CASES OF INTEGRABILITY CORRESPONDING TO THE MOTION OF A PENDULUM IN THE THREE-DIMENSIONAL SPACE

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Abstract

We systematize some results on the study of the equations of spatial motion of dynamically symmetric fixed rigid bodies–pendulums located in a nonconservative force fields. The form of these equations is taken from the dynamics of real fixed rigid bodies placed in a homogeneous flow of a medium. In parallel, we study the problem of a spatial motion of a free rigid body also located in a similar force fields. Herewith, this free rigid body is influenced by a nonconservative tracing force; under action of this force, either the magnitude of the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint, or the center of mass of the body moves rectilinearly and uniformly; this means that there exists a nonconservative couple of forces in the system.

Key words

Rigid body, Pendulum, Resisting Medium, Dynamical Systems With Variable Dissipation, Integrability

1 Model assumptions

Let consider the homogeneous plane circle disk $\mathcal{D}$ (with the center in the point D), the plane of which perpendicular to the holder OD. The disk is rigidly fixed perpendicular to the tool holder OD located on the spherical hinge O, and it flows about homogeneous fluid flow. In this case, the body is a physical (spherical) pendulum. The medium flow moves from infinity with constant velocity $\mathbf{v} = \mathbf{v}_\infty \neq \mathbf{0}$. Assume that the holder does not create a resistance.

We suppose that the total force **S** of medium flow interaction is parallel to the holder, and point N of application of this force is determined by at least the angle of attack α, which is made by the velocity vector $\mathbf{v}_D$ of the point D with respect to the flow and the holder OD (Fig. 1); the total force is also determined by the

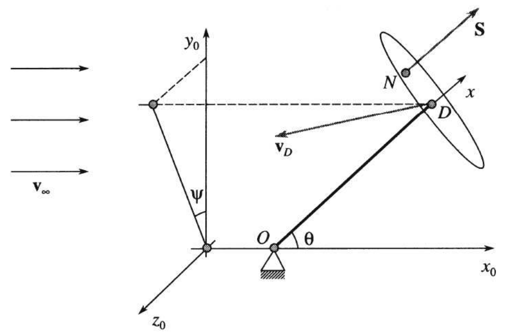

Figure 1. Fixed a pendulum on a spherical hinge in the stream running medium

angle β_1, which is made in the plane of the disk $\mathcal{D}$ (thus, (v, α, β_1) are the spherical coordinates of the tip of the vector $\mathbf{v}_D$), and also the reduced angular velocity $\omega \cong l\Omega/v_D,\ v_D = |\mathbf{v}_D|$ (l is the length of the holder, Ω is the angular velocity of the pendulum). Such conditions arise when one uses the model of streamline flow around spatial bodies [1], [2].

Therefore, the force **S** is directed along the normal to the disk to its side, which is opposite to the direction of the velocity $\mathbf{v}_D$, and passes through a certain point N of the disk such that the velocity vector $\mathbf{v}_D$ and the force of the interaction **S** lie in the plane ODN (see also [2], [3]).

The vector $\mathbf{e} = \mathbf{OD}/l$ determines the orientation of the holder. Then $\mathbf{S} = s(\alpha)v_D^2\mathbf{e}$, where $s(\alpha) = s_1(\alpha)\mathrm{sign}\cos\alpha$, and the resistance coefficient $s_1 \geq 0$ depends only on the angle of attack α. By the axe-symmetry properties of the body–pendulum with respect to the point D, the function $s(\alpha)$ is even.

Let $Dx_1x_2x_3 = Dxyz$ be the coordinate system rigidly attached to the body, herewith, the axis $Dx = Dx_1$ has a direction vector **e**, and the axes $Dx_2 = Dy$ and $Dx_3 = Dz$ lie in the plane of the disk $\mathcal{D}$. In this case, the angle θ is made by the holder and the direction of the over-running medium flow (the axis x_0); and the angle ψ is made by the projection of the holder to the immovable plane y_0z_0 (which perpendicular to the over-running medium flow) and the axis y_0. Obviously, the angles $(\theta, \psi) = (\xi, \eta_1)$ are the spherical coordinates of the point D.

The space of positions of this spherical (physical) pendulum is the two-dimensional sphere

$$\mathbf{S}^2\{(\xi, \eta_1) \in \mathbf{R}^2 :\ 0 \le \xi \le \pi,\ \eta_1 \bmod 2\pi\}, \qquad (1)$$

and its phase space is the tangent bundle of the two-dimensional sphere

$$\begin{gathered} T_*\mathbf{S}^2\{(\dot{\xi}, \dot{\eta_1}; \xi, \eta_1) \in \mathbf{R}^4 : \\ 0 \le \xi \le \pi,\ \eta_1 \bmod 2\pi\}. \end{gathered} \qquad (2)$$

To the angular velocity, we put in correspondence $\Omega = \Omega_1\mathbf{e}_1 + \Omega_2\mathbf{e}_2 + \Omega_3\mathbf{e}_3$ ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the unit vectors of the coordinate system $Dx_1x_2x_3$) the skew-symmetric matrix

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix},\ \tilde{\Omega} \in \mathrm{so}(3).$$

The distance from the center D of the disk to the center of pressure (the point N) has the form

$$|\mathbf{r}_N| = r_N = DN\left(\alpha, \beta_1, \frac{l\Omega}{v_D}\right), \qquad (3)$$

where $\mathbf{r}_N = \{0, x_{2N}, x_{3N}\} = \{0, y_N, z_N\}$ in system $Dx_1x_2x_3 = Dxyz$ (we omit the wave over Ω).

We note, likely in two-dimensional case, that the model used to describe the effects of fluid flow on fixed pendulum is similar to the model constructed for free body and, in further, takes into account of the rotational derivative of the moment of the forces of medium influence with respect to the pendulum angular velocity (see also [3], [4]). An analysis of the problem of the spherical (physical) pendulum in a flow will allow to find the qualitative analogies in the dynamics of partially fixed bodies and free three-dimensional ones.

2 Set of dynamical equations in Lie algebra $\mathrm{so}(3)$

If $\mathrm{diag}\{I_1, I_2, I_2\}$ is the tensor of inertia of the body–pendulum in the coordinate system $Dx_1x_2x_3$ then the general equation of its motion has the following form:

$$\begin{gathered} I_1\dot{\Omega}_1 = 0,\ I_2\dot{\Omega}_2 + (I_1 - I_2)\Omega_1\Omega_3 = \\ = -z_N\left(\alpha, \beta_1, \frac{\Omega}{v_D}\right) s(\alpha)v_D^2, \\ I_2\dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 = \\ = y_N\left(\alpha, \beta_1, \frac{\Omega}{v_D}\right) s(\alpha)v_D^2, \end{gathered} \qquad (4)$$

since the moment of the medium interaction force is determined by the following auxiliary matrix:

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} \\ -s(\alpha)v_D^2 & 0 & 0 \end{pmatrix},$$

where $\{-s(\alpha)v_D^2, 0, 0\}$ is the decomposition of the medium interaction force $\mathbf{S}$ in the coordinate system $Dx_1x_2x_3$.

