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# PHASE TRANSITIONS IN COMPLEX NETWORKS OF ACTIVE AND INACTIVE OSCILLATORS

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**Abstract**

We study phase transitions in mixed populations of interacting active and inactive oscillators on complex networks. As the ratio of inactive oscillators to the total population increases, the macroscopic oscillatory activity of the whole network decreases and eventually stops at a critical ratio. This phase transition, called an aging transition in [Daido and Nakanishi, 2004], has been studied with simple networks so far. To extend the conventional framework, we analyze aging transitions in complex networks including random and scale-free networks. The critical ratio is theoretically derived through appropriate approximations and numerically verified.

**Key words**

Phase transitions, complex networks, coupled oscillators

## 1 Introduction

Complex networks have been intensively studied for this decade, because a diverse range of real networks possess non-regular structures [Strogatz, 2001]. Statistical mechanics of complex networks have been explored with data analyses of social, communication, biological, computer, and power networks [Albert and Barabási, 2002; Newman, 2003; Boccaletti *et al.*, 2006]. Complex network theory is not only useful for understanding the mechanism of structures in real networks but also significant for designing robust and resilient networks [Vespignani, 2010].

One of the major topics in complex newtork theory is robustness and fragility against failures and attacks [Albert *et al.*, 2000; Callaway *et al.*, 2000; Motter and Lai, 2002]. A breakdown of a whole network can be caused by cascading failure, which is often responsible for blackouts and computer system failures. Even if the initial failure serving as a trigger of cascading failure is the same level, the damage of the whole network can be different depending on network topology. For example, it has been known that a scale-free network is unexpectedly robust against random failures but extremely vulnerable to targeted failures, or attacks to hubs with high degrees. These two aspects of robustness and fragility can be caused by the difference in the change of average path length after removal of nodes. Many efforts have been made to elucidate a general property of cascading failure in the framework of percolation theory [Callaway *et al.*, 2000; Newman *et al.*, 2001; Costa, 2004; Buldyrev *et al.*, 2010]. In these models, some damaged nodes are initially removed from a network and other nodes are sequentially removed as a result of cascading failure. The size of the giant component (the largest set of connected nodes) which finally remains is regarded as a macroscopic quantity representing a degree of network functionality. As the fraction of initially removed nodes increases, a phase transition occurs and the giant component vanishes. However, this framework is not applicable to understanding robustness of networks composed of dynamical units.

For getting insights into robustness of biological systems consisting of dynamical elements [Barabási, 2004], we analyze complex networks composed of coupled oscillators in another framework. The tolerance of a network against inactivation of oscillators is examined. As the ratio of inactivated oscillators increases, the global oscillations are diminished. When the ratio increases beyond a certain critical value, the global oscillations terminate. This phase transition is called an aging transition [Daido and Nakanishi, 2004]. The critical ratio can be regarded as an index of robustness of the network: the higher the critical ratio is, the more robust the network is. So far, aging transitions have been investigated with simple network topologies [Daido and Nakanishi, 2004; Daido and Nakanishi, 2007; Pazó and Montbrió, 2006; Daido, 2008; Tanaka *et al.*, 2010; Morino *et al.*].

We explore aging transitions in complex networks of active and inactive oscillators. First, we analytically derive the condition for aging transitions in random networks by using system reduction technique. We clarify that the link density as well as the coupling strength play important roles for the phase transition. Second, we analyze aging transitions in scale-free networks by using approximations with degree-weighted mean fields. The analytical results are verified by numerical simulations.

## 2 Model

We study a network of $N$ coupled oscillators as follows:

$$\dot{z}_j = f_j(z_j) + \frac{K}{N}\sum_{k=1}^{N} A_{jk}(z_k - z_j), \tag{1}$$

where $z_j$ is the complex state variable of the $j$th oscillator, $f_j(z_j) \equiv (\alpha_j + i\Omega - |z_j|^2)z_j$, $K$ is the coupling strength, and $A = (A_{jk})$ is the adjacency matrix where $A_{jk} = 1$ if the $j$th and $k$th oscillators are connected while $A_{jk} = 0$ otherwise. This symmetric matrix determines the network topology, which is simple in the previous studies but complex in our study. The degree of the $j$th node is indicated by $k_j = \sum_{k=1}^{N} A_{jk}$.

