

STABILITY OF DISTRIBUTED PARAMETERS SYNCHRONIZATION SYSTEMS WITH DISTURBANCES

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Abstract

We examine stability properties of nonlinear systems with periodic nonlinearities, referred also to as *synchronization systems* or systems with *cylindric phase space*. An important feature of such systems is their multistability: existence of infinite sequences of stable and unstable equilibria. An important problem regarding dynamics of such systems is the convergence of any solution to one of equilibria (asymptotic stability of the special attractor), referred to as the “gradient-like behavior” or “phase locking”. Using the methods of absolute stability theory taking their origins in the works of V.M. Popov, new frequency-domain criteria for phase locking in synchronization systems are derived. Unlike the existing results, we consider synchronization systems with distributed parameters and disturbances.

Key words

Stability, integral equation, periodic nonlinearity, robustness, pendulum-like system, phase-locked loop

1 Introduction

Systems with periodic nonlinearities arise in abundance in nature and engineering applications and describe a broad class of phenomena, from the pendulum’s swings to dynamics of vibrational units, electric machines and physical oscillators [Baker and Blackburn, 2005, Stoker, 1950, Leonov et al., 1996b, Blekhman, 2000]. An important example of system with periodic nonlinearity is a *phase locked loop* (PLL) and similar control circuits, providing synchronization between the internal controlled oscillator or clock and some exogenous signal [Gardner, 1966, Lindsey,

1972, Margaris, 2004, Best, 2003, Razavi, 2003]. PLL circuits are used for carrier recovery, frequency synthesis and time synchronization; periodic nonlinearities in their mathematical models naturally represent the nonlinear characteristics of phase detectors (comparators). Motivated by applications to PLLs design and synchronization of rotating vibrational units [Blekhman, 2000, Czolczynski et al., 2012], the term *synchronization system* has been coined to denote systems with periodic nonlinearities [Lindsey, 1972, Leonov, 2006].

Systems with periodic nonlinearities typically have infinite sequences of stable and unstable equilibria points, as exemplified by the simplest model of mathematical pendulum. In particular, such a system is *multistable*; it is often convenient to consider their state spaces as toric or cylindric manifolds [Kudrewicz and Wasowicz, 2007, Leonov et al., 1996a]. Many effects in such systems, e.g. oscillations, hidden attractors and “cycle slipping” [Chicone and Heitzman, 2013, Leonov et al., 2015b, Leonov et al., 2015a, Best et al., 2016, Dudkowski et al., 2016] are essentially *non-local* in the sense that they cannot be analyzed via linearization at equilibria points. One of the central problems, concerned with dynamics of synchronization systems, is the convergence of all solutions to equilibria points. This special type of stability is referred to as the *gradient-like* behavior [Leonov, 2006, Duan et al., 2007]; dealing with synchronization systems, it is also called *phase locking*. It excludes, in particular, the possibility of limit cycles and chaotic attractors.

Whereas properties of synchronization systems have been extensively studied in the literature [Kudrewicz and Wasowicz, 2007, Leonov et al., 1996a, Leonov et al., 1996a, Gelig et al., 2004, Leonov, 2006, Duan

et al., 2007, Chicone and Heitzman, 2013, Best et al., 2016], most of the existing results are based on Lyapunov methods, in particular, the Kalman-Yakubovich-Popov lemma [Popov, 1973, Gelig et al., 2004]. In this paper, we develop an alternative approach that stems from Popov’s method of “a priori integral indices” [Popov, 1973, Rasvan, 2006, Yakubovich, 2002] or “integral quadratic constraints (IQC)” [Megretski and Rantzer, 1997]. Unlike Lyapunov method, this approach allows to examine not only ordinary differential equations with periodic nonlinearities, but also a broad class of distributed parameter synchronization systems, including e.g. delays [Wischert et al., 1992, Buckwalter and York, 2002] and non-rational filters [Tripathy et al., 2015]. The prerequisite for the applicability of Popov’s method is the possibility to decompose the system as a feedback interconnection of a linear time-invariant block and a nonlinearity (such a decomposition is also known as the Lur’e form). Stability criteria reduce to a *frequency-domain* condition, involving the transfer function of the linear part, and some nonlinear algebraic conditions, restricting the periodic nonlinearity; in this sense the criteria are “frequency-algebraic”.

Using Popov’s method, we obtain novel criteria of phase-locking, extending the results from the previous works [Leonov et al., 1992, Leonov et al., 1996b, Perkin et al., 2009, Perkin et al., 2012, Smirnova et al., 2015, Smirnova et al., 2016, Smirnova and Proskurnikov, 2016] in two directions. First, we introduce frequency-domain conditions of a novel type, involving the values of the transfer function not on the imaginary axis but rather on an arbitrary vertical line $\{\alpha + i\omega : \omega \in \mathbb{R}\}$. Second, we consider synchronization systems in presence of uncertain *disturbances*. The mitigation of disturbances is an important problem in design of PLLs and other synchronization circuits, see e.g. [Hill and Cantoni, 2000, Cataliotti et al., 2007, Schilling et al., 2010], the existing mathematical results are however very limited and mainly deal with cancellation of harmonic disturbances [Schilling et al., 2010] and cycle slipping effects in presence of random excitations [Tausworthe, 1967, Ascheid and Meyr, 1982]. In this paper, we take an important step in analysis of synchronization systems’ robustness against uncertain non-stochastic disturbances, addressing the case of disturbances with finite “energy”.

