ON SOME SIMPLE MECHANICAL SYSTEMS GOVERNED BY DIFFERENTIAL EQUATIONS WITH FRACTIONAL DERIVATIVES

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Abstract

Three types of suspension of a semi-infinite Bernoulli-Euler beam and a fluid-conveying pipe are considered. It is shown that the environment (i.e. the semi-infinite Bernoulli-Euler beam or the fluid-conveying pipe) adds a fractional derivative into the suspension equation. The eigenvector expansion method based upon transformation of the derived equation into a set of four semidifferential equations is utilised for solving the equations with fractional derivatives.

Key words

fractional derivatives, semi-infinite Bernoulli-Euler beam, fluid-conveying-pipe, half-order differential equation.

1 Introduction

The intent of the presentation is to show that the governing equation for simple mechanical systems may contain fractional derivatives. We consider three types of oscillator to which a semi-infinite Bernoulli-Euler beam is attached. It is shown that if the consideration is limited only to the oscillator, then the environment (i.e. the semi-infinite Bernoulli-Euler beam) adds a fractional derivative into the oscillator equation. Another system governed by a differential equation with fractional derivative is the suspension of the fluid-conveying-pipe. The eigenvector expansion method based upon transformation of the equation into a set of four semidifferential equations is utilised for solving the obtained differential equation with fractional derivatives.

2 Three mechanical systems

Consider a Single-Degree-Of-Freedom system and a semi-infinite Bernoulli-Euler beam $x>0$, which is attached to mass m at $x=0$, see Fig. 1.

The mass m is allowed to perform only vertical displacements and governed by the equation

$$m\frac{d^2y}{dt^2} = -cy + Q\big|_{x=0} + f(t)\,, \qquad (2.1)$$

where y is the absolute displacement of the mass m, $f(t)$ is an external driving force, t is time and $Q\big|_{x=0}$ is the shear force in the beam acting on the mass m.

The equation of the beam bending is as follows

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0,\ 0<x<\infty\,, \qquad (2.2)$$

where $w(x,t)$ is the absolute displacement, EI is the bending stiffness of the beam, ρ is the mass density and A is the cross-sectional area. Kinematics requires the following

$$y(t) = w(0,t)\,. \qquad (2.3)$$

The zero initial conditions are assumed, then the Laplace transformation gives

$$mp^2\bar{y}(p) + c\bar{y}(p) = \bar{Q}(p)\big|_{x=0} + \bar{f}(p), \qquad (2.4)$$

$$EI\frac{d^4\bar{w}}{dx^4} + \rho Ap^2\bar{w} = 0\,,\ 0<x<\infty. \qquad (2.5)$$

The solution of eqn (2.5) bounded at infinity is as follows

$$\bar{w}(x,p) = A_2\exp(\lambda_2 x) + A_4\exp(\lambda_4 x)\,, \qquad (2.6)$$

where the wave numbers are $\lambda_2 = -(1+i)\beta\sqrt{p}$, $\lambda_4 = -(1-i)\beta\sqrt{p}$, $\beta = \sqrt[4]{\rho A/4EI}$ for $\sqrt{p}$. Because of the joint at $x=0$ the bending moment of the beam vanishes, i.e.

$$\bar{M}\big|_{x=0} = -EI\frac{d^2\bar{w}}{dx^2}\bigg|_{x=0} = 2EIi\beta^2 p\left(A_2 - A_4\right) = 0\,.$$

Hence, $A_2 = A_4$ and the shear force $Q\big|_{x=0}$ is needed to be substituted into eqn (2.4), hence

$$\bar{Q}\Big|_{x=0} = -EI\frac{d^3\bar{w}}{dx^3}\Big|_{x=0} = -EI\cdot\left[\lambda_2^3A_2+\lambda_4^3A_4\right] = -4\cdot EI\cdot\beta^3p\sqrt{p}A_2\ \cdot$$

As follows from eqn (2.6) $\bar{w}(0,p)=\bar{y}(p)=2A_2$, that allows the latter equation to be rewritten in the following form

$$\bar{Q}\Big|_{x=0} = -2\cdot EI\cdot\beta^3p\sqrt{p}\bar{y}(p)\ . \qquad (2.7)$$

Inserting the latter equation into eqn (2.4) yields

$$mp^2\bar{y}(p)+2EI\beta^3p\sqrt{p}\bar{y}(p)+c\bar{y}(p)=\bar{f}(p). \quad (2.8)$$

Since the trivial initial conditions were assumed, eqn (2.8) corresponds to the following ordinary differential equation for displacement *y(t)*

$$m\frac{d^2y}{dt^2}+2EI\beta^3\frac{d^{3/2}y}{dt^{3/2}}+cy=f(t). \qquad (2.9)$$

As seen from eqn (2.9), the dynamics of mass *m* is governed by a single differential equation with a fractional derivative.