The single oscillator without coupling (i.e. $K = 0$) is represented by Stuart-Landau equation: $\dot{z} = (\alpha + i\Omega - |z|^2)z$, which is a simple system describing the dynamics near a Hopf bifurcation at $\alpha = 0$. It shows self-oscillatory behavior with amplitude $\sqrt{\alpha}$ and frequency $\Omega$ for $\alpha > 0$, while non-oscillatory behavior after transient damping oscillations for $\alpha < 0$. We suppose that some active oscillators are randomly inactivated with ratio $p$. The set of the inactivated oscillators is denoted by $S_I$ and that of others by $S_A$. We set $\alpha_j = -b < 0$ for $j \in S_I$ and $\alpha_j = a > 0$ for $j \in S_A$, where $a = 1$ and $b = 3$ throughout this article.

The macroscopic oscillations of the whole network is evaluated by the order parameter $|Z|$ where $Z = (1/N)\sum_{j=1}^{N} z_j$. As the ratio of inactive oscillators $p$ increases from zero, the order parameter suddenly falls below a very small threshold value at a critical point $p_c$ due to an aging transition. The threshold is fixed at $10^{-6}$. Numerical integrations were performed by the fourth-order Runge-Kutta method with time step 0.1.

## 3 Results

### 3.1 Random networks

We first consider a random network where the degrees are distributed around the mean degree and the variance of the degree distribution is relatively small. In such networks, the oscillations in each subpopulation of active and inactive oscillators are almost synchronized for a sufficiently large coupling strength. Figure 1 shows that the amplitudes of the oscillators are nearly the same in each group and seem to be independent of the degrees in an Erdős-Rényi random network [Erdős and Rényi, 1960]. Using this property, we perform system reduction and thereby theoretically derive the condition for an aging transition. We introduce the link density $d$, which is the proportion of the total number of links to the possible maximum number of links, i.e. $d = N\langle k\rangle/N(N-1)$, where $\langle k\rangle = (1/N)\sum_{j=1}^{N} k_j$ is the mean degree. The link density can be approximated as $d \simeq \langle k\rangle/N$ for a sufficiently large number of $N$. Then, the number of active oscillators in the neighbors of each oscillator is expected to be $(1-p)\langle k\rangle$ and that of inactive oscillators to be $p\langle k\rangle$. By setting $z_j = A$ for all active oscillators and $z_j = I$ for all inactive oscillators, we obtain

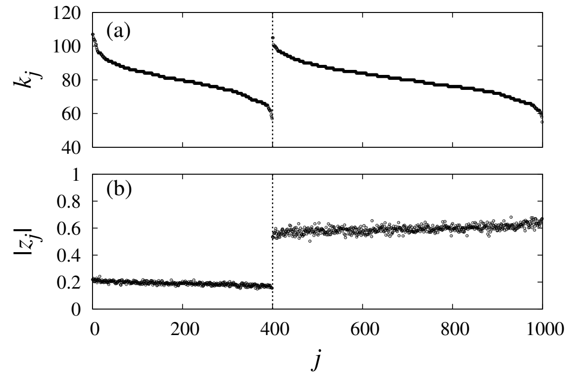


Figure 1. The degrees (a) and the amplitudes (b) of the oscillators in a random network with $N = 1000$, $\langle k\rangle = 80$, $K = 30$, and $p = 0.4$. The vertical line separates inactive (left) and active (right) oscillators. The degrees are sorted in decending order within each group.

$$\dot{A} = (a - Kpd + i\Omega - |A|^2)A + KpdI, \tag{2}$$
$$\dot{I} = (-b - Kqd + i\Omega - |I|^2)I + KqdA, \tag{3}$$

where $q \equiv 1 - p$.

In the limit of $N \to \infty$, the ratio $p$ can be regarded as a real number between 0 and 1. Since an aging transition occurs when the trivial equilibrium point $A = I = 0$ is stabilized as $p$ is increased from 0, a linear stability analysis yields

$$p_c = \frac{a(Kd + b)}{(a+b)Kd}. \tag{4}$$

Figure 2 shows that the theoretical result is in good agreement with the numerical result, although it is influenced by the configuration of a random network for $p$ close to 1. The aging transition points obtained by numerical simulations (the circle with the error bar) and analytical form (4) are plotted in the $(K, p)$-plane. As $K$ goes to the infinity, the critical ratio $p_c$ converges to $a/(a+b)$ as in the globally coupled network [Daido and Nakanishi, 2004]. From the condition that $p_c = 1$, the critical coupling strength is obtained as $K_c = a/d$ with $d$ fixed, below which an aging transition does not

