Lie algebra on synchronization of different systems: a generalized function for Hodgkin-Huxley neurons
EXTENDED ABSTRACT

R. Femat*, J.G. Barajas-Ramírez† and G. Solís-Perales‡
Laboratorio para Biodinámica y Sistemas Alineales
Matemáticas Aplicadas, IPICyT
Apdo. Postal 3-90, Tangamanga 78231
San Luis Potosí, S.L.P., México

Summary: In this contribution two results are taken: (1) The synchronization of noiseless Hodgkin-Huxley (HH) neurons is possible from robust feedback based on Lie algebra approaches and (2) the fact that, from Lie algebra of vector fields, the generalized synchronization of different (triangular form) chaotic systems can be used to derive an explicit synchronization function. Both results are extended to derive the synchronization function in HH neurons despite this systems are not in triangular form. Thus, the Lie algebra of vectors fields permits to establish a theoretical framework for finding the synchronization function in chaotic systems in face they have different model.

Motivation: For several decades many attempts have been addressed to understand the processing of biological information in single neurons and neural networks. Experimental reports [1] suggest that the synchronization plays a very important role in the processing of information by large ensembles of neurons. Recently, it has been demonstrated that a minimal ensemble of two coupled living neurons fire synchronized spiking activity when depolarized by an external DC current [2]. However, total neural mechanisms underlying synchronization are not well understood yet. The Hodgkin-Huxley (HH) neurons are usually used as realistic models of neuronal systems, for studying neuronal synchronization. Some theoretical approaches investigate the synchronization phenomena considering diffusive coupling and the influence of intrinsic noise as a promoter of neuronal activity [3], and studying the synchronization dynamics related to the rhythmic oscillations phenomena (theta and gamma frequency rhythms) in neurons of localized areas of the brain [4]. In addition, the forcing of HH neurons by external stimulus has been widely studied [5] for tonic or periodic currents that trigger the action potential displaying spike activity and refractory dynamics.

The efforts have been focused on the analysis of the time-invariant manifolds related with synchronization [9]-[11]. Two basic approaches have been exploited. On the one hand, chaotic synchronization has been interpreted as the prediction of the chaotic system, i.e., observability approach. In this sense, the reconstruction of the drive system attractor from the response system is interpreted as an observer [11]. An interesting point about observability of the synchronization systems is that differential geometry allows to find an invariant space under vector fields where the attractor can be reconstructed. Synchronization of chaotic systems, on the other hand, has been also studied from the measurable variables (system output) [8]-[10]. In such a case, the synchronization is understood as a stabilization problem. In other words, to compute the control input such that the difference between trajectories of the slave system $x_S(t)$ remains close to the trajectories of the master system $x_M(t)$. That is, the point is to find the invariant space such that the origin of the synchronization error system $\|x_M(t) - x_S(t)\| = 0$ can be stabilized. Both observability and controlability of nonlinear systems are included in the geometrical control theory [13].

*Corresponding author. rfemat@ipicyt.edu.mx
†jgbarajas@ipicyt.edu.mx
‡División de Computación y Electrónica, CUCEI, Universidad de Guadalajara. Av. Revolución 1500, Guadalajara, Jal., México. gualberto.solis@cucei.udg.mx

Here, the goal is to provide some remarks on the diffeomorphism between chaotic attractors by exploiting the Lie algebra of the chaotic systems [12] to compute the synchronization function. In this contribution, two previous results are taken to study the synchronization between HH neurons: the Lie-based approach to the generalized synchronization of different systems [14] and the fact that two HH neurons can be synchronized via robust asymptotic feedback [15] That is, we are interested in the Lie-based geometric properties of the class of dynamical system given by $\dot{x} = \tau_M(x_M) - \tau_S(x_S) - g(x)u$, where $x \in R^n$ is, by definition, $x := x_M - x_S$, and stands for the state vector of the synchronization error system; $\tau_M : R^n \to R^n$, $\tau_S : R^n \to R^n$ and $g : R^n \to R^n$ are smooth vector fields. The product $g(x)u$ is related to the synchronization command. The scalar function $u = u(x)$ can be computed from the construction of accessibility spaces, and is named the control input [13]. Thus, we are interested in finding the output $y = \lambda(x) \triangleq \lambda(x_M - x_S)$ of the above-mentioned dynamical system such that synchronization is attained. Thus the synchronization problem is formulated in terms of geometrical properties of the system along the vector fields $\tau(x, x_M) = \tau_M(x_M) - \tau_S(x_S)$ and $g(x)$ as: *what are the geometrical properties of any synchronization error system such that the scalar function $u = u(x)$ guarantees the existence of the synchronization (output) function $y = \lambda(x)$?* Such problem is addressed via Lie algebra of the vector field related to the synchronization error system.

