PARAMETER UNCERTAINTY IN STATE CONSTRAINED OPTIMAL CONTROL PROBLEMS

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Abstract

This work considers state constrained optimal control problems in the presence of parameter uncertainties and provides necessary conditions in the form of a maximum principle. The uncertain parameter, represented by a vector taking values in a given compact set in a metric space, might affect both the objective function and the dynamics. The necessary conditions obtained here are a generalization of the minimax maximum principle derived earlier for optimal control problems in the sense that, now, we consider state constraints. Moreover, our set of necessary conditions is different due to the fact that they are formulated in the Gamkrelidze framework.

Key words

Optimal Control, Parameter Uncertainty, Necessary Conditions, Maximum Principle, State Constraints.

1 Introduction

In this article we study the minimax optimal control problem with state constraints. This research builds on the work [Vinter, 2005], where the author derived a maximum principle of the Pontryagin type for an optimal control problem involving a set of unknown parameters. We borrow most of the methodology from [Vinter, 2005] but with a critical twist which pays a key role in allowing the incorporation of state constraints: we adopt the Gamkrelidze framework. This is instrumental in overcoming the mathematical technicalities inherent to the presence of measures in the multipliers of the optimality conditions that emerge at the points in time in which the constraints are active. The fact that our multipliers do not involve measures also brings significant computational advantages. On the other hand, the extra smoothness required in our framework, also implies that additional information is provided by our conditions.

One of the distinguishing features of the Maximum Principle for the minimax problem is that the maximum condition involving the extended Pontryagin function is stated as a form distributed over the set of unknown parameters.

The investigation of the Necessary Conditions of Optimality for the minimax control problem becomes significantly more complex due to the presence of a degeneracy effect which is related to the state and state endpoint constraint. This phenomenon has been described in detail in [Vinter, 2005] where a number of examples showing that the degeneracy is generally unavoidable for certain types of endpoint constraints, such as, e.g. equality constraints imposed on the both endpoints. Thus, in order to preserve the meaning of the minimax optimal control problem with endpoint constraints consists in consider special classes of sufficiently large endpoint state constraint sets. A key difficulty arising in the proof is due to the fact that the set of normalized Lagrange multipliers becomes non-convex what prevents the application of certain standard technique to find a measurable Lagrange multiplier.

However, the practical motivation is sufficiently intense to feed the substantial effort required to attempt to overcome these these huge challenges. Uncertainty is extremely pervasive when modeling most real-life systems involving either natural processes or human engineered systems. The optimal control community has been devoting a large effort to investigate various kinds of issues concerning the sensitivity of optimality conditions. However, this is not a remedy if the problem at hand that actually has to be solved does not behave well in this respect. Then, we are left with the challenge of deriving optimality conditions which are meaningful for the worst case of the value of the perturbations.

This article is organized as follows. In the next section we state the problem, list the assumptions on its data, and provide some basic definitions. In section 3, we state our necessary conditions of optimality in the form of a maximum principle and provide, not only a discussion specifying a context in the current state-of-the-art but also an outline of the main arguments underlying the proof. In the final section, some conclusions are extracted and prospective challenges are listed.

2 Problem statement

We start by considering the following minimax optimal control problem

$$
\begin{aligned}
(P)\ \text{Minimize}\ & \max_{w\in C}\{g(x(1),w)\} && (1)\\
& \dot{x}(t)=f(x(t),u(t),w),\ \ [0,1]\text{-a.e.} && (2)\\
& x(0)=x_0,\ \ x(1)\in S && (3)\\
& u\in\mathcal{U},\ \text{and} && (4)\\
& h(x(t),w)\leq 0,\ \forall\, t\in[0,1]. && (5)
\end{aligned}
$$