2 Problem setup. Preliminaries.

Consider a system of integro-differential equations

$$\begin{aligned} \dot{\sigma}(t) &= b(t) + R(\varphi(\sigma(t-h)) + g(t-h)) - \\ &- \int_0^t \gamma(t-\tau)(\varphi(\sigma(\tau)) + g(\tau)) d\tau \quad (t > 0). \end{aligned} \quad (1)$$

Here $\sigma(t) = (\sigma_1(t), \dots, \sigma_l(t))^T$, $\varphi : \mathbb{R}^l \rightarrow \mathbb{R}^l$ and $\varphi(\sigma) = (\varphi_1(\sigma_1), \dots, \varphi_l(\sigma_l))^T$, $g : [-h, +\infty) \rightarrow \mathbb{R}^l$, $b : [0, +\infty) \rightarrow \mathbb{R}^l$, $\gamma : [0, +\infty) \rightarrow \mathbb{R}^{l \times l}$, $R \in \mathbb{R}^{l \times l}$,

$h \geq 0$. The solution of (1) is defined by initial condition

$$\sigma(t)|_{t \in [-h, 0]} = \sigma^0(t). \quad (2)$$

Assume that the following conditions are true:

- 1) the function $b(\cdot)$ is continuous; the function $\gamma(\cdot)$ is piece-wise continuous;
- 2) the functions $b(\cdot)$ and $\gamma(\cdot)$ are exponentially decreasing:

$$|b(t)| + |\gamma(t)| \leq C e^{-rt} \quad (3)$$

for certain $C, r > 0$;

- 3) the function $g(t)$ is differentiable and

$$|g(\cdot)|, |\dot{g}(\cdot)| \in L_2[0, +\infty);$$

- 4) the function $\sigma^0(\cdot)$ is continuous and $\sigma(0+0) = \sigma^0(0)$;

- 5) each map φ_j is Δ_j -periodic ($\varphi_j(\sigma_j + \Delta_j) = \varphi_j(\sigma_j)$) with simple isolated roots; it is C^1 -smooth with

$$\begin{aligned} \alpha_{1j} &:= \inf_{\zeta \in [0, \Delta_j)} \varphi'_j(\zeta); \\ \alpha_{2j} &:= \sup_{\zeta \in [0, \Delta_j)} \varphi'_j(\zeta) \end{aligned} \quad (4)$$

(it is clear that $\alpha_{1j}\alpha_{2j} < 0$).

In this paper we shall establish the conditions for convergence of the solutions of (1). We are going to extend the frequency-domain criteria of gradient-like behavior of phase synchronization system without external disturbance ($g(t) \equiv 0$) [Perkin et al., 2012]. So we need to define the transfer matrix of the linear part of (1):

$$K(p) := -R e^{-ph} + \int_0^\infty \gamma(t) e^{-pt} dt \quad (p \in \mathbb{C}) \quad (5)$$

and introduce the frequency-domain inequality

$$\begin{aligned} \Pi(\omega) \triangleq \Re\{ \varkappa K(i\omega) - (K(i\omega) + i\omega M_1)^* \tau (K(i\omega) + \\ + i\omega M_2) - K(i\omega)^* \varepsilon K(i\omega) \} - \delta \geq 0. \end{aligned} \quad (6)$$

Here $i^2 = -1$, $\varkappa = \text{diag}\{\varkappa_1, \dots, \varkappa_l\}$, $\varepsilon = \text{diag}\{\varepsilon_1, \dots, \varepsilon_l\}$, $\tau = \text{diag}\{\tau_1, \dots, \tau_l\}$ and $\delta = \text{diag}\{\delta_1, \dots, \delta_l\}$ are diagonal matrices of variable parameters, $M_1 = \text{diag}\{m_{11}^{-1}, \dots, m_{1l}^{-1}\}$, $M_2 = \text{diag}\{m_{21}^{-1}, \dots, m_{2l}^{-1}\}$, where

$$m_{1j} \leq \alpha_{1j}, \quad m_{2j} \geq \alpha_{2j}. \quad (7)$$

Notice that m_{ij} ($i = 1, 2; j = 1, \dots, l$) may be either a certain number or ∞ . In the latter case we put

$m_{ij}^{-1} = 0$. For given $n \times n$ -matrices H the notation

$$\Re e H \triangleq \frac{1}{2}(H + H^*), \quad H \in \mathbb{R}^{n \times n} \quad (8)$$

is used, where the symbol $(*)$ means Hermitian conjugation. We shall also need the constants

$$\nu_j := \frac{\int_0^{\Delta_j} \varphi_j(\zeta) d\zeta}{\int_0^{\Delta_j} |\varphi_j(\zeta)| d\zeta}, \quad \nu_{0j} := \frac{\int_0^{\Delta_j} \varphi_j(\zeta) d\zeta}{\int_0^{\Delta_j} \Phi_j(\zeta) |\varphi_j(\zeta)| d\zeta}. \quad (9)$$

where

$$\Phi_j(\zeta) \triangleq \sqrt{(1 - m_{1j}^{-1} \varphi_j'(\zeta)) (1 - m_{2j}^{-1} \varphi_j'(\zeta))}. \quad (10)$$