Main results: The following set of four coupled nonlinear differential equations represents the HH neuron model [6]:

$$
\begin{aligned}
C_m \frac{dV}{dt} &= I_{ext} - g_K n^4 (V - V_K) - g_{Na} m^3 h (V - V_{Na}) - g_l (V - V_{Na}), && (1)\\
\frac{dP_K}{dt} &= \alpha_{P_K}(V)(1 - P_K) - \beta_{P_K}(V) P_K, && (2)\\
\frac{dP_{Na}}{dt} &= \alpha_{P_{Na}}(V)(1 - P_{Na}) - \beta_{P_{Na}}(V) P_{Na}, && (3)\\
\frac{dI_{Na}}{dt} &= \alpha_{I_{Na}}(V)(1 - P_{Na}) - \beta_h(V) I_{Na}, && (4)
\end{aligned}
$$

where variables V, P_K, P_{Na} and I_{Na} represent the membrane potential, the activation of the potassium flow current, the activation and inactivation of the sodium flow current, respectively. C_m is the membrane capacitance, g_K, g_{Na} and g_l are the maximum ionic and leak conductances, while V_K, V_{Na} and V_l stand for the ionic and leak reversal potentials. The external stimulus current can be modeled by the term I_{ext}, usually a tonic or periodic forcing. The explicit form of the functions $\alpha_j(V)$ and $\beta_j(V)$ $(j = n, m, h)$ in Eqs. (2)-(4), and nominal values for the system parameters can be found in [6, 7]. Our starting point is based on the following facts [13]:

- *Fact 1*: Consider an affine nonlinear system $\dot{x} = f(x) + g(x)u$; where $x \in \Omega \subseteq R^n$, $u \in R$, $g, f : R^n \to R^n$ are smooth vector fields. Besides, let us consider that $y = h(x)$ for any smooth function $h(x)$. If involutivity condition is satisfied, then the mappings $\Phi_1 : R^n \to R^\rho$, $x \mapsto z$ and $\Phi_2 : R^n \to R^{n-\rho}$, $x \mapsto (z, \nu)$ are such that the affine nonlinear system can be written in the canonical form

$$
\begin{aligned}
\dot{z}_i &= z_{i+1}, \quad i = 1, 2, ..., \rho - 1,\\
\dot{z}_\rho &= \alpha(z, \nu) + \beta(z, \nu)u, && (5)\\
\dot{\nu} &= \zeta(z, \nu),
\end{aligned}
$$

  and can be derived from Lie derivatives of the output function $h(x)$ along the vector fields $f(x)$ and $g(x)$ as follows

$$
z = \Phi_1(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \ldots \\ L_f^{\rho-1} h(x) \end{pmatrix} \qquad (6)
$$

and

$$\nu=\Phi_2(x)=\left(\begin{array}{c}\phi_{\rho+1}(x)\\ \phi_{\rho+2}(x)\\ \ldots\\ \phi_n(x)\end{array}\right), \tag{7}$$

moreover, it is always possible to chose $\phi_{\rho+1},...,\phi_n$ in such a way that

$$L_g\phi_j(x)=0,\quad \rho+1\leq j\leq n. \tag{8}$$

The Fact 1 is well known in nonlinear control theory. Here it is resumed for clarity in presentation and exploited in neuronal synchronization towards robust feedback synchronization of HH neurons.