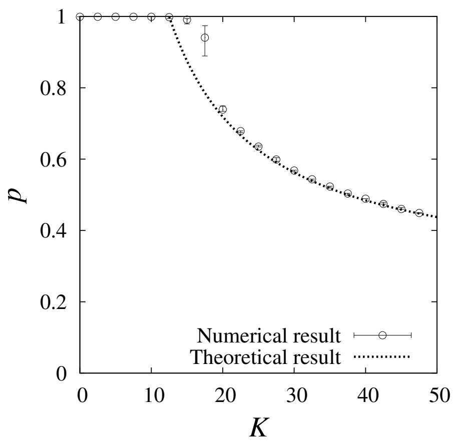


Figure 2. The aging transition points in the Erdős-Rényi random networks with $N = 1000$ and $\langle k \rangle = 80$ (i.e. $d = 0.08$). The circle with the error bar indicates the numerical result for 10 network realizations. The dashed line indicates the theoretical result given by Eq. (4). The critical coupling strength above which an aging transition occurs is given by $K_c = a/d = 12.5$.

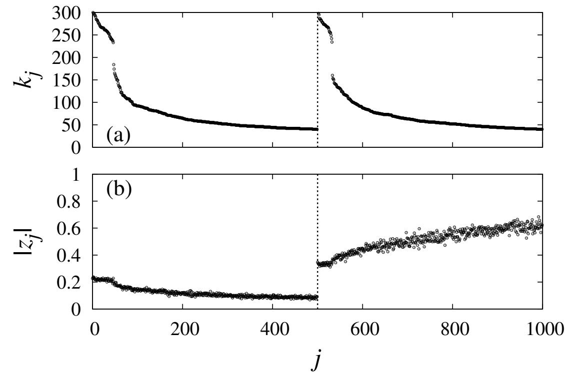


Figure 3. The degrees (a) and the amplitudes (b) of the oscillators in a scale-free network with $N = 1000$, $\langle k \rangle = 80$, $K = 30$, and $p = 0.5$. The vertical line separates inactive (left) and active (right) oscillators. The degrees are sorted in descending order within each group.

take place until $p = 1$. A similar diagram can be depicted in the $(d, p)$-plane when $K$ is fixed. The critical density is given as $d_c = a/K$, below which an aging transition does not occur. In the case of $d = 1$, the network corresponds to the globally coupled one and Eq. (4) is reduced to the result derived in the previous study [Daido and Nakanishi, 2004].

In random networks, it is feasible to assume that the oscillators in each subpopulation of active and inactive elements are synchronized. Since they are regarded as identical oscillators in each group, the system reduction based on this assumption is successful. A similar analysis is possible for a network where the degree distribution is relatively homogeneous, e.g. small-world networks. For such networks, the robustness is dependent on the link density as well as the coupling strength.

### 3.2 Scale-free networks

In contrast to random networks, the degree distribution of a scale-free network is highly heterogeneous. A scale-free network is composed of a small number of nodes with many degrees and a large number of nodes with few degrees. This network heterogeneity largely influences the oscillatory dynamics and the aging transition. As shown in Fig. 3, the amplitudes of oscillations depend on the degree of oscillators. Therefore, the system reduction technique in the previous subsection is not valid for scale-free networks.

We consider mean fields for each subpopulation of active and inactive oscillators, because active and inactive oscillators behave differently even if the degree is the same. We use a degree-weighted mean field approximation for system reduction, which was employed for analysis of scale-free networks of oscillators [Nakao and Mikhailov, 2009]. Considering the local field:

$$h_j = \sum_{k=1}^{N} A_{jk} z_k, \qquad (5)$$

Eq. (1) is rewritten as follows:

$$\dot{z}_j = f_j(z_j) + \frac{K}{N}(h_j - k_j z_j). \qquad (6)$$

For a sufficiently large number of $N$, the number of active oscillators in the neighbors of the $j$th oscillator is expected to be $(1-p)k_j$ and that of inactive oscillators to be $pk_j$. Here we introduce degree-weighted mean fields for active and inactive subpopulations, respectively, as follows:

$$H_A(t) = \frac{\sum_{j \in S_A} k_j z_j(t)}{\sum_{j \in S_A} k_j}, \qquad (7)$$

$$H_I(t) = \frac{\sum_{j \in S_I} k_j z_j(t)}{\sum_{j \in S_I} k_j}. \qquad (8)$$