3 Stability criteria.

Theorem 1. Suppose there exist positive definite matrices $\varkappa, \delta, \tau, \varepsilon$, matrices M_1 and M_2 , and numbers $a_j \in [0, 1]$ ($j = 1, \dots, l$) such that the following conditions are fulfilled:

- 1) for all $\omega \geq 0$ the frequency-domain inequality (6) is true;
- 2) the quadratic forms

$$Q_j(x, y, z) := \varepsilon_j x^2 + \delta_j y^2 + \tau_j z^2 + \varkappa_j a_j \nu_j x y + \varkappa_j (1 - a_j) \nu_{0j} y z \quad (j = 1, \dots, l) \quad (11)$$

are positive definite.

Then

$$\dot{\sigma}(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty \quad (12)$$

and

$$\sigma_j(t) \rightarrow q_j \quad \text{as } t \rightarrow +\infty, \quad (13)$$

where $\varphi_j(q_j) = 0$ ($j = 1, \dots, l$).

Proof. The proof is based on Popov's method of a priori integral indices [Popov, 1961, Popov, 1973, Rasvan, 2006, Yakubovich, 2002] which is traditionally used for stability investigation of integral and integro-differential equations. A prerequisite for Popov's method applicability is the existence of an integral quadratic constraint, satisfied by the nonlinearities.

Let $\eta(t) = \varphi(\sigma(t))$ and $\xi(t) = \eta(t) + g(t)$. Introduce the auxiliary function

$$\mu(t) \triangleq \begin{cases} 0, & t < 0, \\ t, & 0 \leq t \leq 1, \\ 1, & t > 1. \end{cases} \quad (14)$$

For $T \geq 1$ and a given solution of (1) consider the functions

$$\xi_T \triangleq \begin{cases} \mu(t)\xi(t), & t < T, \\ \xi(T)e^{c(T-t)} & t \geq T \quad (c > 0); \end{cases} \quad (15)$$

$$\sigma_0(t) = b(t) + R\xi(t-h)(1 - \mu(t-h)) - \int_0^t (1 - \mu(\tau))\gamma(t-\tau)\xi(\tau) d\tau, \quad (16)$$

$$\sigma_T(t) = R\xi_T(t-h) - \int_0^t \gamma(t-\tau)\xi_T(\tau) d\tau.$$

It is clear that

$$\dot{\sigma}(t) = \sigma_0(t) + \sigma_T(t) \quad \text{for } t \in [0, T]. \quad (17)$$

Consider a set of functionals ($T \geq 1$)

$$J_T \triangleq \int_0^\infty \{ \sigma_T^* \varkappa \xi_T + \xi_T^* \delta \xi_T + \sigma_T^* \varepsilon \sigma_T + (\sigma_T - M_1 \dot{\xi}_T)^* \tau (\sigma_T - M_2 \dot{\xi}_T) \} dt. \quad (18)$$

Due to Plancherel theorem we have

$$J_T = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \{ \mathfrak{F}(\sigma_T)^* \varkappa \mathfrak{F}(\xi_T) + \mathfrak{F}(\xi_T)^* \delta \mathfrak{F}(\xi_T) + \mathfrak{F}(\sigma_T)^* \varepsilon \mathfrak{F}(\sigma_T) + (\mathfrak{F}(\sigma_T) - M_1 \mathfrak{F}(\dot{\xi}_T))^* \tau (\mathfrak{F}(\sigma_T) - M_2 \mathfrak{F}(\dot{\xi}_T)) \} d\omega, \quad (19)$$

where \mathfrak{F} stands for the Fourier transform. Notice that

$$\mathfrak{F}(\sigma_T)(i\omega) = -K(i\omega)\mathfrak{F}(\xi_T)(i\omega) \quad (20)$$

and

$$\mathfrak{F}(\dot{\xi}_T)(i\omega) = i\omega \mathfrak{F}(\xi_T)(i\omega). \quad (21)$$

So

$$J_T = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathfrak{F}^*(\xi_T)(i\omega) \Pi(\omega) \mathfrak{F}(\xi_T)(i\omega) d\omega \quad (22)$$

and by virtue of (6)

$$J_T \leq 0. \quad (23)$$

On the other hand the following equality is true

$$J_T = \rho_T + J_0 + J_{1T} + J_{2T} + J_{3T}, \quad (24)$$

where

$$\rho_T \triangleq \int_0^T \{\dot{\sigma}^* \varkappa \xi + \xi^* \delta \xi + \dot{\sigma}^* \varepsilon \dot{\sigma} + (\dot{\sigma} - M_1 \dot{\xi})^* \tau (\dot{\sigma} - M_2 \dot{\xi})\} dt; \quad (25)$$