Another mechanical system governed by the differential equation with a fractional derivative is obtained from the above system if the semi-infinite Bernoulli-Euler beam is clamped to a rigid mass *m*, rather than it is simply supported. In this case the governing equation is as follows (the derivation is omitted as it is fully analogous to the previous one)

$$m\frac{d^2y}{dt^2}+4EI\beta^3\frac{d^{3/2}y}{dt^{3/2}}+cy=f(t). \qquad (2.10)$$

The third mechanical system consists of a disk attached to an angular spring and a semi-infinite Bernoulli-Euler beam, Fig. 2. The beam is supposed to be clamped to the disc in such a way that the angle of rotation of the disk and that of the beam at x=0 coincide. The differential equation of the disk is given as

$$J\frac{d^2\varphi}{dt^2}+2\beta EI\frac{d^{1/2}\varphi}{dt^{1/2}}+k\varphi=m(t). \qquad (2.11)$$

where *J* is the moment of the mass inertia, *k* is the angular stiffness of the spring and *m(t)* is the external driving moment. Again, a differential equation with a fractional derivative is obtained however, in contrast to eqs. (2.9)-(2.11), now we have the fractional derivative of the order 1/2.

3 Mechanical system with a pipeline conveying fluid

We consider a pipeline conveying a heavy fluid. The suspension is assumed to be modeled by a spring of stiffness *c* and a dashpot *b*. The mass of the suspension is *m* and *v* denotes the velocity of the fluid, see Fig. 3.

Omitting the derivation of the differential equation for mass *m* we demonstrate the final result for the case of massless pipe, heavy fluid and low velocities

$$m\frac{d^2y}{dt^2}+4EI\beta^3\frac{d^{3/2}y}{dt^{3/2}}+(b-\rho Av)\frac{dy}{dt}+cy=f(t). \quad (3.1)$$

where *f(t)* denotes the distributed external force in the transversal direction.

4 An example of solving the derived differential equation with fractional derivative

We take eqn (3.1) in the case *f(t)=0* and solve it by means of the eigenvector expansion method suggested by Suarez and Shokooh for solving differential equation with fractional derivatives. This method is based upon the transformation of the equations of motions into a set of four first-order semidifferential equations. To this end, we introduce non-dimensional time $\tau=kt$ where $k=\sqrt{c/m}$ denotes the natural frequency of the suspension. After a little algebra we obtain the equation in terms of the non-dimensional variables

$$D^2y+2\delta D^{3/2}y+\varepsilon Dy+y=0,\quad D=\frac{d}{d\tau}. \qquad (4.1)$$

Here two coefficients

$$\delta=2\frac{EI\beta^3}{m\sqrt{k}}\ \text{and}\ \varepsilon=\frac{(\rho Av-b)}{mk} \qquad (4.2)$$

are responsible for stability or instability of the pipe with the fluid. Figure 4 displays the impulse response function for two set of parameters. Both stable and unstable modes are observable.

5 Conclusion

It is shown that some mechanical systems are governed by differential equations with fractional derivatives. The eigenvector expansion method is used for solving the obtained equations with fractional derivatives and deriving closed-form solutions. For example, this closed form solution is appropriate for obtaining simple formula for the critical velocity for systems conveying fluids.

References

Suarez, L. E. and Shokooh, A. (1997). ‘‘An Eigenvector Expansion Method for the Solution of Motion Containing Fractional Derivatives,’’ *ASME J. Appl. Mech.*, **64**, pp. 629–635.