Since the dimension of the Lie algebra $\mathrm{so}(3)$ is equal to 3, the system of equations (4) is a group of dynamical equations on $\mathrm{so}(3)$, and, simply speaking, the motion equations.

We see, that in the right-hand side of Eq. (4), first of all, it includes the angles α, β_1, therefore, this system of equations is not closed. In order to obtain a complete system of equations of motion of the pendulum, it is necessary to attach several sets of kinematic equations to the dynamic equation on the Lie algebra $\mathrm{so}(3)$.

2.1 Cyclic first integral

We immediately note that the system (4), by the existing dynamic symmetry

$$I_2 = I_3, \qquad (5)$$

possesses the cyclic first integral

$$\Omega_1 \equiv \Omega_1^0 = \text{const}. \qquad (6)$$

In this case, further, we consider the dynamics of our system at zero level:

$$\Omega_1^0 = 0. \qquad (7)$$

Under conditions (5)–(7), the system (4) has the form of unclosed system of two equations:

$$\begin{gathered} I_2\dot{\Omega}_2 = -z_N\left(\alpha, \beta_1, \frac{\Omega}{v_D}\right) s(\alpha)v_D^2, \\ I_2\dot{\Omega}_3 = y_N\left(\alpha, \beta_1, \frac{\Omega}{v_D}\right) s(\alpha)v_D^2. \end{gathered} \qquad (8)$$

3 First set of kinematic equations

In order to obtain a complete system of equations of motion, it needs the set of kinematic equations which relate the velocities of the point D (i.e., the formal center of the disk $\mathcal{D}$) and the over-running medium flow:

$$\begin{gathered} \mathbf{v}_D = v_D \cdot \mathbf{i}_v(\alpha, \beta_1) = \\ = \tilde{\Omega}\begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} + (-v_\infty)\mathbf{i}_v(-\xi, \eta_1), \end{gathered} \qquad (9)$$

where

$$\mathbf{i}_v(\alpha, \beta_1) = \begin{pmatrix} \cos\alpha \\ \sin\alpha\cos\beta_1 \\ \sin\alpha\sin\beta_1 \end{pmatrix}. \tag{10}$$

The equation (9) expresses the theorem of addition of velocities in projections on the related coordinate system $Dx_1x_2x_3$.

Indeed, the left-hand side of Eq. (9) is the velocity of the point D of the pendulum with respect to the flow in the projections on the related with the pendulum coordinate system $Dx_1x_2x_3$. Herewith, the vector $\mathbf{i}_v(\alpha, \beta_1)$ is the unit vector along the axis of the vector $\mathbf{v}_D$. The vector $\mathbf{i}_v(\alpha, \beta_1)$ has the spherical coordinates $(1, \alpha, \beta_1)$, which determines the decomposition (10).

The right-hand side of the Eq. (9) is the sum of the velocities of the point D when you rotate the pendulum (the first term), and the motion of the flow (the second term). In this case, in the first term, we have the coordinates of the vector $\mathbf{OD} = \{l, 0, 0\}$ in the coordinate system $Dx_1x_2x_3$.

We explain the second term of the right-hand side of Eq. (9) in more detail. We have in it the coordinates of the vector $(-\mathbf{v}_\infty) = \{-v_\infty, 0, 0\}$ in the immovable space. In order to describe it in the projections on the related coordinate system $Dx_1x_2x_3$, we need to make a (reverse) rotation of the pendulum at the angle $(-\xi)$ that is algebraically equivalent to multiplying the value $(-v_\infty)$ on the vector $\mathbf{i}_v(-\xi, \eta_1)$.

Thus, the first set of kinematic equations (9) has the following form in our case:

$$\begin{gathered} v_D\cos\alpha = -v_\infty\cos\xi, \\ v_D\sin\alpha\cos\beta_1 = l\Omega_3 + v_\infty\sin\xi\cos\eta_1, \\ v_D\sin\alpha\sin\beta_1 = -l\Omega_2 + v_\infty\sin\xi\sin\eta_1. \end{gathered} \tag{11}$$

4 Second set of kinematic equations

We also need a set of kinematic equations which relate the angular velocity tensor $\tilde{\Omega}$ and coordinates $\dot{\xi}, \dot{\eta}_1, \xi, \eta_1$ of the phase space (2) of pendulum studied, i.e., the tangent bundle $T_*\mathbf{S}^2\{\dot{\xi}, \dot{\eta}_1; \xi, \eta_1\}$.

We draw the reasoning style allowing arbitrary dimension. The desired equations are obtained from the following two sets of relations. Since the motion of the body takes place in a Euclidean space $\mathbf{E}^n, n = 3$ formally, at the beginning, we express the tuple consisting of a phase variables Ω_2, Ω_3, through new variable z_1, z_2 (from the tuple z). For this, we draw the following turn by the angle η_1:

$$\begin{pmatrix} \Omega_2 \\ \Omega_3 \end{pmatrix} = T_{1,2}(\eta_1)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{12}$$

where

$$T_{1,2}(\eta_1) = \begin{pmatrix} \cos\eta_1 & -\sin\eta_1 \\ \sin\eta_1 & \cos\eta_1 \end{pmatrix}.$$

In other words, the relations

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = T_{1,2}(-\eta_1)\begin{pmatrix} \Omega_2 \\ \Omega_3 \end{pmatrix}$$

hold, i.e.,

$$\begin{gathered} z_1 = \Omega_2\cos\eta_1 + \Omega_3\sin\eta_2, \\ z_2 = -\Omega_2\sin\eta_1 + \Omega_3\cos\eta_2. \end{gathered}$$

Then we substitute the following relationship instead of the variable z:

$$z_2 = \dot{\xi},\ z_1 = -\dot{\eta}_1\tfrac{\sin\xi}{\cos\xi}. \tag{13}$$

Thus, two sets of Eqs. (12) and (13) give the second set of kinematic equations:

$$\begin{gathered} \Omega_2 = -\dot{\xi}\sin\eta_1 - \dot{\eta}_1\tfrac{\sin\xi}{\cos\xi}\cos\eta_1, \\ \Omega_3 = \dot{\xi}\cos\eta_1 - \dot{\eta}_1\tfrac{\sin\xi}{\cos\xi}\sin\eta_1. \end{gathered} \tag{14}$$

We see that three sets of the relations (8), (11), and (14) form the closed system of equations.

These three sets of equations include the following functions:

$$y_N\left(\alpha, \beta_1, \frac{\Omega}{v_D}\right),\ z_N\left(\alpha, \beta_1, \frac{\Omega}{v_D}\right),\ s(\alpha).$$

In this case, the function s is considered to be dependent only on α, and the functions y_N, z_N may depend on, along with the angles α, β_1, generally speaking, the reduced angular velocity $\omega \cong l\Omega/v_D$.