$$J_{1T} \triangleq - \int_0^T \{\sigma_0^* \varkappa \xi_T + 2\sigma_0^* \varepsilon \dot{\sigma} - \sigma_0^* (\varepsilon + \tau) \sigma_0 + \sigma_0^* \tau (\dot{\sigma} - M_2 \dot{\xi}_T) + (\dot{\sigma} - M_1 \dot{\xi}_T)^* \tau \sigma_0\} dt; \quad (26)$$

$$J_0 \triangleq \int_0^1 \{(\mu - 1) \dot{\sigma}^* \varkappa \xi + (\mu^2 - 1) \xi^* \delta \xi - (\mu \dot{\xi} - \dot{\xi})^* M_1 \tau \dot{\sigma} - \dot{\sigma}^* \tau M_2 (\mu \dot{\xi} - \dot{\xi}) + \mu \dot{\xi}^* M_1 \tau M_2 \mu \dot{\xi} - \dot{\xi}^* M_1 \tau M_2 \dot{\xi}\} dt; \quad (27)$$

$$J_{2T} \triangleq \int_T^\infty \{\sigma_T^* \varkappa \xi(T) e^{c(T-t)} + \xi(T)^* \delta \xi(T) e^{2c(T-t)} + c^2 \xi(T)^* M_1 \tau M_2 \xi(T) e^{2c(T-t)} - c e^{c(T-t)} \xi(T)^* (M_1 + M_2) \tau \sigma_T\} dt; \quad (28)$$

$$J_{3T} \triangleq \int_T^\infty \sigma_T^* (\varepsilon + \tau) \sigma_T dt. \quad (29)$$

Notice that from the assumptions 1)–5) it follows that $\eta(t)$, $\dot{\sigma}(t)$, $\dot{\eta}(t)$ are bounded on \mathbb{R}_+ ,

$$g(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty, \quad (30)$$

and

$$|\sigma_0(t)| < C_1 e^{-rt} \quad (C_1 > 0). \quad (31)$$

It is easy to see that

$$|J_{1T}| \leq C_2, \quad (32)$$

where C_2 does not depend on T .

It is evident that $\sigma_T(t)$ is also bounded on $\mathbb{R}_+ \forall t \geq T$ uniformly with respect to T . So

$$|J_{2T}| < C_3, \quad (33)$$

where C_3 does not depend on T . Then inequalities (23), (32), (33) together with (24) imply that

$$\rho_T < C_0, \quad \forall T > 1, \quad (34)$$

where C_0 does not depend on T .

As $\xi(t)$ is the sum of two functions we can divide the functional ρ_T into two parts:

$$\rho_T = \int_0^T G_1(\dot{\sigma}, \eta, \dot{\eta}) dt + \int_0^T G_2(\dot{\sigma}, \eta, \dot{\eta}, g, \dot{g}) dt. \quad (35)$$

where

$$G_1(\dot{\sigma}, \eta, \dot{\eta}) \triangleq \dot{\sigma}^* \varkappa \eta + \eta^* \delta \eta + \dot{\sigma}^* \varepsilon \dot{\sigma} + (\dot{\sigma} - M_1 \dot{\eta})^* \tau (\dot{\sigma} - M_2 \dot{\eta}); \quad (36)$$

$$G_2(\dot{\sigma}, \eta, \dot{\eta}, g, \dot{g}) \triangleq \dot{\sigma}^* \varkappa g + 2g^* \delta \eta + g^* \delta g - \dot{g}^* M_1 \tau (\dot{\sigma} - M_2 \dot{\eta}) - (\dot{\sigma} - M_1 \dot{\eta})^* \tau M_2 \dot{g} + \dot{g}^* M_1 \tau M_2 \dot{g}. \quad (37)$$

It is obvious that

$$\rho_T \geq \int_0^T G_1(\dot{\sigma}, \eta, \dot{\eta}) dt - \int_0^T |G_2(\dot{\sigma}, \eta, \dot{\eta}, g, \dot{g})| dt. \quad (38)$$

Let us evaluate the integral $\int_0^T |G_2| dt$. Notice that

$$\int_0^T g^* \delta g dt \geq 0 \quad (39)$$

and

$$\int_0^T |\dot{g}^* M_1 \tau M_2 \dot{g}| dt \leq C_4, \quad \forall T > 0. \quad (40)$$

On the other hand for any $\varepsilon_1 > 0$ the following inequalities are valid:

$$|\dot{\sigma}^* \varkappa g| \leq \varepsilon_1 |\dot{\sigma}|^2 + \frac{1}{4\varepsilon_1} (\max_{j=1, \dots, l} \varkappa_j^2) |g|^2; \quad (41)$$

$$|\eta^* \delta g| \leq \varepsilon_1 |\eta|^2 + \frac{1}{4\varepsilon_1} \left(\max_{j=1, \dots, l} \delta_j^2 \right) |g|^2; \quad (42)$$

$$\begin{aligned} |(\dot{\sigma} - M_1 \dot{\eta})^* \tau M_2 \dot{g}| &\leq \varepsilon_1 |\dot{\sigma} - M_1 \dot{\eta}|^2 + \\ &+ \frac{1}{4\varepsilon_1} \left(\max_{j=1, \dots, l} (\tau_j m_{2j}^{-1})^2 \right) |\dot{g}|^2; \end{aligned} \quad (43)$$

$$\begin{aligned} |g^* M_1 \tau (\dot{\sigma} - M_2 \dot{\eta})| &\leq \varepsilon_1 |\dot{\sigma} - M_2 \dot{\eta}|^2 + \\ &+ \frac{1}{4\varepsilon_1} \left(\max_{j=1, \dots, l} (\tau_j m_{1j}^{-1})^2 \right) |\dot{g}|^2. \end{aligned} \quad (44)$$