From now on, $\dot{x}=\dfrac{dx}{dt}$, for $t\in[0,1]$, x is the state variable with values in $\mathbb{R}^n$, u is the control functions from a given set $\mathcal{U}:=\{u\in L_\infty([0,1];\mathbb{R}^m): u(t)\in\Omega\}$, where Ω is some given compact set, $w\in\mathbb{R}^l$ is the so-called unknown parameter taking values in a given compact set C, and the set S is compact. The maps $g:\mathbb{R}^n\times\mathbb{R}^l\to\mathbb{R}$, $f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^n$, and $h:\mathbb{R}^n\times\mathbb{R}^l\to\mathbb{R}^k$ are continuous in u, and w, and is Lipschitz continuous and sufficiently smooth in x. The control function is a measurable function $u:[0,1]\to\Omega$. Note that the unknown parameter $w\in C$ does not depend on time.

The control process (x,u) comprises the function u and the family $\{x(t;w)\}_{w\in C}$ of arcs, satisfying for each $w\in C$, the equation

$$\dot{x}(t;w)=f(x(t;w),u(t),w)\ \ \mathcal{L}\text{-a.e. in }[0,1],$$

with $x(0;w)=x_0$.

The control process (x,u) is feasible if the endpoint constraints $x(1;w)\in S$ as well as the state constraints $h(x(t;w),w)\leq 0\ \forall\, t\in[0,1]$ are satisfied for each $w\in C$.

The control process $(\bar{x},\bar{u})$ is said to be optimal if and only if

$$\max_{w\in C}\{g(x(1;w),w)\}\geq\max_{w\in C}\{g(\bar{x}(1;w),w)\}$$

for all feasible processes (x,u).

Thus, we have to consider first the maximization of the cost over the set of all unknown parameters to get the worst case performance for any given control strategy u. Then, we minimize over all control functions u in order to obtain the optimal solution that guarantees a minimum cost no matter how bad is the impact of the worst case value of the unknown parameter w.

3 The Necessary Conditions of Optimality

In this section, we state the Necessary Conditions of Optimality in the form of maximum principle for the above problem. Given not only its apparent complexity, but also the difficulty in ensuring the critical property of nondegeneracy, we will provide the bridges to the simpler cases in which the set C is a finite set of points and, ultimately, to the conventional optimal control problem if C is reduced the a single point.

Our maximum principle is stated in the R.V. Gamkrelidze framework (for details, see Chapter 6 in [Pontryagin, Boltyanskij, Gamkrelidze, Mishchenko, 1962] and also [Gamkrelidze, 1960; Arutyunov, Karamzin, Pereira, 2011]).

Let us define some notation.

Consider the extended Pontryagin (also known by pseudo-Hamiltonian) function

$$
\begin{aligned}
&\bar{H}(x,u,w,\psi,\mu)\\
&\qquad:=\langle\psi,f(x,u,w)\rangle-\langle\mu,\Gamma(x,u,w)\rangle,
\end{aligned}
$$

where $\Gamma(x,u,w):=D_xh(x,w)f(x,u,w)$ is the so-called Gamkrelidze function, denoting D_x the Jacobian of the vector field h with respect to x.

Throughout the article, $N_S(x)$ designates the limiting normal cone to the set S at point x in the sense proposed by B. Mordukhovich, [Mordukhovich, 2006]. To the best of our knowledge, this mathematical object appeared for the first time in [Mordukhovich, 1976]. The limiting subdifferential, defined via N_S, is denoted as usual by the symbol ∂.

The necessity to avoid the degeneracy effect, see [Vinter, 2005], forces us to restrict our consideration to the case when the set S is given by a certain number of smooth functional inequalities, i.e.,

$$S:=\{x\in\mathbb{R}^n: e_j(x)\leq 0, j=1,\ldots,r\}.$$

In this instance, $N_S(\bar{x})$ is the cone defined by all the nonnegative linear combinations of all $\nabla_x e_j(\bar{x})$ - i.e., the gradients of the e_j's at $\bar{x}$ - for which $e_j(\bar{x})=0$.