5 Problem on free body motion under assumption of tracing force

Parallel to the present problem of the motion of the fixed body, we study the spatial motion of the free axially symmetric rigid body with the frontal plane buttend (the circle disk $\mathcal{D}$) in the resistance force fields under the quasi-stationarity conditions [4], [5] with the same model of medium interaction.

If (v, α, β_1) are the spherical coordinates of the velocity vector of the center D of disk $\mathcal{D}$ lying on the axis of symmetry of a body, $\Omega = \{\Omega_1, \Omega_2, \Omega_3\}$ are the projections of its angular velocity on the axes of the coordinate system $Dx_1x_2x_3$ related to the body (in this case, the axis of symmetry CD coincides with the axis $Dx_1 = Dx$, C is the center of mass), and the axes $Dx_2 = Dy$ and $Dx_3 = Dz$ lie in the hyperplane of the disk; $I_1,\ I_2,\ I_3 = I_2,\ m$ are characteristics of inertia and mass, then the dynamical part of the equations of motion in which the tangent forces of the interaction of the body with the medium are absent, has the form

$$\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha + \Omega_2 v\sin\alpha\sin\beta_1 -$$

$$-\Omega_3 v\sin\alpha\cos\beta_1 + \sigma(\Omega_2^2+\Omega_3^2) = \frac{F_x}{m},$$

$$\dot{v}\sin\alpha\cos\beta_1 + \dot{\alpha}v\cos\alpha\cos\beta_1 -$$

$$-\dot{\beta}_1 v\sin\alpha\sin\beta_1 + \Omega_3 v\cos\alpha -$$

$$-\Omega_1 v\sin\alpha\sin\beta_1 - \sigma\Omega_1\Omega_2 - \sigma\dot{\Omega}_3 = 0,$$

$$\dot{v}\sin\alpha\sin\beta_1 + \dot{\alpha}v\cos\alpha\sin\beta_1 +$$

$$+\dot{\beta}_1 v\sin\alpha\cos\beta_1 + \Omega_1 v\sin\alpha\cos\beta_1 - \qquad (15)$$

$$-\Omega_2 v\cos\alpha - \sigma\Omega_1\Omega_3 + \sigma\dot{\Omega}_2 = 0,$$

$$I_1\dot{\Omega}_1 = 0,$$

$$I_2\dot{\Omega}_2 + (I_1 - I_2)\Omega_1\Omega_3 = -z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha)v^2,$$

$$I_2\dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 = y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha)v^2,$$

where $F_x = -S$, $S = s(\alpha)v^2$, $\sigma = CD$, in this case $(0, y_N(\alpha,\beta_1,\Omega/v), z_N(\alpha,\beta_1,\Omega/v))$ are the coordinates of the point N of application of the force **S** in the coordinate system $Dx_1x_2x_3 = Dxyz$ related to the body.

The first part of three equations of the system (15) describe the motion of the center of a mass in the three-dimensional Euclidean space $\mathbf{E}^3$ in the projections on the coordinate system $Dx_1x_2x_3$. And the second part of three equation of the system (15) is obtained from the theorem on the change of the angular moment of a rigid body in the König axis.

Thus, the direct product $\mathbf{R}^1\times\mathbf{S}^2\times\mathrm{so}(3)$ of the three-dimensional manifold and the Lie algebra $\mathrm{so}(3)$ is the phase space of sixth-order system (15) of the dynamical equations. Herewith, since the medium influence force dos not depend on the position of the body in a plane, the system (15) of the dynamical equations *is separated from the system of kinematic equations* and may be studied independently (see also [4], [6]).

5.1 Cyclic first integral

We immediately note that the system (15), by the existing dynamic symmetry

$$I_2 = I_3, \qquad (16)$$

possesses the cyclic first integral

$$\Omega_1 \equiv \Omega_1^0 = \text{const}. \qquad (17)$$

In this case, further, we consider the dynamics of our system at zero level:

$$\Omega_1^0 = 0. \qquad (18)$$

5.2 Nonintegrable constraint

If we consider *a more general problem* on the motion of a body under the action of a certain tracing force **T** passing through the center of mass and providing the fulfillment of the equality

$$v \equiv \text{const}, \qquad (19)$$

during the motion (see also [7], [8]), then F_x in system (15) must be replaced by $T - s(\alpha)v^2$.

As a result of an appropriate choice of the magnitude T of the tracing force, we can achieve the fulfillment of Eq. (19) during the motion. Indeed, if we formally express the value T by virtue of system (15), we obtain (for $\cos\alpha \neq 0$):

$$T = T_v(\alpha,\beta_1,\Omega) = m\sigma(\Omega_2^2+\Omega_3^2) +$$

$$+s(\alpha)v^2\left[1 - \frac{m\sigma}{I_2}\frac{\sin\alpha}{\cos\alpha}\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1 +\right.\right.$$

$$\left.\left.+y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1\right]\right]. \qquad (20)$$

This procedure can be viewed from two standpoints. First, a transformation of the system has occurred at the presence of the tracing (control) force in the system which provides the corresponding class of motions (19). Second, we can consider this procedure as a procedure that allows one to reduce the order of the system. Indeed, system (15) generates an independent fourth-order system of the following form:

$$\begin{gathered}
\dot{\alpha}v\cos\alpha\cos\beta_1 - \dot{\beta}_1 v\sin\alpha\sin\beta_1 + \\
+\Omega_3 v\cos\alpha - \sigma\dot{\Omega}_3 = 0, \\
\dot{\alpha}v\cos\alpha\sin\beta_1 + \dot{\beta}_1 v\sin\alpha\cos\beta_1 - \\
-\Omega_2 v\cos\alpha + \sigma\dot{\Omega}_2 = 0, \\
I_2\dot{\Omega}_2 = -z_N\left(\alpha,\beta_1,\tfrac{\Omega}{v}\right)s(\alpha)v^2, \\
I_2\dot{\Omega}_3 = y_N\left(\alpha,\beta_1,\tfrac{\Omega}{v}\right)s(\alpha)v^2,
\end{gathered} \qquad (21)$$

where the parameter v is supplemented by the constant parameters specified above.