The identities

$$\begin{aligned} \dot{\sigma}_j - m_{ij}^{-1} \dot{\eta}_j &= \dot{\sigma}_j \left(1 - m_{ij}^{-1} \frac{d\varphi_j(\zeta)}{d\zeta} \right) \\ (i = 1, 2; j = 1, \dots, l) \end{aligned} \quad (45)$$

imply that

$$|\dot{\sigma} - M_i \dot{\eta}|^2 \leq C_5 |\dot{\sigma}|^2. \quad (46)$$

Introduce the functional

$$\begin{aligned} \rho_{0T} &\triangleq \int_0^T \{ \dot{\sigma}^* \varkappa \eta + \eta \delta_0 \eta + \dot{\sigma}^* \varepsilon_0 \dot{\sigma} + \\ &+ (\dot{\sigma} - M_1 \dot{\eta})^* \tau (\dot{\sigma} - M_2 \dot{\eta}) \} dt \end{aligned} \quad (47)$$

with diagonal matrices $\varepsilon_0 = \text{diag} \{ \varepsilon_{01}, \dots, \varepsilon_{0l} \}$, $\delta_0 = \text{diag} \{ \delta_{01}, \dots, \delta_{0l} \}$ defined by the formulas

$$\delta_0 = \delta - 2\varepsilon_1 E_l; \quad \varepsilon_0 = \varepsilon - (1 + 2C_5) \varepsilon_1 E_l. \quad (48)$$

From estimate (38) and inequalities (39)–(46) and assumption 3) it follows that

$$\rho_T \geq \rho_{0T} - C_6, \quad (49)$$

where a positive constant C_6 depends on ε_1 but does not depend on T .

Let us choose $\varepsilon_1 > 0$ so small that the quadratic forms

$$\begin{aligned} Q_{0j}(x, y, z) &:= \varepsilon_{0j} x^2 + \delta_{0j} y^2 + \tau_j z^2 + \\ &+ \varkappa_j \nu_j a_j x y + \varkappa_j \nu_{0j} (1 - a_j) y z \quad (j = 1, \dots, l) \end{aligned} \quad (50)$$

are positive definite. Once the number ε_1 is defined, the estimates (34) and (49) allow to affirm that

$$\rho_{0T} \leq C_7, \quad (51)$$

where C_7 does not depend on T .

Now we can apply special technique destined for stability investigation of control systems with periodic nonlinearities. Its main idea is to separate from periodic functions in ρ_{0T} the functions with null average on the period. We repeat here the argument of paper [Perkin et al., 2012]. New periodic functions are determined:

$$\begin{aligned} F_j(\zeta) &\triangleq \varphi_j(\zeta) - \nu_j |\varphi_j(\zeta)|, \\ \Psi_j(\zeta) &\triangleq \varphi_j(\zeta) - \nu_{0j} |\varphi_j(\zeta)| \Phi_j(\zeta). \end{aligned} \quad (52)$$

It is evident that

$$\int_0^{\Delta_j} F_j(\zeta) d\zeta = \int_0^{\Delta_j} \Psi_j(\zeta) d\zeta = 0. \quad (53)$$

The functional ρ_{0T} is decomposed into two summands:

$$\begin{aligned} \rho_{0T} &= \int_0^T \sum_{j=1}^l Q_{0j}(\dot{\sigma}_j, |\eta_j|, \Phi_j(\sigma_j) \dot{\sigma}_j) dt + \\ &+ \sum_{j=1}^l \left(\varkappa_j \int_{\sigma_j(0)}^{\sigma_j(T)} (a_j F_j(\zeta) + (1 - a_j) \Psi_j(\zeta)) d\zeta \right). \end{aligned} \quad (54)$$

It follows from (51) and (53) that

$$\int_0^T \sum_{j=1}^l Q_{0j}(\dot{\sigma}_j, |\eta_j|, \Phi_j(\sigma_j) \dot{\sigma}_j) dt < C_9, \quad (55)$$

where C_9 does not depend on T . Since all the quadratic forms Q_{0j} are positive definite the estimate (55) implies that

$$\begin{aligned} \dot{\sigma}_j(t) &\in L_2[0, +\infty), \\ \varphi_j(\sigma_j(t)) &\in L_2[0, +\infty) \quad (j = 1, \dots, l). \end{aligned} \quad (56)$$

Any $\varphi_j(\sigma(t))$ is uniformly continuous on $[0, +\infty)$. So according to Barbalat lemma [Popov, 1973] it tends to zero as $t \rightarrow +\infty$. Then $\sigma_j(t)$ tends to a zero of $\varphi_j(\zeta)$ as $t \rightarrow +\infty$. Since the functions $\dot{\sigma}_j(t)$ are uniformly continuous on $[0, +\infty)$, they tend to zero as $t \rightarrow +\infty$. Theorem 1 is proved. \square

Next we present a modification of Theorem 1. The following assertion contains the same frequency–domain condition but another algebraic requirements on variable parameters.

