathcal{X}_0 \subseteq E(a_0, \tilde{Q}(\sigma)) + o(\sigma)B(0,1), \qquad (17)$$

where $\sigma^{-1}o(\sigma) \to 0$ for $\sigma \to +0$,

$$\tilde{Q}(\sigma) = \mathrm{diag}\{(p^{-1}+1)\sigma^2(a_i^0)^2 + (p+1)r^2(\sigma)\},$$
$$a_0 = \{a_i^0\}, \quad r(\sigma) = \sigma \max_z \|z\| \left(\|Q_0^{-\frac{1}{2}} z\|_{l_1}\right)^{-1},$$

Here p is the unique positive root of the equation $\sum_{i=1}^n 1/p + \alpha_i = n/p(p+1)$, where $\alpha_i \geq 0$ $(i = 1, ..., n)$ being the roots of the following equation $\prod_{i=1}^n \left(\sigma^2(a_i^0)^2 - \alpha r^2(\sigma)\right) = 0$.

Then an external ellipsoidal estimate of the trajectory tube $\mathcal{X}(t)$ of the system (8) may be found by applying the following new result.

Theorem 6. *For the trajectory tube $\mathcal{X}(t)$ of the system (8) and for all $\sigma > 0$ the following inclusion holds*

$$\mathcal{X}(t_0+\sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)) + o(\sigma)B(0,1), \quad (18)$$

where $\sigma^{-1}o(\sigma) \to 0$ for $\sigma \to +0$,

$$a^+(\sigma) = a^*(t_0+\sigma),$$
$$Q^+(\sigma) = (p^{-1}+1)\tilde{Q}(\sigma) + (p+1)Q^*(t_0+\sigma).$$

Here $a^*(t_0+\sigma)$, $Q^*(t_0+\sigma)$, $\tilde{Q}(\sigma)$ are defined in Theorem 3, Theorem 5 and p is the unique positive root of the equation $\sum_{i=1}^n 1/p + \alpha_i = n/p(p+1)$, where $\alpha_i \geq 0$ $(i = 1, ..., n)$ being the roots of the following equation $|\tilde{Q}(\sigma) - \alpha Q^*(t_0+\sigma)| = 0$.

Proof. From the Theorem 2 and formula (11) it follows that

$$\mathcal{X}(t_0+\sigma) \subseteq (I + \sigma(A^0 + \mathcal{A}^1)) * \mathcal{X}_0 + \\ + \sigma\mathcal{A}^2 * \mathcal{X}_0 + o(\sigma)B(0,1),$$

where $\sigma^{-1}o(\sigma) \to 0$ for $\sigma \to +0$. From Theorem 3 and Theorem 5 we have estimates

$$(I + \sigma(A^0 + \mathcal{A}^1)) * \mathcal{X}_0 \subseteq \\ \subseteq E(a^*(t_0+\sigma), Q^*(t_0+\sigma)) + o(\sigma)B(0,1),$$
$$\sigma\mathcal{A}^2 * \mathcal{X}_0 \subseteq E(0, \tilde{Q}(\sigma)) + o(\sigma)B(0,1).$$

Apply the procedure of external ellipsoidal estimate of sum of two ellipsoids given in [Chernousko, 1994; Kurzhanski and Valyi, 1997] we obtain the result of the Theorem 6. □

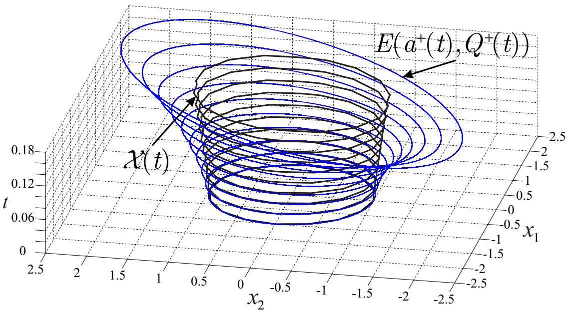

Figure 1. Trajectory tube $\mathcal{X}(t)$ and its ellipsoidal estimating tube $E(a^+(t), Q^+(t))$ for the bilinear control system with uncertain initial states.

The following algorithm is based on Theorem 6 and may be used to produce the external ellipsoidal estimates for the reachable sets of the system (8).

Algorithm 1. The time segment $[t_0, T]$ is subdivided into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + i\sigma$ ($i = 1, \ldots, m$), $\sigma = (T - t_0)/m$, $t_m = T$.

- For given $\mathcal{X}_0 = E(a_0, Q_0)$ and $\mathbf{A}_0 = B(0,1)$ we find the external estimate $E(a^+(\sigma), Q^+(\sigma))$ by Theorem 6 such that

$$\mathcal{X}(t_1) = \mathcal{X}(t_0+\sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)).$$

- Consider the system on the next subsegment $[t_1, t_2]$ with $E(a^+(\sigma), Q^+(\sigma))$ as the initial ellipsoid at instant t_1.