The system (21) is equivalent to the system

$$\begin{gathered}
\dot\alpha v\cos\alpha+\\
+v\cos\alpha\left[\Omega_3\cos\beta_1-\Omega_2\sin\beta_1\right]+\\
+\sigma\left[-\dot\Omega_3\cos\beta_1+\dot\Omega_2\sin\beta_1\right]=0,\\
\dot\beta_1 v\sin\alpha-\\
-v\cos\alpha\left[\Omega_2\cos\beta_1+\Omega_3\sin\beta_1\right]+\\
+\sigma\left[\dot\Omega_2\cos\beta_1+\dot\Omega_3\sin\beta_1\right]=0,\\
\dot\Omega_2=-\frac{v^2}{I_2}z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha),\\
\dot\Omega_3=\frac{v^2}{I_2}y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha).
\end{gathered}\tag{22}$$

We introduce new quasi-velocities in our system:

$$\begin{pmatrix}\Omega_2\\ \Omega_3\end{pmatrix}=T_{1,2}(\beta_1)\begin{pmatrix}z_1\\ z_2\end{pmatrix},\tag{23}$$

where

$$T_{1,2}(\beta_1)=\begin{pmatrix}\cos\beta_1 & -\sin\beta_1\\ \sin\beta_1 & \cos\beta_1\end{pmatrix}.$$

In other words, the following relations

$$\begin{pmatrix}z_1\\ z_2\end{pmatrix}=T_{1,2}(-\beta_1)\begin{pmatrix}\Omega_2\\ \Omega_3\end{pmatrix}\tag{24}$$

hold, i.e.,

$$\begin{gathered}
z_1=\Omega_2\cos\beta_1+\Omega_3\sin\beta_2,\\
z_2=-\Omega_2\sin\beta_1+\Omega_3\cos\beta_2.
\end{gathered}\tag{25}$$

We can see from (22) that the system cannot be solved uniquely with respect to $\dot\alpha,\ \dot\beta_1$ on the manifold

$$O=\Big\{(\alpha,\beta_1,\Omega_2,\Omega_3)\in\mathbf{R}^4:$$

$$\alpha=\frac{\pi}{2}k,\ k\in\mathbf{Z}\Big\}.\tag{26}$$

Thus, formally speaking, the uniqueness theorem is violated on manifold (26). Moreover, the indefiniteness occurs for even k because of the degeneration of the spherical coordinates (v,α,β_1), and an obvious violation of the uniqueness theorem for odd k occurs since the first equation of (22) is degenerate for this case.

This implies that system (21) outside of the manifold (26) (and only outside it) is equivalent to the following system:

$$\begin{gathered}
\dot\alpha=-z_2+\frac{\sigma v}{I_2}\frac{s(\alpha)}{\cos\alpha}\times\\
\times\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1+\right.\\
\left.+y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1\right],\\
\dot z_2=\frac{v^2}{I_2}s(\alpha)\times\\
\times\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1+\right.\\
\left.+y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1\right]-\\
-z_1^2\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_2}\frac{s(\alpha)}{\sin\alpha}z_1\times\\
\times\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1-\right.\\
\left.-y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right],\\
\dot z_1=z_1z_2\frac{\cos\alpha}{\sin\alpha}+\left[-\frac{v^2}{I_2}s(\alpha)+\frac{\sigma v}{I_2}\frac{s(\alpha)}{\sin\alpha}z_2\right]\times\\
\times\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1-\right.\\
\left.-y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right],\\
\dot\beta_1=z_1\frac{\cos\alpha}{\sin\alpha}+\frac{\sigma v}{I_2}\frac{s(\alpha)}{\sin\alpha}\times\\
\times\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1-\right.\\
\left.-y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right].
\end{gathered}\tag{27}$$

In the sequel, the dependence on the variables $(\alpha,\beta_1,\Omega/v)$ must be treated as the composite dependence on $(\alpha,\beta_1,z_1/v,z_2/v)$ by virtue of (25).

The uniqueness theorem is violated for system (22) on the manifold (26) for odd k in the following sense: regular phase trajectories of system (27) pass through almost all points of the manifold (26) for odd k and intersect the manifold (26) at a right angle, and also there exists a phase trajectory that completely coincides with the specified point at all time instants. However, these trajectories are different since they correspond to different values of the tracing force.

5.3 Constant velocity of the center of mass

If we consider *a more general problem* on the motion of a body under the action of a certain tracing force $\mathbf{T}$ passing through the center of mass and providing the fulfillment of the equality

$$\mathbf{V}_C\equiv\mathbf{const}\tag{28}$$

during the motion ($\mathbf{V}_C$ is the velocity of the center of mass), then F_x in system (15) must be replaced by zero since the nonconservative couple of the forces acts on the body: $T-s(\alpha)v^2\equiv 0$.

Obviously, we must choose the value of the tracing force T as follows:

$$T=T_v(\alpha,\beta_1,\Omega)=s(\alpha)v^2,\ \mathbf{T}\equiv-\mathbf{S}.\tag{29}$$

The choice (29) of the magnitude of the tracing force T is a particular case of the possibility of separation of an independent fourth-order subsystem after a certain transformation of the system (15).

Indeed, let the following condition hold for T:

$$T=T_v(\alpha,\beta_1,\Omega)=$$

$$= \sum_{i,j=0,\ i\leq j}^{3} \tau_{i,j}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\Omega_i\Omega_j = \tag{30}$$

$$= T_1\left(\alpha,\beta_1,\frac{\Omega}{v}\right)v^2,\ \Omega_0 = v.$$

At the beginning, we introduce new quasi-velocities (23)–(25).

We rewrite the system (15) for the cases (16)–(18) in the form

$$\dot{v} + \sigma(z_1^2+z_2^2)\cos\alpha -$$

$$-\sigma\frac{v^2}{I_2}s(\alpha)\sin\alpha\left[y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 +\right.$$

$$\left.+z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right] =$$

$$= \frac{T_1\left(\alpha,\beta_1,\frac{\Omega}{v}\right)v^2 - s(\alpha)v^2}{m}\cos\alpha,$$

$$\dot{\alpha}v + z_2 v - \sigma(z_1^2+z_2^2)\sin\alpha -$$

$$-\sigma\frac{v^2}{I_2}s(\alpha)\cos\alpha\left[y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 +\right.$$

$$\left.z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right] = \tag{31}$$

$$= \frac{s(\alpha)v^2 - T_1\left(\alpha,\beta_1,\frac{\Omega}{v}\right)v^2}{m}\sin\alpha,$$

$$\dot{\Omega}_3 = \frac{v^2}{I_2}y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha),$$

$$\dot{\Omega}_2 = -\frac{v^2}{I_2}z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha),$$

$$\dot{\beta}_1\sin\alpha - z_1\cos\alpha -$$

$$-\frac{\sigma v}{I_2}s(\alpha)\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 -\right.$$

$$\left.y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right] = 0.$$

If we introduce the new dimensionless phase variables and the differentiation by the formulas $z_k = n_1 v Z_k,\ k = 1,2,\ <\cdot>= n_1 v <'>,\ n_1 > 0,\ n_1 =$ const, system (31) has the following form:

$$v' = v\Psi(\alpha,\beta_1,Z_1,Z_2), \tag{32}$$

$$\alpha' = -Z_2 + \sigma n_1(Z_1^2+Z_2^2)\sin\alpha +$$

$$+\frac{\sigma}{I_2 n_1}s(\alpha)\cos\alpha\left[y_N\left(\alpha,\beta_1,n_1 Z\right)\cos\beta_1 +\right.$$

$$\left.+z_N\left(\alpha,\beta_1,n_1 Z\right)\sin\beta_1\right] -$$

$$-\frac{T_1\left(\alpha,\beta_1,n_1 Z\right) - s(\alpha)}{mn_1}\sin\alpha, \tag{33}$$