Let us introduce the functions

$$P_j(\zeta; \alpha, \beta) \triangleq \sqrt{1 + \frac{\alpha}{\beta} \Phi_j^2(\zeta)}, \quad (57)$$

where $\alpha > 0$ and $\beta > 0$ are parameters, and the constants

$$\nu_{1j}(\alpha, \beta) := \frac{\int_0^{\Delta_j} \varphi_j(\zeta) d\zeta}{\int_0^{\Delta_j} |\varphi_j(\zeta)| P_j(\zeta; \alpha, \beta) d\zeta}. \quad (58)$$

Theorem 2. *Suppose there exist positive definite matrices \varkappa , δ , τ , ε , matrices M_1 and M_2 such that for all $\omega \geq 0$ frequency-domain inequality (6) holds and algebraic inequalities*

$$2\sqrt{\varepsilon_j \delta_j} > |\nu_{1j}(\tau_j, \varepsilon_j)| \varkappa_j \quad (j = 1, \dots, l) \quad (59)$$

are true.

Then the conclusion of Theorem 1 is valid.

Proof. The proof of this theorem is based on the proof of Theorem 1. We repeat all the argument of Theorem 1 up to the formula (49) and choose $\varepsilon_1 > 0$ so small that the inequalities

$$2\sqrt{\varepsilon_{0j} \delta_{0j}} > |\nu_{1j}(\tau_j, \varepsilon_{0j})| \varkappa_j \quad (j = 1, \dots, l) \quad (60)$$

are true. The inequality (49) implies the inequality (51). Then we determine new periodic functions

$$Y_j(\zeta) \triangleq \varphi_j(\zeta) - \nu_{1j}(\tau_j, \varepsilon_{0j}) |\varphi_j(\zeta)| P_j(\zeta; \tau_j, \varepsilon_{0j}). \quad (61)$$

Notice that

$$\int_0^{\Delta_j} Y_j(\zeta) d\zeta = 0. \quad (62)$$

With the help of Y_j and the quadratic forms

$$W_j(x, y) = \varepsilon_{0j} x^2 + \delta_{0j} y^2 + \varkappa_j \nu_{1j}(\tau_j, \varepsilon_{0j}) xy \quad (63)$$

the functional ρ_{0T} can be decomposed as follows

$$\begin{aligned} \rho_{0T} = & \int_0^T \sum_{j=1}^l W_j(\dot{\sigma}_j P_j(\sigma_j; \tau_j, \varepsilon_{0j}), |\eta_j|) dt + \\ & + \sum_{j=1}^l \int_{\sigma_j(0)}^{\sigma_j(T)} \varkappa_j Y_j(\zeta) d\zeta. \end{aligned} \quad (64)$$

It follows from (51) and (62) that

$$\int_0^T \sum_{j=1}^l W_j(\dot{\sigma}_j P_j(\sigma_j; \tau_j, \varepsilon_{0j}), \eta_j) dt < C_{10}, \quad (65)$$

where C_{10} does not depend on T . In virtue of (60) each W_j is positive definite. Then (65) implies (56). Thus Theorem 2 is proved.

Example. Theorem 1 was applied to the phase-locked loop (PLL) with integrating filter and sine-shaped characteristic of phase detector. For the system $l = 1$, $n = 1$,

$$K(p) = \frac{T}{Tp + 1} \quad (T > 0), \quad (66)$$

$$\psi_1(\zeta) = \sin \zeta - \beta \quad (\beta \in (0, 1)). \quad (67)$$

The mathematical description of this PLL coincides with that of mathematical pendulum

$$\ddot{\sigma} + a\dot{\sigma} + (\sin \sigma - \beta) = 0 \quad (a = \frac{1}{T}). \quad (68)$$

By computer simulation, for various coefficients T the values β_T such that the system (66), (67) with $\beta \leq \beta_T$ is stable were established. The latter were compared with frontier values of stability domain on the plane $\{T^2, \beta\}$ computed by qualitatively-numerical methods [Belyustina et al., 1970]. It turned out that Theorem 1 guarantees no less than 80% of genuine stability domain. \square

4 Extended frequency-domain criteria

In this section we suppose that $R = 0$. We assume also that the functions $b(\cdot)$ and $\gamma(\cdot)$ are differentiable on $[0, +\infty)$ and

$$|\dot{b}(t)| + |\dot{\gamma}(t)| \leq C e^{-rt}. \quad (69)$$

Theorem 3. *Suppose there exist positive definite matrices \varkappa , δ , ε , matrices M_i ($i = 1, 2$), matrix τ with $\tau_j \geq 0$, and numbers $a_j \in [0, 1]$ such that the following conditions are fulfilled:*

1) for a certain $\lambda \in [0; \frac{\tau}{2})$ and all $\omega \geq 0$ the frequency-domain inequality

$$\begin{aligned} \Pi_\lambda(\omega) \triangleq & \Re\{\varkappa K(i\omega - \lambda) - (K(i\omega - \lambda) + \\ & + (i\omega - \lambda)M_1)^* \tau (K(i\omega - \lambda) + (i\omega - \lambda)M_2) - \\ & - K(i\omega - \lambda)^* \varepsilon K(i\omega - \lambda)\} - \delta \geq 0. \end{aligned} \quad (70)$$

is true;