We assume that the local field is approximated by using the degree-weighted mean fields as follows:

$$h_j(t) \simeq (1-p)k_j H_A(t) + pk_j H_I(t). \qquad (9)$$

With this assumption, the oscillators with the same degree are viewed as identical ones and Eq. (6) is rewritten as follows:

$$\dot{z}_j = f_j(z_j) + \frac{Kk_j}{N}((1-p)H_A(t) + pH_I(t) - z_j). \qquad (10)$$

From numerical simulation of the coupled system, we notice that all the oscillators exhibit phase synchronization of oscillations with frequency $\Omega$. Thus, we suppose that the state variables are written as $z_j(t) = r_j(t)\exp(i(\Omega t+\theta))$, where $r_j$ is the amplitude and $\theta$ is the phase delay. Then, the mean fields for the two subpopulations are respectively described as follows:

$$H_A(t) = R_A(t)e^{i(\Omega t+\theta)}, \tag{11}$$
$$H_I(t) = R_I(t)e^{i(\Omega t+\theta)}, \tag{12}$$

where

$$R_A = \frac{\sum_{j\in S_A} k_j r_j}{\sum_{j\in S_A} k_j}, \tag{13}$$
$$R_I = \frac{\sum_{j\in S_I} k_j r_j}{\sum_{j\in S_I} k_j}. \tag{14}$$

By substituting Eqs. (11)-(12) into Eq. (10), we obtain the following evolution equation:

$$\dot{r}_j = (\alpha_j - \frac{Kk_j}{N} - r_j^2)r_j + \frac{Kk_j}{N}((1-p)R_A + pR_I). \tag{15}$$

Once $R_A$ and $R_I$ are given, the amplitude of the oscillations in a stationary state is obtained as $r_j^*(R_A, R_I)$ from Eq. (15). The self-consistency of the mean field approximation requires that the mean fields in Eqs. (13)-(14) calculated from these stationary amplitudes are consistent with the originally given ones. Namely, it follows that

$$R_A = G_A(R_A, R_I) \equiv \frac{\sum_{j\in S_A} k_j r_j^*(R_A, R_I)}{\sum_{j\in S_A} k_j}, \tag{16}$$
$$R_I = G_I(R_A, R_I) \equiv \frac{\sum_{j\in S_I} k_j r_j^*(R_A, R_I)}{\sum_{j\in S_I} k_j}. \tag{17}$$

There exists a stable solution with $R_A, R_I > 0$ before the aging transition, while the origin $R_A = R_I = 0$ is stable after the aging transition. Therefore, the change of the stability of the origin corresponds to the critical transition point. The condition can be discussed by the eigenvalues of the linearized matrix at the origin, described as follows:

$$J_0 = \left[\begin{array}{cc} \frac{\partial G_A(R_A,R_I)}{\partial R_A} & \frac{\partial G_A(R_A,R_I)}{\partial R_I} \\ \frac{\partial G_I(R_A,R_I)}{\partial R_A} & \frac{\partial G_I(R_A,R_I)}{\partial R_I} \end{array}\right]\Bigg|_{R_A=R_I=0} \tag{18}$$

Now let us calculate the components of $J_0$. First we derive $\partial r_j^*/\partial R_A$ and $\partial r_j^*/\partial R_I$ at $R_A = R_I = 0$. From Eq. (15), the stationary state before an aging transition is a positive real solution of the following cubic equation:

$$r_j^3 - \beta_j r_j - \delta_j = 0, \tag{19}$$

where

$$\beta_j = \alpha_j - Kk_j/N, \tag{20}$$
$$\delta_j = \frac{Kk_j}{N}((1-p)R_A + pR_I). \tag{21}$$

It should be noted that $\beta_j$ depends on the oscillator type (active or inactive) and the degree of the $j$th node. It is obvious that $\delta_j > 0$ for $R_A, R_I > 0$ because $0 < p < 1$. The number of real solutions of the cubic equation is different according to the sign of $\beta_j$. Therefore, we separately consider the cases of $\beta_j < 0$, $\beta_j = 0$, and $\beta_j < 0$.