$$Z_2' = \frac{s(\alpha)}{I_2 n_1^2}\left[1-\sigma n_1 Z_2\sin\alpha\right]\left[y_N\left(\alpha,\beta_1,n_1 Z\right)\cos\beta_1 +\right.$$

$$\left.+z_N\left(\alpha,\beta_1,n_1 Z\right)\sin\beta_1\right] -$$

$$-Z_1^2\frac{\cos\alpha}{\sin\alpha} + \sigma n_1 Z_2(Z_1^2+Z_2^2)\cos\alpha -$$

$$-\frac{\sigma}{I_2 n_1}Z_1\frac{s(\alpha)}{\sin\alpha}\left[z_N\left(\alpha,\beta_1,n_1 Z\right)\cos\beta_1 -\right.$$

$$\left.-y_N\left(\alpha,\beta_1,n_1 Z\right)\sin\beta_1\right] -$$

$$-Z_2 \frac{T_1(\alpha, \beta_1, n_1 Z) - s(\alpha)}{m n_1} \cos\alpha, \qquad (34)$$

$$Z_1' = \frac{1}{I_2 n_1^2} \frac{s(\alpha)}{\sin\alpha} [\sigma n_1 Z_2 \sin\alpha - 1] \times$$

$$\times [z_N(\alpha, \beta_1, n_1 Z) \cos\beta_1 - y_N(\alpha, \beta_1, n_1 Z) \sin\beta_1] +$$

$$+ Z_1 Z_2 \frac{\cos\alpha}{\sin\alpha} + \sigma n_1 Z_1 (Z_1^2 + Z_2^2) \cos\alpha -$$

$$- \frac{\sigma}{I_2 n_1} Z_1 s(\alpha) \sin\alpha \, [z_N(\alpha, \beta_1, n_1 Z) \sin\beta_1 +$$

$$+ y_N(\alpha, \beta_1, n_1 Z) \cos\beta_1] -$$

$$-Z_1 \frac{T_1(\alpha, \beta_1, n_1 Z) - s(\alpha)}{m n_1} \cos\alpha, \qquad (35)$$

$$\beta_1' = Z_1 \frac{\cos\alpha}{\sin\alpha} +$$

$$+ \frac{\sigma}{I_2 n_1} \frac{s(\alpha)}{\sin\alpha} [z_N(\alpha, \beta_1, n_1 Z) \cos\beta_1 -$$

$$- y_N(\alpha, \beta_1, n_1 Z) \sin\beta_1], \qquad (36)$$

$$\Psi(\alpha, \beta_1, Z_1, Z_2) = -\sigma n_1 (Z_1^2 + Z_2^2) \cos\alpha +$$

$$+ \frac{\sigma}{I_2 n_1} s(\alpha) \sin\alpha \, [y_N(\alpha, \beta_1, n_1 Z) \cos\beta_1 +$$

$$+ z_N(\alpha, \beta_1, n_1 Z) \sin\beta_1] +$$

$$+ \frac{T_1(\alpha, \beta_1, n_1 Z) - s(\alpha)}{m n_1} \cos\alpha.$$

We see that the independent fourth-order subsystem (33)–(36) can be substituted into the fifth-order system (32)–(36) and can be considered separately on its own four-dimensional phase space.

In particular, if condition (29) holds, then the method of separation of an independent fourth-order subsystem is also applicable.

6 Case where the moment of nonconservative forces is independent of the angular velocity

We take the function $\mathbf{r}_N$ as follows (the disk $\mathcal{D}$ is given by the equation $x_{1N} \equiv 0$):

$$\mathbf{r}_N = \begin{pmatrix} 0 \\ x_{2N} \\ x_{3N} \end{pmatrix} = R(\alpha)\mathbf{i}_N, \qquad (37)$$

were

$$\mathbf{i}_N = \mathbf{i}_v \left(\frac{\pi}{2}, \beta_1\right)$$

(see (10)).

In our case

$$\mathbf{i}_N = \begin{pmatrix} 0 \\ \cos\beta_1 \\ \sin\beta_1 \end{pmatrix}.$$

Thus, the equalities $x_{2N} = R(\alpha)\cos\beta_1$, $x_{3N} = R(\alpha)\sin\beta_1$ hold and show that for the considered system, the moment of the nonconservative forces is independent of the angular velocity (it depends only on the angles α, β_1).

And so, for the construction of the force field, we use the pair of dynamical functions $R(\alpha), s(\alpha)$; the information about them is of a qualitative nature. Similarly to the choice of the Chaplygin analytical functions (see [1], [2]), we take the dynamical functions s and R as follows:

$$R(\alpha) = A\sin\alpha, \ s(\alpha) = B\cos\alpha, \ A, B > 0. \qquad (38)$$

6.1 Reduced systems

Theorem 1. *The simultaneous equations (4), (11), (14), under conditions (5)–(7), (37), (38) can be reduced to the dynamical system on the tangent bundle (2) of the two-dimensional sphere (1).*

Indeed, if we introduce the dimensionless parameter and the differentiation by the formulas

$$b_* = l n_0, \ n_0^2 = \frac{AB}{I_2}, \ < \cdot > = n_0 v_\infty <'>, \qquad (39)$$

then the obtained equations have the following form ($b_* > 0$):

$$\begin{gathered} \xi'' + b_* \xi' \cos\xi + \sin\xi\cos\xi - \eta_1'^2 \frac{\sin\xi}{\cos\xi} = 0, \\ \eta_1'' + b_* \eta_1' \cos\xi + \xi' \eta_1' \frac{1+\cos^2\xi}{\cos\xi\sin\xi} = 0. \end{gathered} \qquad (40)$$

After the transition from the variables z (about the variables z see (13)) to the variables w

$$w_2 = -\frac{1}{n_0 v_\infty} z_2 - b_* \sin\xi, \ w_1 = -\frac{1}{n_0 v_\infty} z_1, \qquad (41)$$

system (40) is equivalent to the system

$$\left.\begin{array}{c}\xi' = -w_2 - b_* \sin\xi, \\ w_2' = \sin\xi\cos\xi - w_1^2 \frac{\cos\xi}{\sin\xi}, \\ w_1' = w_1 w_2 \frac{\cos\xi}{\sin\xi},\end{array}\right\} \tag{42}$$

$$\eta_1' = w_1 \frac{\cos\xi}{\sin\xi}, \tag{43}$$

on the tangent bundle

$$T_*\mathbf{S}^2\{(w_2, w_1; \xi, \eta_1) \in \mathbf{R}^4 :$$

$$0 \le \xi \le \pi,\ \eta_1 \bmod 2\pi\} \tag{44}$$

of the two-dimensional sphere $\mathbf{S}^2\{(\xi, \eta_1) \in \mathbf{R}^2 :\ 0 \le \xi \le \pi,\ \eta_1 \bmod 2\pi\}$.

We see that the independent third-order subsystem (42) (due to cyclicity of the variable η_1) can be substituted into the fourth-order system (42), (43) and can be considered separately on its own three-dimensional manifold.