2)

$$\tau_j m_{1j}^{-1} m_{2j}^{-1} = 0, \quad \forall j = 1, 2, \dots, l; \quad (71)$$

3) the quadratic forms

$$\begin{aligned} Q'_j(x, y, z) := & \varepsilon_j x^2 + \delta_j y^2 + \tau_j z^2 + \bar{\varkappa}_j a_j \nu_j xy + \\ & + \bar{\varkappa}_j (1 - a_j) \nu_{0j} yz \quad (j = 1, \dots, l), \end{aligned} \quad (72)$$

where $\bar{\varkappa}_j = \varkappa_j + 2\lambda\tau_j(m_{1j}^{-1} + m_{2j}^{-1})$ and $a_j = 1$ if $\tau_j = 0$, are positive definite.

Then the conclusion of Theorem 1 is true.

Proof. Throughout the proof we use the notation

$$[f]^s(t) \triangleq f(t)e^{st}. \quad (73)$$

We shall also use the functions defined in the previous section. The whole scheme of the proof is alike that of Theorem 1, though the "cut-function" $\xi_T(t)$ is replaced here by the "cut-function"

$$\xi_T^1(t) = \begin{cases} \mu(t)\xi(t), & t \leq T, \\ 0, & t > T. \end{cases} \quad (74)$$

Let

$$\sigma_T^1(t) = - \int_0^t \gamma(t-\tau)\xi_T^1(\tau) d\tau. \quad (75)$$

Consider the functions $[\xi_T^1]^\lambda(t)$ and $[\sigma_T^1]^\lambda(t)$. We have

$$[\sigma_T^1]^\lambda(t) = - \int_0^t [\gamma]^\lambda(t-\tau)[\xi_T^1]^\lambda(\tau) d\tau. \quad (76)$$

It is obvious that

$$\dot{\sigma}(t) = \sigma_0(t) + \sigma_T^1(t) \quad \text{for } t \in [0, T]. \quad (77)$$

It is also clear that $\forall T > 1$

$$|[\xi_T^1]^\lambda|, |[\sigma_T^1]^\lambda(t)| \in L_2[0, +\infty) \cap L_1[0, +\infty), \quad (78)$$

and

$$|[\dot{\xi}_T^1]^\lambda|, |[\dot{\sigma}_T^1]^\lambda(t)| \in L_2[0, +\infty) \cap L_1[0, +\infty). \quad (79)$$

It is easy to demonstrate that

$$|[\dot{\sigma}_0]^\lambda(t)|, |[\sigma_0]^\lambda(t)| \in L_2[0, +\infty) \cap L_1[0, +\infty). \quad (80)$$

Notice that

$$\mathfrak{F}([\sigma_T^1]^\lambda) = -K(i\omega - \lambda)\mathfrak{F}([\xi_T^1]^\lambda) \quad (81)$$

Consider the functional

$$\begin{aligned} J_{\lambda T}^1 \triangleq & \int_0^\infty \{([\sigma_T^1]^\lambda)^*(\varkappa + \lambda M_1 \tau + \\ & + \lambda \tau M_2)[\xi_T^1]^\lambda + ([\sigma_T^1]^\lambda)^*(\varepsilon + \tau)[\sigma_T^1]^\lambda + \\ & + (\frac{d}{dt}([\sigma_T^1]^\lambda))^* \tau M_2 [\xi_T^1]^\lambda + ([\xi_T^1]^\lambda)^* M_1 \tau \frac{d}{dt}[\sigma_T^1]^\lambda + \\ & + ([\xi_T^1]^\lambda)^* \delta([\xi_T^1]^\lambda)\} dt. \end{aligned} \quad (82)$$

Due to Plancherel theorem

$$J_{\lambda T}^1 = -\frac{1}{2\pi} \int_0^\infty \{\mathfrak{F}^*([\xi_T^1]^\lambda)\Pi_\lambda(\omega)\mathfrak{F}([\xi_T^1]^\lambda)\} d\omega. \quad (83)$$

So it follows from (70) that

$$J_{\lambda T}^1 \leq 0. \quad (84)$$

On the other hand by virtue of (77), (78), (79) and the special form of ξ_T^1 one can conclude that

$$J_{\lambda T}^1 \geq \rho_{\lambda T}^1 + J_{\lambda T}^0 \quad (85)$$

where

$$\begin{aligned} \rho_{\lambda T}^1 \triangleq & \int_0^T \{([\dot{\sigma}]^\lambda)^*(\varkappa + \lambda M_1 \tau + \\ & + \lambda \tau M_2)[\xi]^\lambda + ([\dot{\sigma}]^\lambda)^*(\varepsilon + \tau)[\dot{\sigma}]^\lambda + \\ & + (\frac{d}{dt}([\dot{\sigma}]^\lambda))^* \tau M_2 [\xi]^\lambda + ([\xi]^\lambda)^* M_1 \tau \frac{d}{dt}([\dot{\sigma}]^\lambda) + \\ & + ([\xi]^\lambda)^* \delta([\xi]^\lambda)\} dt, \end{aligned} \quad (86)$$

$$|J_{\lambda T}^0| \leq C_{12}, \quad (87)$$

and C_{12} does not depend on T .