- *Fact 2*: If exists the map $\Phi=(\Phi_1,\Phi_2):R^n\rightarrow R^n$, $x\mapsto(z,\nu)$ derived from (6) and (7), then there exists the inverse $\Phi^{-1}(\Phi(x))=x\in\Omega\subset R^n$. This fact is proved since $h(x)$, $L_fh(x)$, ..., $L_f^{\rho-1}h(x)$ and $\phi_{\rho+1}(x)$, ..., $\phi_n(x)$ are linearly independent at any x in the neighborhood $U\subset\Omega\subseteq R^n$ of the point x^0 in Ω.

Firstly, let us define the state variables by $x=(V,P_K,P_{Na},I_{Na})\in\Omega\subset R^4$. Then, by assuming the output function is an scalar given by the membrane potential (i.e., $h(x)=x_1$), the dynamical model for a HH neuron can be written in nonlinear affine form $\dot{x}=f(x)+g(x)u$, with

$$f(x)=\left(\begin{array}{c}\{1/C_m[I_{ext}-g_Kx_2^4\,(x_1-V_K)\\ -g_{Na}x_3^3x_4\,(x_1-V_{Na})-g_l\,(x_1-V_{Na})]\}\\ \alpha_n(x_1)(1-x_2)-\beta_n(x_1)x_2\\ \alpha_m(x_1)(1-x_3)-\beta_m(x_1)x_3\\ \alpha_h(x_1)(1-x_4)-\beta_h(x_1)x_4\end{array}\right),\quad g(x)=\left(\begin{array}{c}1\\0\\0\\0\end{array}\right), \tag{9}$$

and the output $y=h(x)$ is given by the membrane potential, *i. e.*, $h(x)=x_1$. By computing the Lie derivatives of the output function along the vector fields (9), we obtain

$$z=\Phi_1(x)=h(x)=x_1 \tag{10}$$

and

$$\nu=\Phi_2(x)=\left(\begin{array}{c}\phi_1(x)\\ \phi_2(x)\\ \phi_3(x)\end{array}\right), \tag{11}$$

where ϕ_1,ϕ_2 and ϕ_3 are solutions of the differential equations given by (8); that is, for this specific case, we have

$$\frac{\partial\phi_j}{\partial x_1}+\frac{\partial\phi_j}{\partial x_2}+\frac{\partial\phi_j}{\partial x_3}=0 \tag{12}$$

where $j=1,2,3$. If the PDE's (12) have solution there exists a diffeomorphic function $\Phi=(\Phi_1,\Phi_2)^T:\Omega\rightarrow R^4,x\mapsto(z,\nu)$, such that the dynamics of the synchronization error can be written in the form (5). As matter of fact, a solution for (12) is found for any $(\phi_1,\phi_2,\phi_3)\neq(\phi_1(x_1),\phi_2(x_1),\phi_3(x_1))$; hence the function Φ exists. Finally, if there exists a synchronization force u such that the system of the synchronization error is asymptotically stable, the synchronization function becomes $x_s=\Psi\triangleq\Phi_S^{-1}\circ\Phi_M(x_M)$, where subscripts M and S denote the mater and slave neurons, respectively [14]. It should be noted that, generally speaking, the solution of (5) is not unique. Hence the synchronization function $\Psi:\Omega_S\rightarrow\Omega_S$, $x_M\mapsto x_S$ is a smooth vector field whose components $\phi_1,\phi_2\phi_3$ are exist but can be uncertain.

In order to show this fact, we consider the following solution: $\phi_{(j,k)}=x_{j,k}$, where $j=2,3,4$ and $k=M,S$. Assuming the output is the membrane potential; i.e., $h(x)=x_1$. Then, the model of the Master and slacve HH neurons can be transformed into

$$\begin{aligned}\dot{z}_1 &= 1/C_m[I_{ext}-g_K\nu_1^4\,(z_1-V_K)-g_{Na}\nu_2^3\nu_4\,(z_1-V_{Na})\\ &\quad -g_l\,(z_1-V_{Na})]+u\,, &(13)\\ \dot{\nu}_1 &= \alpha_n\,(z_1)\,(1-\nu_1)-\beta_n\,(z_1)\,\nu_1\,, &(14)\\ \dot{\nu}_2 &= \alpha_m\,(z_1)\,(1-\nu_2)-\beta_m\,(z_1)\,\nu_2\,, &(15)\\ \dot{\nu}_3 &= \alpha_h\,(z_1)\,(1-\nu_3)-\beta_h\,(z_1)\,\nu_3\,. &(16)\end{aligned}$$