6.2 Complete list of the first integrals

We turn now to the integration of the desired fourth-order system (42), (43) (without any simplifications, i.e., in the presence of all coefficients).

First, we compare the third-order system (42) with the nonautonomous second-order system

$$\begin{array}{c}\frac{dw_2}{d\xi} = \frac{\sin\xi\cos\xi - w_1^2\cos\xi/\sin\xi}{-w_2 - b_*\sin\xi}, \\ \frac{dw_1}{d\xi} = \frac{w_1 w_2 \cos\xi/\sin\xi}{-w_2 - b_*\sin\xi}.\end{array} \tag{45}$$

Using the substitution $\tau = \sin\xi$, we rewrite system (45) in the algebraic form:

$$\begin{array}{c}\frac{dw_2}{d\tau} = \frac{\tau - w_1^2/\tau}{-w_2 - b_*\tau}, \\ \frac{dw_1}{d\tau} = \frac{w_1 w_2/\tau}{-w_2 - b_*\tau}.\end{array} \tag{46}$$

Further, if we introduce the uniform variables by the formulas $w_k = u_k\tau,\ k = 1, 2$, we reduce system (46) to the following form:

$$\tau\frac{du_2}{d\tau} = \frac{1 - u_1^2 + u_2^2 - bu_2}{-u_2 - b_*},\ \tau\frac{du_1}{d\tau} = \frac{2u_1u_2 - bu_1}{-u_2 - b_*}. \tag{47}$$

We compare the second-order system (47) with the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{1 - u_1^2 + u_2^2 + b_*u_2}{2u_1u_2 + b_*u_1}, \tag{48}$$

which can be easily reduced to the exact differential equation

$$d\left(\frac{u_2^2 + u_1^2 + b_*u_2 + 1}{u_1}\right) = 0.$$

Therefore, Eq. (48) has the first integral

$$\frac{u_2^2 + u_1^2 + b_*u_2 + 1}{u_1} = C_1 = \text{const}, \tag{49}$$

which in the old variables has the form

$$\Theta_1(w_2, w_1; \xi) =$$

$$= \frac{w_2^2 + w_1^2 + b_*w_2\sin\xi + \sin^2\xi}{w_1\sin\xi} = C_1 = \tag{50}$$

$$= \text{const}.$$

Remark 2. *We consider system (42) with variable dissipation with zero mean (see [6], [7]), which becomes conservative for $b_* = 0$:*

$$\begin{array}{c}\xi' = -w_2, \\ w_2' = \sin\xi\cos\xi - w_1^2\frac{\cos\xi}{\sin\xi}, \\ w_1' = w_1 w_2\frac{\cos\xi}{\sin\xi}.\end{array} \tag{51}$$

It has two analytical first integrals of the form

$$w_2^2 + w_1^2 + \sin^2\xi = C_1^* = \textit{const}, \tag{52}$$

$$w_1\sin\xi = C_2^* = \textit{const}. \tag{53}$$

It is obvious that the ratio of the first integrals (52) and (53) is also a first integral of system (51). However, for $b_* \neq 0$, both functions

$$w_2^2 + w_1^2 + b_*w_2\sin\xi + \sin^2\xi \tag{54}$$

and (53) are not first integrals of system (42), but their ratio (i.e., the ratio of the functions (54) and (53)) is a first integral of system (42) for any b_*.

Later on, we find the obvious form of the additional first integral of the third-order system (42). For this, at the beginning, we transform the invariant relation (49) for $u_1 \neq 0$ as follows:

$$\left(u_2 + \frac{b_*}{2}\right)^2 + \left(u_1 - \frac{C_1}{2}\right)^2 = \frac{b_*^2 + C_1^2}{4} - 1. \tag{55}$$

We see that the parameters of the given invariant relation must satisfy the condition

$$b_*^2 + C_1^2 - 4 \geq 0, \tag{56}$$

and the phase space of system (42) is stratified into a family of surfaces defined by Eq. (55).

Thus, by virtue of relation (49), the first equation of system (47) has the form

$$\tau \frac{du_2}{d\tau} = \frac{2(1 + b_* u_2 + u_2^2) - C_1 U_1(C_1, u_2)}{-u_2 - b_*},$$

$$U_1(C_1, u_2) = \frac{1}{2}\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 + b_* u_2 + 1)}\},$$

and the integration constant C_1 is chosen from condition (56).

Therefore, the quadrature for the search of an additional first integral of system (42) has the form

$$\int \frac{d\tau}{\tau} = \int \frac{(-b_* - u_2)du_2}{A}, \tag{57}$$

$$A = 2(1 + b_* u_2 + u_2^2) -$$

$$-C_1\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 + b_* u_2 + 1)}\}/2.$$

Obviously, the left-hand side up to an additive constant is equal to $\ln|\sin\xi|$. If

$$u_2 + \frac{b_*}{2} = r_1, \; b_1^2 = b_*^2 + C_1^2 - 4,$$

then the right-hand side of Eq. (57) has the form

$$-\frac{1}{4}\int \frac{d(b_1^2 - 4r_1^2)}{(b_1^2 - 4r_1^2) \pm C_1\sqrt{b_1^2 - 4r_1^2}} +$$

$$+b_* \int \frac{dr_1}{(b_1^2 - 4r_1^2) \pm C_1\sqrt{b_1^2 - 4r_1^2}} =$$

$$= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2 - 4r_1^2}}{C_1} \pm 1\right| \pm \frac{b_*}{2} I_1, \tag{58}$$

where

$$I_1 = \int \frac{dr_3}{\sqrt{b_1^2 - r_3^2}(r_3 \pm C_1)}, \tag{59}$$

$$r_3 = \sqrt{b_1^2 - 4r_1^2}.$$

In the calculation of integral (59), the following three cases are possible.

I. $b_* > 2$.

$$I_1 = -\frac{1}{2\sqrt{b_*^2 - 4}} \times$$

$$\times \ln\left|\frac{\sqrt{b_*^2 - 4} + \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \pm \frac{C_1}{\sqrt{b_*^2 - 4}}\right| +$$

$$+\frac{1}{2\sqrt{b_*^2 - 4}}\ln\left|\frac{\sqrt{b_*^2 - 4} - \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \mp \frac{C_1}{\sqrt{b_*^2 - 4}}\right| + \tag{60}$$

$$+\text{const}.$$

II. $b_* < 2$.

$$I_1 = \frac{1}{\sqrt{4 - b_*^2}} \arcsin \frac{\pm C_1 r_3 + b_1^2}{b_1(r_3 \pm C_1)} + \text{const}. \tag{61}$$

III. $b_* = 2$.

$$I_1 = \mp \frac{\sqrt{b_1^2 - r_3^2}}{C_1(r_3 \pm C_1)} + \text{const}. \tag{62}$$

When we return to the variable

$$r_1 = \frac{w_2}{\sin\xi} + \frac{b_*}{2}, \tag{63}$$

we obtain the final form for the value I_1:

I. $b_* > 2$.

$$I_1 =$$

$$= -\frac{1}{2\sqrt{b_*^2 - 4}}\ln\left|\frac{\sqrt{b_*^2 - 4} \pm 2r_1}{\sqrt{b_1^2 - 4r_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{b_*^2 - 4}}\right| +$$

$$+\frac{1}{2\sqrt{b_*^2-4}}\ln\left|\frac{\sqrt{b_*^2-4}\mp 2r_1}{\sqrt{b_1^2-4r_1^2}\pm C_1}\mp\frac{C_1}{\sqrt{b_*^2-4}}\right|+ \quad (64)$$

$$+\text{const.}$$

II. $b_* < 2$.

$$I_1=\frac{1}{\sqrt{4-b_*^2}}\arcsin\frac{\pm C_1\sqrt{b_1^2-4r_1^2}+b_1^2}{b_1(\sqrt{b_1^2-4r_1^2}\pm C_1)}+ \quad (65)$$

$$+\text{const.}$$

III. $b_* = 2$.

$$I_1=\mp\frac{2r_1}{C_1(\sqrt{b_1^2-4r_1^2}\pm C_1)}+\text{const.} \quad (66)$$

Thus, we have found an additional first integral for the third-order system (42), i.e., we have a complete set of first integrals that are transcendental functions of the phase variables.

Remark 3. *In the expression of the found first integral, we must formally substitute the left-hand side of the first integral (49) instead of* C_1.

Then the obtained additional first integral has the following structure similar to the transcendental first integral from the planar dynamics):

$$\Theta_2(w_2,w_1;\xi)=$$

$$=G\left(\sin\xi,\frac{w_2}{\sin\xi},\frac{w_1}{\sin\xi}\right)=C_2=\text{const.} \quad (67)$$

Thus, we have already found two independent first integrals for the integration of the fourth-order system (42), (43). For its complete integrability, it suffices to find one additional first integral, which “attaches” Eq. (43).

Since

$$\frac{du_1}{d\tau}=\frac{u_1(2u_2+b_*)}{(-b_*-u_2)\tau},\ \frac{d\eta_1}{d\tau}=\frac{u_1}{(-b_*-u_2)\tau},$$

we have

$$\frac{du_1}{d\eta_1}=2u_2+b_*.$$

Obviously, for $u_1\neq 0$, the following equality holds:

$$u_2=\frac{1}{2}\left(-b_*\pm\sqrt{b_1^2-4\left(u_1-\frac{C_1}{2}\right)^2}\right),$$

$$b_1^2=b_*^2+C_1^2-4,$$

then integration of the quadrature

$$\eta_1+\text{const}=\pm\int\frac{du_1}{\sqrt{b_1^2-4\left(u_1-\frac{C_1}{2}\right)^2}}$$

yields the invariant relation

$$2(\eta_1+C_3)=$$

$$=\pm\arcsin\frac{2u_1-C_1}{\sqrt{b_*^2+C_1^2-4}},\ C_3=\text{const.}$$

In other words, the equality

$$\sin[2(\eta_1+C_3)]=\pm\frac{2u_1-C_1}{\sqrt{b_*^2+C_1^2-4}}$$

holds and, returning to the old variables, we obtain

$$\sin[2(\eta_1+C_3)]=\pm\frac{2w_1-C_1\sin\xi}{\sqrt{b_*^2+C_1^2-4}\sin\xi}.$$

In principle, in order to obtain an additional invariant relation that “attaches” Eq. (43), we could stop on the last equation. In this case, we must formally substitute the left-hand side of the first integral (49) into the last expression instead of C_1.

But we perform some transformations which allow to obtain the following explicit form of the additional first integral (in this case, we use Eq. (49)):

$$\text{tg}^2[2(\eta_1+C_3)]=\frac{(u_1^2-u_2^2-b_*u_2-1)^2}{u_1^2(4u_2^2+4b_*u_2+b_*^2)}.$$

Returning to the old coordinates, we obtain an additional invariant relation of the form

$$\text{tg}^2[2(\eta_1+C_3)]=$$

$$=\frac{(w_1^2-w_2^2-b_*w_2\sin\xi-\sin^2\xi)^2}{w_1^2(4w_2^2+4b_*w_2\sin\xi+b_*^2\sin^2\xi)},$$

or, finally,

$$\Theta_3(w_2, w_1; \xi, \eta_1) =$$

$$= -\eta_1 \pm \frac{1}{2}\text{arctg}\frac{w_1^2 - w_2^2 - b_* w_2 \sin\xi - \sin^2\xi}{w_1(2w_2 + b_* \sin\xi)} = \tag{68}$$

$$= C_3 = \text{const}.$$

Therefore, in the considered case, the system of dynamical equations (42), (43) has three first integrals expressing by relations (50), (67), (68), which are the transcendental functions of its phase variables (in the sense of the complex analysis) and are expressed as a finite combination of elementary functions (in this case, we use the expressions (63)–(66)).

Theorem 4. *Three sets of relations (4), (11), (14) under conditions (5)–(7), (37), (38) possess three the first integrals (the complete set), which are the transcendental function (in the sense of complex analysis) and are expressed as a finite combination of elementary functions.*

6.3 Topological analogies

Now we present two groups of analogies related to the system (15), which describes the motion of a free body in the presence of a tracking force.

The first group of analogies deals with the case of the presence the nonintegrable constraint (19) in the system. In this case the dynamical part of the motion equations under certain conditions is reduced to a system (27).

Under conditions (37), (38) the system (27) has the form

$$\begin{array}{c} \alpha' = -w_2 + b\sin\alpha, \\ w_2' = \sin\alpha\cos\alpha - w_1^2\frac{\cos\alpha}{\sin\alpha}, \\ w_1' = w_1 w_2 \frac{\cos\alpha}{\sin\alpha}, \end{array} \tag{69}$$

$$\beta_1' = w_1\frac{\cos\alpha}{\sin\alpha}, \tag{70}$$

if we introduce the dimensionless parameter, the variables, and the differentiation analogously to (39):

$$b = \sigma n_0,\ n_0^2 = \frac{AB}{I_2},\ z_k = n_0 v w_k,\ k = 1, 2, \tag{71}$$

$$< \cdot >= n_0 v <'> .$$

Theorem 5. *System (69), (70) (for the case of a free body) is equivalent to the system (42), (43) (for the case of a fixed pendulum).*

Indeed, it is sufficient to substitute

$$\xi = \alpha,\ \eta_1 = \beta_1,\ b_* = -b. \tag{72}$$

Corollary 6. *1. The angle of attack α and the angle of sliding β_1 for a free body are equivalent to the angles of a body deviation $\xi = \theta$ and $\eta_1 = \psi$, respectively, of a fixed pendulum (Fig. 1).*