Theorem 1. *Suppose that (x^*,u^*) is a solution to (P). Then, there exist a Radon probability measure Λ on C and, for each $w\in C$, a set of Lagrange multipliers $\lambda(w)\in[0,1]$, $\zeta(w)\in N_S(x^*(1;w))$, $p(\cdot;w)\in$*

$AC([0,1];\mathbb{R}^n)$, and $\mu(\cdot;w) \in BV([0,1];\mathbb{R}^k)$, which non-trivial, i.e.,

$$\lambda(w) + |\zeta(w)| + \sum_{j=1}^{k} \mu^j(0;w) = 1,\ \Lambda\text{-a.e.},$$

such that

$$-\dot{p}(t;w) = \nabla_x \bar{H}(x^*(t;w), u^*(t), w, p(t;w), \mu(t;w)) \quad \mathcal{L}\text{-a.e. in } [0,1]$$

$$-p(1;w) = \lambda(w)\nabla_x g(x^*(1;w), w) + \zeta(w),\ w \in C$$

$$\max_{u\in\Omega} \int_C \bar{H}(x^*(t;w), u, w, p(t;w), \mu(t;w))d\Lambda = \int_C \bar{H}(x^*(t;w), u^*(t), w, p(t;w), \mu(t;w))d\Lambda \quad \mathcal{L}\text{-a.e. in } [0,1]$$

where the function μ satisfies:

a) the set $\mu^j([a,b];w)$ is a singleton, if $h^j(x^*(t;w),w) < 0\ \forall t \in [a,b]$;
b) each function $\mu^j(\cdot;w)$ is monotonically decreasing; and
c) for each j, $\mu^j(1;w) = 0$,

and the Lagrange multipliers λ, and μ are Λ-measurable, p is Λ-measurable for all $t \in [0,1]$, and μ is Λ-measurable for a.a. $t \in (0,1]$, and for $t = 0$.

We note that this maximum principle degenerates whenever there exists $w \in C$ such that $h^j(x_0, w) = 0$ for some j. Indeed, to see this, we just need to consider the Radon measure $\Lambda = \delta_w$ and the Lagrange multipliers

$$(\lambda(w), \nu(w), p(\cdot;w), \mu(\cdot;w)) = (0, 0, 0, \bar{\mu}(\cdot, w))$$

where $\bar{\mu}^i(\cdot, w) = 0$ if $i \neq j$, and

$$\bar{\mu}^j(t;w) = \begin{cases} 1, & t = 0, \\ 0, & t > 0. \end{cases}$$

Thus, for control processes whose trajectories have a starting point lying on the boundary of the state constraint set, the above results hold trivially true. Moreover, the necessary conditions of optimality may also degenerate at the terminal point unless some compatibility assumptions are imposed, see [Arutyunov, Karamzin, Pereira, 2011]. Then, a natural question arises: how to overcome the degeneracy phenomenon?

Several approaches have been developed in the literature w.r.t. the classic problem statement, that is when C is a singleton, see, e.g., [Arutyunov, Tynyanskij, 1984; Dubovitskij, Dubovitskij, 1985; Ferreira, Vinter, 1994; Arutyunov, Aseev, 1997; Ferreira, Fontes, Vinter, 1999; Arutyunov, 2000; Arutyunov, Karamzin, Pereira, 2005; A. V. Arutyunov and D. Yu. Karamzin, 2015]. The same approaches can be applied to the study of minimax state constrained control problem as well.

Now, we just provide a brief outline of the key ideas underlying the proof of this result. It consists in considering an increasing sequence of finite sets $\{C_N\}$ converging adequately to C, and constructing a sequence of auxiliary optimal control problems $\{(P_N)\}$ with (P_N) similar to (P) but with the set C replaced by C_N. It is shown that, by using standard variational arguments, notably Ekeland's variational principle, [Ekeland, 1974], Theorem 2 below can be applied to the resulting sequence of new auxiliary optimal control problems to provide a sequence of multipliers converging to the ones satisfying the conditions of our result. Further details will be available in [Karamzin, Oliveira, Pereira, Silva, 2015].