Then, both master and slave neurons are separately transformed and the maps $\Phi_M(x_M)$ and $\Phi_S(x_S)$ are derived to get

$$\begin{pmatrix} z_M \\ \nu_M \end{pmatrix} = \begin{pmatrix} \Phi_{1M}(x_M) \\ \Phi_{2M}(x_M) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_S \\ \nu_S \end{pmatrix} = \begin{pmatrix} \Phi_{1S}(x_S) \\ \Phi_{2S}(x_S) \end{pmatrix}, \tag{17}$$

from where each HH neuron can be transformed into (5) which implies that there is a driving signal synchronizing the master behavior onto slave neuron [15]. Then, if stability holds and neurons are minimum-phase systems, $(z_S, \nu_S) \to (z_M, \nu_S^*)$ for $t > t_0 \geq 0$ and initial conditions $(z(0), \nu(0)) = (\Phi_1(x(0)), \Phi_2(x(0)))$ in physical domain. Note that ν_S^* is a stable manifold which can correspond to the stable manifold of the master neuron ν_M^*. In this case complete synchronization is achieved. In case $\nu_S^* \neq \nu_M^*$, the partial state synchronization is attained [16].

Preliminary conclusions and discussion: The composition $\Phi_S^{-1}(\Phi_1(x_M); \nu_S^*) = x_S \in \Omega \subset R^n$, where Ω denotes the physical domain. In particular, if $\nu_S^* \equiv \nu_M^*$ for all time $t > t_0 \geq 0$, where t_0 stands for time of turning on the control, then $x_S = \Phi_S^{-1}(\Phi_{1M}(x_M), \Phi_{2M}(x_M))$. Since HH neurons (1)-(4) are minimum-phase systems, the GS yields the following function $x_S = \left(h_S^{-1}(h_M(x_M)), x_{2S}^*, x_{3S}^*, x_{4S}^*\right)^T$. It should be noted that the explicit expression for the synchronization function Ψ depends on the solution of (12). Then, the partial state synchronization is depicted but a question arises: how can we know about the geometric or algebraic properties of the synchronization function depends on multiple solution of (12)? This question is under study and it will be reported elsewhere.

References

[1] A. K. Kreiter, W. Singer, J. Neurosci., 16 (1996) 238196.

[2] R. C. Elson, A. I. Selverston, R. Huerta, N. F. Rulkov, M. I. Rabinovich, H. D. Abarbanel, Phys. Rev. Lett., 87 (1999) 5692.

[3] J. M. Casado, J. P. Baltanás, Phys. Rev. E, 68 (2003) 061917.

[4] G. B. Ermentrout, N. Kopell, Proc. Natl. Acad. Sci., 95 (1998) 1259.

[5] Y. Yu, W. Wang, J. Wang, F. Liu, Phys. Rev. E, 63 (2001) 21907.

[6] A. L. Hodgkin, A. F. Huxley, J. Physiol. (London) 117 (1952) 500.

[7] C. Koch, Biophysics of Computation: Information Processing in Single Neurons (Oxford University Press, New York, 1998).

[8] R. Femat and J. Alvarez-Ramirez, Phys. Letts. A, 236, (1997), 307.

[9] S. Boccaletti, L.M Pecora and A. Pelaez, Phys. Rev. E, 63 (2001) 066219.

[10] K. Josić, Nonlinearity,13,(2000) 1321.

[11] H. Nijmeijer, and M. Y. Mareels, IEEE Trans. Circuits and Syst. I., 44 (1997) 882.

[12] L. Kocarev, U. Parlitz, and B. Hu, Chaos, Solitons and Fractals, 9 (1998) 1359.

[13] H. Nijmeijer and A. van der Schaft, *Nonlinear control dynamical systems*, Springer-Verlag, 1990.

[14] R. Femat, L. Kocarev, L. van Gerven, M.E. Monsivais-Pérez, Phys. Letts A, 342 (2005) 247.

[15] O. Cornejo-Pérez, R. Femat, Chaos Solitons and Fractals, 25 (2005) 43.

[16] R. Femat, G. Solís-Perales, Phys. Letts A, 262 (1999) 50.