2. *The distance $\sigma = CD$ for a free body corresponds to the length of a holder $l = OD$ of a fixed pendulum.*
3. *The first integrals of the system (69), (70) can be automatically obtained through the Eqs. (50), (67), (68) after substitutions (72) (see also [8], [9]):*

$$\Theta_1'(w_2, w_1; \alpha) =$$

$$= \frac{w_2^2 + w_1^2 - bw_2\sin\alpha + \sin^2\alpha}{w_1\sin\alpha} = \tag{73}$$

$$= C_1 = \mathit{const}.$$

$$\Theta_2'(w_2, w_1; \alpha) =$$

$$= G\left(\sin\alpha, \frac{w_2}{\sin\alpha}, \frac{w_1}{\sin\alpha}\right) = \tag{74}$$

$$= C_2 = \mathit{const}.$$

$$\Theta_3'(w_2, w_1; \alpha, \beta_1) =$$

$$= -\beta_1 \pm \frac{1}{2}\mathit{arctg}\frac{w_1^2 - w_2^2 + bw_2\sin\alpha - \sin^2\alpha}{w_1(2w_2 - b\sin\alpha)} = \tag{75}$$

$$= C_3 = \mathit{const}.$$

The second group of analogies deals with the case of a motion with the constant velocity of the center of mass of a body, i.e., when the property (28) holds. In this case the dynamical part of the motion equations under certain conditions is reduced to a system (32)–(36).

Then, under conditions (28), (37), (38), (71) ($w_k \leftrightarrow Z_k$) the reduced dynamical part of the motion equations (system (33)–(36)) has the form of analytical system

$$\alpha' = -Z_2 + b(Z_1^2 + Z_2^2)\sin\alpha + b\sin\alpha\cos^2\alpha,$$

$$Z_2' = \sin\alpha\cos\alpha - Z_1^2\frac{\cos\alpha}{\sin\alpha} + bZ_2(Z_1^2 + Z_2^2)\cos\alpha - \quad (76)$$

$$-bZ_2\sin^2\alpha\cos\alpha,$$

$$Z_1' = Z_1Z_2\frac{\cos\alpha}{\sin\alpha} + bZ_1(Z_1^2 + Z_2^2)\cos\alpha -$$

$$-bZ_1\sin^2\alpha\cos\alpha,$$

$$\beta_1' = Z_1\frac{\cos\alpha}{\sin\alpha}, \quad (77)$$

in this case, we choose the constant n_1 as follows: $n_1 = n_0$.

If the problem on the first integrals of the system (69), (70) is solved using Corollary 6, the same problem for the system (76), (77) can be solved by the following Theorem 7.

At the beginning, we note that one of the first integrals of the system (76), (77) has the following form (see [10]):

$$\Theta_1''(Z_2, Z_1; \alpha) =$$

$$= \frac{Z_2^2 + Z_1^2 - bZ_2\sin\alpha + \sin^2\alpha}{Z_1\sin\alpha} = \quad (78)$$

$$= C_1 = \text{const}.$$

Later on, we study an additional first integral of the third-order system (76) using, in this case, the first integral (78). For this we introduce the following notations and new variables (comp. with [11], [12]):

$$\tau = \sin\alpha,\ Z_k = u_k\tau,\ k = 1, 2,\ p = \frac{1}{\tau^2}. \quad (79)$$

Then the problem on explicit form of the desired first integral reduces to solving of the linear inhomogeneous equation:

$$\frac{dp}{du_2} = \frac{2(u_2 - b)p + 2b(1 - U_1^2(C_1, u_2) - u_2^2)}{1 - bu_2 + u_2^2 - U_1^2(C_1, u_2)}, \quad (80)$$

$$U_1(C_1, u_2) = \frac{1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}\right\},$$

in this case, an additive constant C_1 can be chosen as follows: $b^2 + C_1^2 - 4 \geq 0$.

The last fact means that we can find another transcendental first integral in the explicit form (i.e., as a finite combination of quadratures). Here, the general solution of Eq. (80) depends on an arbitrary constant C_2. We omit the calculation, but note that the general solution of the linear homogeneous equation obtained from (80) even in the particular case $b = C_1 = 2$ has the following solution:

$$p = p_0(u_2) = C[\sqrt{1 - (u_2 - 1)^2} \pm 1]\times$$

$$\times\exp\left[\sqrt{\frac{1 \mp \sqrt{1 - (u_2 - 1)^2}}{1 \pm \sqrt{1 - (u_2 - 1)^2}}}\right],\ C = \text{const}.$$

Then the desired additional first integral has the following structural form (which is similar to the transcendental first integral from the plane-parallel dynamics):

$$\Theta_2''(Z_2, Z_1; \alpha) =$$

$$= G\left(\sin\alpha, \frac{Z_2}{\sin\alpha}, \frac{Z_1}{\sin\alpha}\right) = C_2 = \text{const}, \quad (81)$$

in this case, we use the notations and substitutions (79).

Thus, for the integration of the fourth-order system (76), (77), we have found two independent first integrals. For the complete its integration, it suffices to find one (additional) first integral that “attaches” Eq. (77).

The desired first integral can be obtained by the following relation:

$$\sin[2(\beta_1 + C_3)] = \pm\frac{2Z_1 - C_1\sin\alpha}{\sqrt{b^2 + C_1^2 - 4}\sin\alpha}.$$

In principle, in order to obtain an additional invariant relation that “attaches” Eq. (77), we could stop on the last equation. In this case, we must formally substitute the left-hand side of the first integral (78) into the last expression instead of C_1.

But we perform some transformations which allow to obtain the following final explicit form of the additional first integral:

$$\Theta_3''(Z_2, Z_1; \alpha, \beta_1) =$$

$$= -\beta_1 \pm \frac{1}{2}\mathrm{arctg}\frac{Z_1^2 - Z_2^2 + bZ_2 \sin\alpha - \sin^2\alpha}{Z_1(2Z_2 - b\sin\alpha)} = \tag{82}$$

$$= C_3 = \mathrm{const}.$$

Theorem 7. *Three first integrals (78), (81), (82) of the system (76), (77) are the transcendental functions of its own phase variables and are expressed as a finite combination of elementary functions.*

Theorem 8. *Three first integrals (78), (81), (82) of the system (76), (77) are equivalent to three first integrals (73), (74), (75) of the system (69), (70).*

Indeed, the couples of the first integrals (78), (73) and (82), (75) coincides, if we substitute $b = -b_*$. And finally, we need to identify the phase variables $Z_k,\ k = 1, 2$, for the system (76), (77) with the phase variables $w_k,\ k = 1, 2$, of the system (69), (70). Because of their cumbersome character, the similar arguments concerning of the couples of the first integrals (81), (74), we do not represent.

7 Conclusions

Thus, we have the following topological and mechanical analogies in the sense explained above.

(1) A motion of a fixed physical pendulum on a spherical hinge in a flowing medium (nonconservative force fields).

(2) A spatial free motion of a rigid body in a nonconservative force field under a tracing force (in the presence of a nonintegrable constraint).

(3) A spatial composite motion of a rigid body rotating about its center of mass, which moves rectilinearly and uniformly, in a nonconservative force field.

On more general topological analogues, see also [8], [9], [12], [13].

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