Before stating the necessary conditions of optimality for the problem (P_N), let us observe that if $N = 1$, i.e., $C = \{w_0\}$ for some w_0, then our optimal control control problem (P) becomes a standard one. Clearly, in this case, we may simply remove the variable w in the statement of problem (P), and, then, the Necessary Conditions of Optimality turn out to be the already well-known ones, see [Gamkrelidze, 1960; Arutyunov, Karamzin, Pereira, 2011]. Now, note that, for this particular case, the conditions can be improved in that an arbitrary closed endpoint constraint set S can be considered. From now on, we consider (P_N), the instance o problem (P) with, not only C replaced by C_N but also this more general set S. Its conditions can be stated as follows:

Theorem 2. *Let (x^*, u^*) is a solution to (P_N).*
Then, there exists a Radon probability measure $\Lambda(w) = \sum_{i=1}^{N} c_i \delta_{w_i}(w)$ on C, where the N numbers c_i are nonnegative and satisfy $\sum_{i=1}^{N} c_i > 0$, and, for each $w \in C$, a set of Lagrange multipliers $\lambda(w) \in [0,1]$, $p(\cdot;w) \in AC([0,1];\mathbb{R}^n)$, and $\mu(\cdot;w) \in BV([0,1];\mathbb{R}^k)$, which are non-trivial, i.e.,

$$\lambda(w) + \max_{t\in[0,1]} |p(t;w)| + |\mu(0;w)| = 1,\ \Lambda\text{-a.e.},$$

such that the conditions of Theorem 1 are satisfied for all $w \in C_N$.

In order to see that (P_N) can be easily converted into a conventional optimal control problem, i.e., the case $N = 1$, with an nonsmooth cost functional and a state variable with dimension $n \times N$, just consider the following notation:

$$\bar{x} = \text{col}(x^1, x^2, \cdots, x^N),$$

$$\begin{aligned}
&\bar{f}(x,u) = \text{col}(f^1(x^1,u), f^2(x^2,u), \cdots, f^N(x^N,u)),\\
&\bar{g}(\bar{x}(1)) = \max_{i=1,\dots,N} \{g^i(x^i(1))\},\\
&\bar{h}(\bar{x}(\cdot)) = \max_{i=1,\dots,N} h^i(x^i(\cdot)),\\
&\bar{S} = S^N, \text{ and}\\
&\{\bar{x}_0\} = \{x_0\}^N.
\end{aligned}$$

Here, the N power of a set is interpreted as the N times Cartesian product of the set, being, for $i = 1,\dots,N$, $g^i(x) = g(x, w_i)$, $f^i(x,u) = f(x,u,w_i)$, and x^i the solution to $\dot{x}(t) = f^i(x(t), u(t))$ for some $u \in \mathcal{U}$ with $x(0) = x_0$. Remark that this new extended optimal control control exhibits the same structure and is equivalent to (P_N).

For the sake of completeness, we recall the Maximum Principle in the Gamkrelidze framework already derived in [Gamkrelidze, 1960; Arutyunov, Karamzin, Pereira, 2011] for this conventional optimal control problem (P_1)

Theorem 3. *Let (x^*, u^*) be a solution to problem (P_1).*

Then, there exists a set of Lagrange multipliers $\lambda \in [0,1]$, $p \in AC([0,1];\mathbb{R}^n)$, and $\mu \in BV([0,1];\mathbb{R}^k)$, which are non-trivial, i.e.,

$$\lambda + \max_{t\in[0,1]} |p(t)| + |\mu(0)| = 1,$$

such that

$$\begin{aligned}
&-\dot{p}(t) = \nabla_x \bar{H}(x^*(t), u^*(t), p(t), \mu(t))\\
&\qquad\qquad\qquad\qquad \mathcal{L}\text{-a.e. in } [0,1]\\
&-p(1) \in \lambda \nabla_x g(x^*(1)) + N_S(x^*(1)),\\
&\max_{u\in\Omega} \bar{H}(x^*(t), u, p(t), \mu(t)) =\\
&\qquad\qquad \bar{H}(x(t)^*, u(t)^*, p(t), \mu(t)),\\
&\qquad\qquad\qquad\qquad \mathcal{L}\text{-a.e. in } [0,1],
\end{aligned}$$