The following steps repeat the previous iteration.

At the end of the process we will get the external estimate of the tube $\mathcal{X}(\cdot)$ of the system (8) with accuracy tending to zero when $m \to \infty$.

The following example illustrates the Algorithm 1.
Example 1. Consider the following system

$$\begin{cases} \dot{x}_1 = a_1 x_1 + x_2, \\ \dot{x}_2 = a_2 x_2 + c(t)\, x_1, \end{cases} \quad 0 \leq t \leq 0.18,$$

where $x_0 \in \mathcal{X}_0 = B(0,1)$, $c(t)$ is an unknown but bounded measurable function with $|c(t)| \leq 1$, the uncertain bounded matrix function $A(t) \in \mathcal{A}$ where

$$\mathcal{A} = \Big\{ A(t) : A(t) = \mathrm{diag}\{a_1, a_2\}, \\ a_1^2 + a_2^2 \leq 1,\ t \in [0, 0.18] \Big\}.$$

The trajectory tube $\mathcal{X}(t)$ and its external ellipsoidal estimate $E(a^+(t), Q^+(t))$ calculated by Algorithm 1 are given in Figure 1.

4.2 Bilinear Impulsive Control System

Consider the bilinear impulsive control system (1) with restrictions (2)–(6)

$$dx(t) = A(t)x(t)dt + B(t)du(t),$$
$$x(t_0 - 0) = x_0 \in \mathcal{X}_0 = E(a_0, Q_0), \quad t \in [t_0, T]$$
$$A(t) \in \mathcal{A}, \quad u \in \mathcal{U}.$$

Let us introduce a new time variable [Rishel, 1965] $\eta = \eta(t)$ and a new state coordinate $\tau = \tau(\eta)$

$$\eta(t) = t + \int_{t_0}^{t} du(s), \quad \tau(\eta) = \inf\{t \mid \eta(t) \geq \eta\}.$$

Consider the following auxiliary equation

$$\frac{d}{d\eta}\begin{pmatrix} z \\ \tau \end{pmatrix} \in H(\tau, z), \tag{19}$$
$$z(t_0) = z_0 \in \mathcal{X}_0 = E(a_0, Q_0),$$
$$\tau(t_0) = t_0, \quad t_0 \leq \eta \leq T + \mu,$$
$$H(\tau, z) = \bigcup_{0 \leq \nu \leq 1} \left\{ (1-\nu)\begin{pmatrix} A(\tau)z \\ 1 \end{pmatrix} + \nu \begin{pmatrix} B(\tau) \\ 0 \end{pmatrix} \right\}.$$

Denote by $w = \{z, \tau\}$ the extended state vector of the system (19) and by $W(\eta) = W(\eta; t_0, \mathcal{X}_0 \times \{t_0\}, \mathcal{A})$ $(t_0 \leq \eta \leq T + \mu)$ the reachable set of the system (19).

Theorem 7. *The following inclusion holds for $\sigma > 0$*

$$W(t_0 + \sigma) \subseteq W(t_0, \sigma) + o(\sigma)B^{n+1}(0, 1), \tag{20}$$
$$\lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0,$$
$$W(t_0, \sigma) = \bigcup_{0 \leq \nu \leq 1} W(t_0, \sigma, \nu),$$
$$W(t_0, \sigma, \nu) = \begin{pmatrix} E\big(a^+(t_0, \sigma, \nu), Q^+(t_0, \sigma, \nu)\big) \\ t_0 + \sigma(1-\nu) \end{pmatrix},$$
$$a^+(t_0, \sigma, \nu) = \tilde{a}^*(\sigma, \nu) + \sigma\nu B(t_0),$$
$$Q^+(t_0, \sigma, \nu) = (q^{-1} + 1)\sigma^2(1-\nu)^2\tilde{Q}(\sigma) + \\ + (q+1)\tilde{Q}^*(\sigma, \nu),$$

where $\tilde{Q}(\sigma)$ is defined in Theorem 5 and functions $\tilde{a}^*(\sigma, \nu)$, $\tilde{Q}^*(\sigma, \nu)$ calculated as $a^*(t)$, $Q^*(t)$ in Theorem 3 but when we replace matrix A^0 in (13), (14) *by $\tilde{A}^0 = (1-\nu)A^0$. Here $q = q(\sigma, \nu)$ is the unique positive root of the equation $\sum_{i=1}^{n} 1/q + \lambda_i = n/q(q+1)$, with $\lambda_i = \lambda_i(\sigma, \nu) \geq 0$ $(i = 1, ..., n)$ being the roots of the equation $|\sigma^2(1-\nu)^2\tilde{Q}(\sigma) - \lambda\tilde{Q}^*(\sigma, \nu)| = 0$.*