Consider now the functional

$$\begin{aligned} \rho_{\lambda T} = & \int_0^T \{\dot{\sigma}^*(\varkappa + 2\lambda(M_1 \tau + \tau M_2))\xi + \xi^* \delta \xi + \\ & + \dot{\sigma}^* \varepsilon \dot{\sigma} + (\dot{\sigma} - M_1 \dot{\xi})^* \tau (\dot{\sigma} - M_2 \dot{\xi})\} dt \end{aligned} \quad (88)$$

We have by virtue of (71)

$$\begin{aligned} \rho_{\lambda T} = & \int_0^T \{\dot{\sigma}^*(\varkappa + 2\lambda(M_1 \tau + \tau M_2))\xi + \xi^* \delta \xi + \\ & + \dot{\sigma}^*(\varepsilon + \tau)\dot{\sigma} - \dot{\sigma}^* \tau M_2 \dot{\xi} - \dot{\xi}^* M_1 \tau \dot{\sigma}\} dt. \end{aligned} \quad (89)$$

With the help of equalities

$$\begin{aligned} \int_0^T \dot{\sigma}^* \tau M_2 \dot{\xi} dt &= \dot{\sigma}^* \tau M_2 \xi \Big|_0^T - \\ &- \int_0^T \ddot{\sigma}^* \tau M_2 \xi dt, \end{aligned} \quad (90)$$

$$\begin{aligned} \int_0^T \xi^* M_1 \tau \dot{\sigma} dt &= \dot{\xi}^* M_1 \tau \dot{\sigma} \Big|_0^T - \\ &- \int_0^T \xi M_1 \tau \ddot{\sigma} dt, \end{aligned} \quad (91)$$

we get

$$\rho_{\lambda T} \leq I_{\lambda T} + C_{13}, \quad (92)$$

where

$$\begin{aligned} I_{\lambda T} = & \int_0^T \{ \dot{\sigma}^*(\varkappa + 2\lambda(M_1\tau + \tau M_2))\xi + \xi^*\delta\xi + \\ & + \dot{\sigma}^*(\varepsilon + \tau)\dot{\sigma} + \ddot{\sigma}^*\tau M_2\xi + \xi^*M_1\tau\ddot{\sigma} \} dt. \end{aligned} \quad (93)$$

and C_{13} does not depend on T . Further

$$\begin{aligned} I_{\lambda T} = & \int_0^T e^{-2\lambda t} \{ ([\dot{\sigma}]^\lambda)^*(\varkappa + 2\lambda(M_1\tau + \tau M_2))[\xi]^\lambda + \\ & + ([\xi]^\lambda)^*\delta[\xi]^\lambda + ([\dot{\sigma}]^\lambda)^*(\varepsilon + \tau)[\dot{\sigma}]^\lambda + \\ & + \left(\frac{d}{dt}([\dot{\sigma}]^\lambda) - \lambda([\dot{\sigma}]^\lambda)^*\tau M_2[\xi]^\lambda + \right. \\ & \left. + ([\xi]^\lambda)^*M_1\tau\left(\frac{d}{dt}([\dot{\sigma}]^\lambda) - \lambda([\dot{\sigma}]^\lambda)\right) \} dt = \rho_{\lambda T}^1, \end{aligned} \quad (94)$$

where $T' \in [0, T]$. From (84), (87) and (92) it follows that

$$\rho_{\lambda T} \leq C_{14}, \quad (95)$$

where C_{14} does not depend on T .

The formula (95) is alike the formula (34). So from this we can repeat the argument of Theorem 1.

Theorem 4. *Suppose there exist positive definite matrices \varkappa , δ , ε , matrices M_1 and M_2 , and nonnegative matrix τ such that following conditions are fulfilled:*

- 1) *for a certain $\lambda \in [0; \frac{\tau}{2})$ and all $\omega \geq 0$ the frequency-domain inequality (70) is true;*
- 2) $\tau_j m_{1j}^{-1} m_{2j}^{-1} = 0, \forall j = 1, 2, \dots, l;$
- 3)

$$2\sqrt{\varepsilon_j \delta_j} > |\nu_{1j}(\tau_j, \varepsilon_j)| \bar{\varkappa}_j \quad (j = 1, \dots, l), \quad (96)$$

where $\bar{\varkappa}_j = \varkappa_j + 2\lambda\tau_j(m_{1j}^{-1} + m_{2j}^{-1})$.

Then the conclusion of Theorem 1 is valid.

The proof of Theorem 4 follows from that of Theorem 3 and of Theorem 2.

5 Conclusion

In this paper we consider the infinite-dimensional synchronization system with uncertain disturbances. The system has a denumerable set of stable and unstable equilibria. We offer a number of novel frequency-domain criteria which guarantee that any solution of the system converges to one of equilibria. The techniques used stem from Popov's method of "a priori integral indices". New types of Popov functionals destined for systems with periodic nonlinearities are exploited.

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