If $\beta_j < 0$, then the cubic equation has only one real root because the discriminant $D = 4\beta_j^3 - 27\delta_j^2$ is negative. The real root is described as follows:

$$r_j^* = (X+Y)^{1/3} - (Y-X)^{1/3}, \tag{22}$$

where

$$X = \frac{\delta_j}{2}, \tag{23}$$
$$Y = \sqrt{\left(\frac{\delta_j}{2}\right)^2 - \left(\frac{\beta_j}{3}\right)^3}. \tag{24}$$

Since $X < Y$, $\delta_j = 2X = (X+Y) - (Y-X) > 0$. Hence, $(X+Y)^{1/3} > (Y-X)^{1/3}$ and thereby $r_j^* > 0$. A linear stability analysis shows that this positive solution is stable. By differentiating Eq. (21) and Eq. (22), we obtain the following derivatives:

$$\left.\frac{\partial r_j^*}{\partial R_A}\right|_{R_A=R_I=0} = \left.\frac{\partial r_j^*}{\partial \delta_j}\right|_{\delta_j=0} \cdot \left.\frac{\partial \delta_j}{\partial R_A}\right|_{R_A=R_I=0} = -\frac{1}{\beta_j}\cdot\frac{(1-p)Kk_j}{N}, \tag{25}$$
$$\left.\frac{\partial r_j^*}{\partial R_I}\right|_{R_A=R_I=0} = \left.\frac{\partial r_j^*}{\partial \delta_j}\right|_{\delta_j=0} \cdot \left.\frac{\partial \delta_j}{\partial R_I}\right|_{R_A=R_I=0} = -\frac{1}{\beta_j}\cdot\frac{pKk_j}{N}. \tag{26}$$

If $\beta_j = 0$, then the cubic equation has only one positive real root represented as follows:

$$r_j^* = \delta_j^{1/3}. \tag{27}$$

However, the derivatives $\partial r_j^*/\partial R_A$ and $\partial r_j^*/\partial R_I$ diverge in the limit of $R_A, R_I \to 0$. Therefore, the mean field approximation does not work in this case.

If $\beta_j > 0$, the cubic equation has three real roots for $\delta_j$ close to 0 because the discriminant $D = 4\beta_j^3 - 27\delta_j^2$ is positive. The roots are described as follows:

$$r_j^* = \omega^m (X + iY')^{1/3} + \omega^{3-m}(X - iY')^{1/3} \quad (m = 0, 1, 2), \quad (28)$$

where $Y' = iY$ is a positive real value and $\omega = e^{2\pi i/3}$. By introducing $e^{i\Theta} \equiv (X + iY')/\sqrt{X^2 + Y'^2}$, they are calculated as $r_j^* = 2\sqrt{\beta_j/3}\cos((\Theta + 2m\pi)/3)$ $(m = 0, 1, 2)$. In the limit of $R_A, R_I \to 0$ ($\delta_j \to 0$), the three roots approach $2\sqrt{\beta_j/3}\cos(\pi/6 + 2m\pi/3)$ $(m = 0, 1, 2)$, respectively. The only positive root given by the above root with $m = 2$, corresponding to the stationary amplitude, remains positive even if the given mean fields vanish. Hence, the self-consistency of the mean field approximation conflicts with the presumption that an aging transition occurs at a ratio $p$ in the range of $0 < p < 1$.

Form above discussions, we assume $\beta_j < 0$ for all $j$ in what follows so that the mean field approximation works well. This assumption is satisfied if the minimum degree of the active oscillator populations is larger than $aN/K$. From Eq. (16) and Eqs. (25)-(26), the (1,1)th entry of the linearized matrix $J_0$ is obtained as follows:

$$\left.\frac{\partial G_A}{\partial R_A}\right|_{R_A=R_I=0} = \frac{1}{\sum_{j\in S_A} k_j}\left(\sum_{j\in S_A} k_j \left.\frac{\partial r_j^*}{\partial R_A}\right|_{R_A=R_I=0}\right) = \frac{(1-p)K}{\sum_{j\in S_A} k_j}\left(\frac{1}{N}\sum_{j\in S_A}\frac{k_j^2}{Kk_j/N - \alpha_j}\right). \quad (29)$$