Proof. The above generalization is based on a combination of the techniques described above and the results of [Filippova and Matviychuk, 2011; Filippova and Matviychuk, 2015]. □

Remark 2. To find the estimate of the reachable set $W(t_0 + \sigma)$ we introduce small parameter $\varepsilon > 0$ and embed the degenerate ellipsoid $W(t_0, \sigma, \nu)$ in nondegenerate ellipsoid $E\big(w_\varepsilon(t_0, \sigma, \nu), O_\varepsilon(t_0, \sigma, \nu)\big)$:

$$W(t_0, \sigma, \nu) \subseteq E\big(w_\varepsilon(t_0, \sigma, \nu), O_\varepsilon(t_0, \sigma, \nu)\big),$$
$$w_\varepsilon(t_0, \sigma, \nu) = \begin{pmatrix} a^+(t_0, \sigma, \nu) \\ t_0 + \sigma(1-\nu) \end{pmatrix},$$
$$O_\varepsilon(t_0, \sigma, \nu) = \begin{pmatrix} Q^+(t_0, \sigma, \nu) & 0 \\ 0 & \varepsilon^2 \end{pmatrix}.$$

Thus, for all small $\varepsilon > 0$ we get

$$W(t_0 + \sigma) \subseteq W(t_0, \sigma) \subseteq W_\varepsilon(t_0, \sigma),$$
$$W_\varepsilon(t_0, \sigma) = \bigcup_{0 \leq \nu \leq 1} E\big(w_\varepsilon(t_0, \sigma, \nu), O_\varepsilon(t_0, \sigma, \nu)\big)$$

and $\lim_{\varepsilon \to +0} h(W(t_0, \sigma), W_\varepsilon(t_0, \sigma)) = 0$. The passage to the family of nondegenerate ellipsoids enables one to use the algorithms of [Vzdornova and Filippova, 2006] and construct an external estimate of the union of the ellipsoids $W_\varepsilon(t_0, \sigma) \subset E_\varepsilon(w^+(\sigma), O^+(\sigma))$.

The following lemma explains the reason of construction of the auxiliary differential inclusion (19).

Lemma 1. [Filippova and Matviychuk, 2011] *The set $\mathcal{X}(T) = \mathcal{X}(T, t_0, \mathcal{X}_0)$ is the projection of $W(T + \mu)$ at the subspace of variables z: $\mathcal{X}(T) = \pi_z W(T + \mu)$.*

The next iterative algorithm based on Theorem 7.

Algorithm 2. The time segment $[t_0, T + \mu]$ is subdivided into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + i\sigma$ $(i = 1, \ldots, m)$, $\sigma = (T + \mu - t_0)/m$, $t_m = T + \mu$. Subdivide the segment $[0, 1]$ into subsegments $[\nu_j, \nu_{j+1}]$ where $\nu_j = jh_*$, $h_* = 1/k$, $\nu_0 = 0$, $\nu_k = 1$ $(j = 1, \ldots, k)$.

- For the given $\mathcal{X}_0 = E(a_0, Q_0)$ define by Theorem 7 sets $W(\sigma, \nu_j)$ $(j = 0, \ldots, k)$.
- Fix the small parameter $\varepsilon > 0$ and find ellipsoid $E_\varepsilon(w_1(\sigma), O_1(\sigma))$ in $\mathbb{R}^{n+1}$ such that $W(\sigma, \nu_j) \subseteq E_\varepsilon(w_1(\sigma), O_1(\sigma))$ $(j = 0, \ldots, k)$. At this step we find the ellipsoidal estimate for the union of a finite family of ellipsoids [Filippova and Matviychuk, 2011; Matviychuk, 2012].
- Find the projection of $E(a_1, Q_1) = \pi_z E_\varepsilon(w_1(\sigma), O_1(\sigma))$ by Lemma 1.
- Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, Q_1)$ as the initial ellipsoid at instant t_1.

The following steps are repeated previous iteration.

At the end of the process we will get the external estimate $E(a^+(T), Q^+(T))$ of the reachable sets $\mathcal{X}(T)$ of the impulsive control systems (1) with uncertain matrix of the system and uncertain initial state basing on the special structure of the data $\mathcal{A}$, $\mathcal{U}$ and $\mathcal{X}_0$.

5 Conclusion

The problem of state estimation of the reachable sets for uncertain impulsive control systems for which we assume that the initial state is unknown but bounded with given constraints and the matrix in the linear part of state velocities is also unknown but bounded is considered here.

The modified state estimation approach which uses the special constraints on the controls and uncertainty and allows to construct the external ellipsoidal estimates of reachable sets is presented. This method is based on results of ellipsoidal calculus developed earlier for some classes of uncertain systems.

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