Appendix: Figures

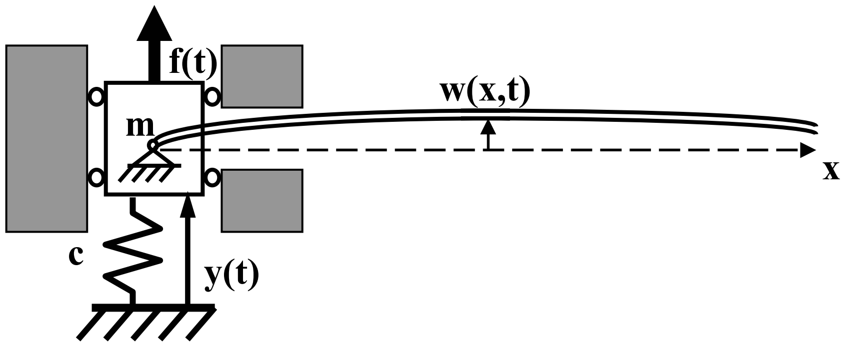


Figure 1. Schematics of the first model

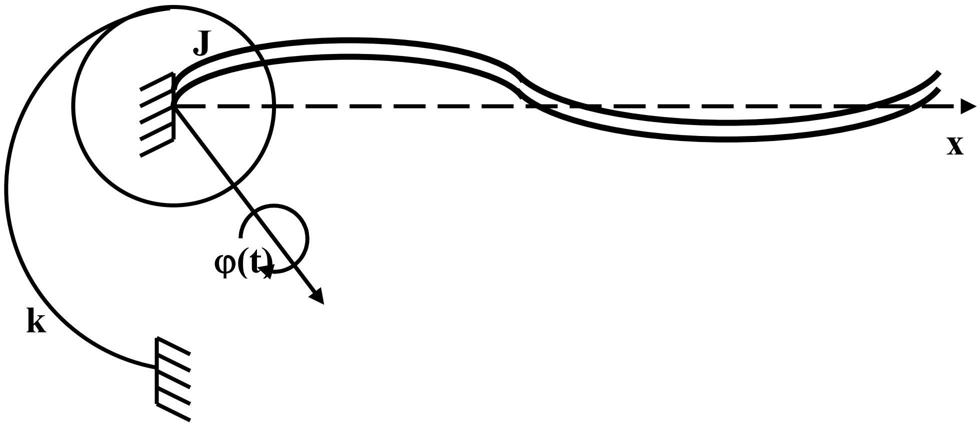


Figure 2. Schematics of the third model

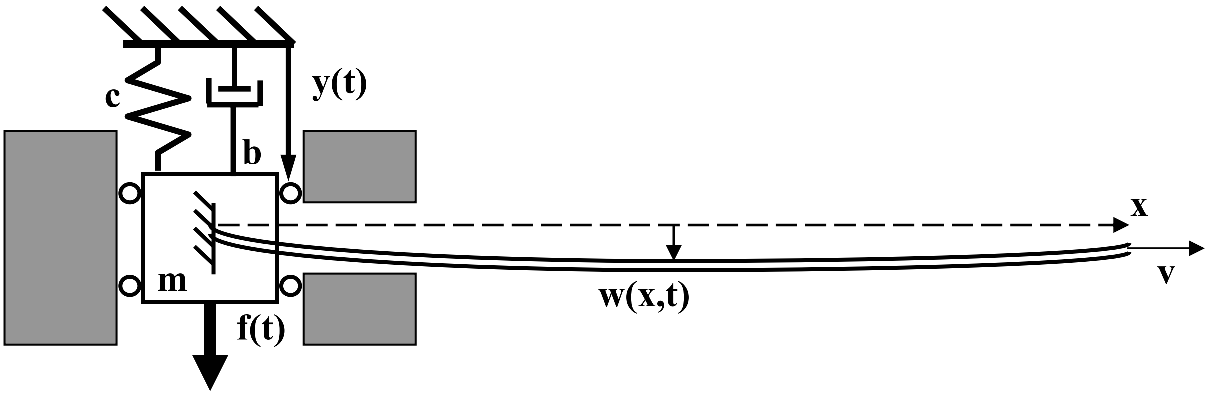
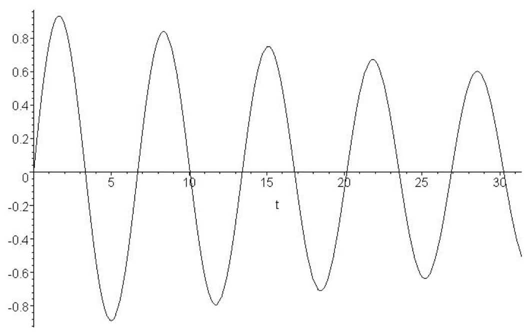


Figure 3. Schematics of the fluid-conveying-pipeline
and its suspension



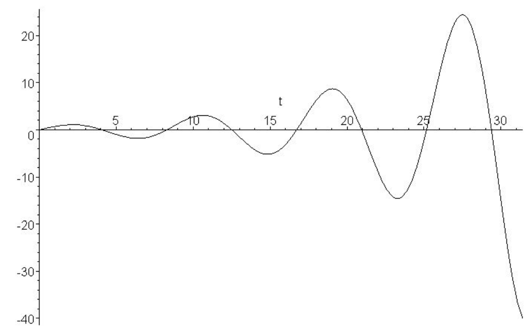


Figure 4. Displacement y versus time for $\delta = 0.1$,
$\varepsilon = 0.1$ (left) and $\delta = 0.5$, $\varepsilon = 1$ (right)