The configuration of the active and inactive oscillators is determined independently of the degree distribution. Therefore, the total number of links owned by active oscillators can be approximated as $\sum_{j\in S_A} k_j \simeq (1-p)N\langle k\rangle$ by using the mean degree $\langle k\rangle$ of the whole network. When $N$ is sufficiently large, using the link density $d \simeq \langle k\rangle/N$, it is further approximated as $\sum_{j\in S_A} k_j \simeq (1-p)dN^2$. Thus, the derivative in Eq. (29) can be approximated as follows:

$$\left.\frac{\partial G_A}{\partial R_A}\right|_{R_A=R_I=0} \simeq \frac{1}{d}\left(\frac{1}{N}\sum_{j\in S_A}\frac{d_j^2}{d_j - \alpha_j/K}\right), \quad (30)$$

where $d_j = k_j/N$ is the normalized degree. Note that the average of the normalized degrees is equivalent to the link density in the limit of $N \to \infty$. Similarly, we can approximate all the other entries of the linearized matrix $J_0$. For simplicity of description, we define the following function:

$$F(K, \alpha) = \frac{1}{N}\sum_{j=1}^{N}\frac{d_j^2}{d_j - \alpha/K}, \quad (31)$$

which is independent of the ratio $p$. When $N$ is sufficiently large, the following approximations hold:

$$\frac{1}{N}\sum_{j\in S_A}\frac{d_j^2}{d_j - \alpha_j/K} \simeq (1-p)F(K, a), \quad (32)$$

$$\frac{1}{N}\sum_{j\in S_I}\frac{d_j^2}{d_j - \alpha_j/K} \simeq pF(K, -b). \quad (33)$$

Substitution of the above equations into Eq. (30) and into the corresponding equations for the other components yields

$$J_0 = \begin{bmatrix} \frac{(1-p)F(K,a)}{d} & \frac{pF(K,a)}{d} \\ \frac{(1-p)F(K,-b)}{d} & \frac{pF(K,-b)}{d} \end{bmatrix}. \quad (34)$$

The characteristic equation of this matrix is given by

$$\chi(\lambda) = \lambda^2 - \left(\frac{(1-p)F(K,a)}{d} + \frac{pF(K,-b)}{d}\right)\lambda.$$

The condition that the solution at $R_A = R_I = 0$ loses its stability is given by $\chi(1) = 0$. By solving this equation with respect to $p$, we finally get the critical ratio as follows:

$$p_c = \frac{F(K,a) - d}{F(K,a) - F(K,-b)}. \quad (35)$$

To verify the theoretical result, we generate a truncated scale-free network by the preferential attachment rule [Barabási and Albert, 1999; Albert and Barabási, 2002], where the minimum degree is nearly a half of the mean degree. Figure 4 shows a phase diagram in the ($K$, $p$)-plane, where the aging transition points obtained by numerical simulations and the aging transition curve represented by the analytical form (35) are plotted. The theoretical result is in good agreement with the numerical result. The critical coupling strength is given as $K_c \sim aN/k_{min}$ where $k_{min} = \mathrm{min_j k_j}$, below which $p_c = 1$. When $K$ approaches $K_c$ from above, $F(K, a)$ becomes large enough to neglect the other terms in Eq. (35) and accordingly $p_c$ approaches 1.

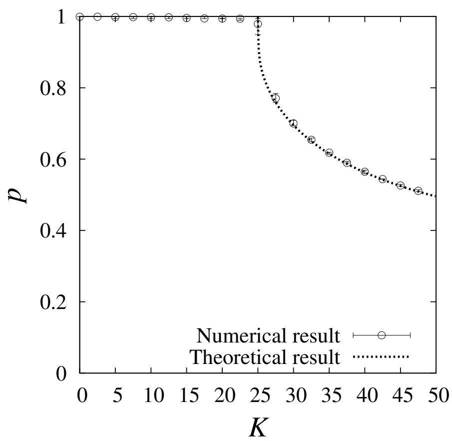


Figure 4. The aging transition points in the Barabási-Albert scale-free networks with $N = 1000$ and $\langle k \rangle = 80$ (i.e. $d = 0.08$). The circle with the error bar indicates the numerical result for 10 network realizations. The dashed line indicates the theoretical result given by Eq. (35).

## 4 Conclusions

We have studied phase transitions in mixed populations of active and inactive oscillators on complex networks. As the ratio of inactive oscillators increases, the order parameter indicating the global activity of the whole network decreases. A phase transition occurs at a critical ratio when the order parameter vanishes. We have theoretically derived the critical ratio using system reduction. For random networks, by regarding the oscillators to be identical in each subpopulation, we have obtained the reduced model which well explains the phase transition. The result has shown that the link density as well as the coupling strength play important roles for the transition. For scale-free networks, the degree-weighted mean fields work very well to approximate the phase transition curve. Currently the scaling property near the transition point is under investigation. The approximation methods that we have introduced would be useful to understand phase transitions, synchronization, and collective behavior in complex networks of coupled oscillators.

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