where the map $\mu : [0,1] \to \mathbb{R}^k$ satisfies the following properties:

a) if $h^j(x^*(t)) < 0\, \forall\, t \in [a,b]$, then the set $\mu^j([a,b])$ is a singleton;
b) each function μ^j is monotonically decreasing; and
c) for each j, $\mu^j(1) = 0$.

Albeit the arguments underlying the proof of this result - based on straightforward application of variational analysis, [Mordukhovich, 2006; Vinter, 2000] - are well known, for the sake of completeness, we will provide an outline here. The proof consists in constructing a sequence of auxiliary optimal control problems whose state constraints and endpoint state constraints are removed by adding to the cost functional an appropriate nonsmooth penalty function, [Arutyunov, Karamzin, 2015; Karamzin, 2002; Arutyunov, Karamzin, Pereira, 2015], and for which there corresponds a sequence of almost minimizers that converge to the solution to the original problem (P). Then, Ekeland's variational principle, [Ekeland, 1974], is applied and we obtain another sequence of optimal control problems whose sequence of solutions and associated set of multipliers converge, respectively, to the solution of (P) and to an associated multiplier that satisfies the maximum principle. Further details will be available in [Karamzin, Oliveira, Pereira, Silva, 2015].

In [Arutyunov, Karamzin, Pereira, 2011; Arutyunov, Karamzin, Pereira, 2011], it is shown that there is a relation between the necessary conditions in Dubovitskii-Milyutin form, [Dubovitskij, Milyutin, 1963] and those in Gamkrelidze form, [Gamkrelidze, 1960]. However, it is important to remark that the necessary conditions of optimality in Theorem 3 are convenient to investigate the continuity of the Lagrange multiplier μ in many regards, such as, e.g., [Arutyunov, 2012; Arutyunov, Karamzin, Pereira, 2014; Arutyunov, 2012; Karamzin, 2007; E.V. Zakharov, D.Yu. Karamzin, 2015; A. V. Arutyunov and D. Yu. Karamzin, 2015]. This type of assumptions are very important in many engineering applications.

We remark that it is a trivial matter to extend the Theorem 1 for the case for which f depends on the time variable t, being this dependence merely measurable, (see the assumptions on the data in [Arutyunov, Karamzin, Pereira, 2011]).

4 Conclusions and future work

In this article the authors present, discuss and outline the proof of necessary conditions of optimality in the form of a Maximum Principle for mini-max optimal control problems, that is, we provide a characterization of the minimum cost of the optimal control problem for the worst case of a certain finite dimensional parameter constrained to a given set. The motivation for this type of results is huge since it provides a kind of robust optimality in very usual engineering and natural system contexts which, typically, are plagued by modeling uncertainty and unmodeled perturbations.

As it was shown in the previous section with a counterexample, the Maximum Principle in Theorem 1 may degenerate. These arguments also show that the same feature is exhibited by the optimality conditions of Theorems 2 and 3. An important line of development is to improve the derived results in order to ensure nondegeneracy. One way to overcome this drawback is to use a method inspired on the one of A.V. Arutyunov in [Arutyunov, 2000], based on a time-transversality condition. The main idea is to assume, for a while, that the time interval $[0,1]$ in problem (P) is not fixed, and now it is $[t_0, 1]$, where the left time-endpoint t_0 is free. In this context, we let the optimal process to be the same, that is (x^*, u^*) defined on $[0,1]$, with $t_0^* = 0$. Then, the necessary conditions of optimality can be supplemented by a time-transversality condition which enables to ensure the nondegeneracy.

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