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GELIG'S AVERAGING METHOD FOR LOCAL STABILIZATION OF A CLASS OF NONLINEAR SYSTEMS BY A PULSE-WIDTH MODULATED CONTROL

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Abstract

A stabilization problem for a nonlinear system with a sector bound nonlinearity and a pulse-width modulated (PWM) feedback is considered. The linear matrix inequalities (LMI) technique is used to estimate the domain of attraction for the zero equilibrium of the closed system.

Key words

Nonlinear systems, hybrid systems, networked systems.

1 Introduction

The subject of this study is a controlled nonlinear system with a pulse-width modulator in the feedback. The system comprises a nonlinear continuous-time subsystem with a sector bound nonlinearity and a modulator, which transforms a continuous signal into a train of rectangular pulses that are emitted at a given constant frequency, but their widths vary (are modulated). The pulse signal can take only three discrete values -1, 0 and 1.

The idea of a PWM control goes back to the end of the XIXth century (see [Gouy, 1897]). Because of its simple engineering implementation, PWM gained a great popularity and has been a subject of many mathematical studies (see, among others, [Skoog and Blankenship, 1970; Kuntsevich and Chekhovoi, 1970; Kuntsevich and Chekhovoi, 1971; Tsypkin and Popkov, 1973; Kipnis, 1992; Gelig and Churilov, 1998; Yuan et al., 1998; Zhusubaliyev and Mosekilde, 2003; Massioni et al., 2019]). The two main mathematical approaches can be mentioned. In the case when the continuous part of a system is linear, the reduction to discrete-time equations is frequently applied (see, e. g., [Kadota and Bourne, 1961; Hou and Michel, 2001; De Koning, 2003; Asai, 2006; Tomita and Asai, 2006; Almér et al., 2007; Síra-Ramirez et al., 2015]). Another approach relies on averaging of the impulsive signal in sampling periods (see [Andeen, 1960a; Andeen, 1960b; Síra-Ramirez, 1989; Taylor, 1992; Sakamoto and Hori, 2002; Sakamoto et al., 2002]). The Gelig's method of averaging based on the absolute stability theory was proposed and developed in [Gelig, 1982; Gelig and Churilov, 1993b; Gelig and Churilov, 1996; Gelig and Churilov, 1998; Gelig, 2009]). Further it was refined with the help of the integral quadratic constraints (IQC) theory, see [Chaudenson et al., 2013; Chaudenson, 2013; Fetzer and Scherer, 2016; Fetzer, 2017]. However, the PWM systems considered in the latter works had linear continuous parts.

Since a PWM control signal is bounded, we can rarely achieve stabilization for all the initial data. However, the stabilization problem can be solved locally, in a vicinity of the zero point (cf. [Asai, 2006; Tomita and Asai, 2006]). A domain of attraction of the zero equilibrium is estimated using the computational technique of linear matrix inequalities (LMI. see [Boyd et al., 1994]).

In this paper we will explore stability of a PWM system with the help of the Gelig's version of the averaging method. We develop further the technique proposed previously in [Churilov, 2019c] to extend it to the case of unsaturated behavior.

2 Problem Setting

Consider a controlled nonlinear system

$$\dot{x} = Ax(t) + B_0 f(t) + Bu(t), \tag{1}$$

$$\eta(t) = C_0 x(t), \quad \sigma(t) = Cx(t). \tag{2}$$

Here A, B_0, B, C_0, C are constant matrix coefficients of sizes $m\times m, m\times 1, m\times 1, 1\times m, 1\times m$, respectively, $x(t)$ is an m-dimensional state vector, $\eta(t)$, $\sigma(t)$ are scalar intrinsic functions, $u(t)$ is a scalar control function.

Assume that $f(t) = \varphi_0(\eta(t), t)$, where the nonlinearity $\varphi_0(\cdot, \cdot)$ satisfies a sector bound

$$\mu_1 \leqslant \frac{\varphi_0(\eta, t)}{\eta} \leqslant \mu_2, \quad \forall \eta \neq 0,\ \forall t \geqslant 0, \tag{3}$$

where μ_1, μ_2 are some constants. Thus for any t the graph of $\varphi_0(\cdot, t)$ lies in a conic sector in the plane.

The control function $u(t)$ is defined as PWM signal. Let $T > 0$ be a constant sampling period, and

$$u(t) = \begin{cases} u_n, & nT \leqslant t < nT + \tau_n, \\ 0, & nT + \tau_n \leqslant t < (n+1)T, \end{cases} \tag{4}$$

$n = 0, 1, \ldots$. Here

$$u_n = \operatorname{sgn} \sigma(nT), \tag{5}$$

$$\tau_n = \begin{cases} \frac{T}{\sigma_*}\,|\sigma(nT)|, & |\sigma(nT)| \leqslant \sigma_*, \\ T, & |\sigma(nT)| \geqslant \sigma_*, \end{cases} \tag{6}$$

where σ_* is a given threshold. Thus the function in the right-hand side of (6) has regions of linearity and of saturation. The control law (5), (6) implements a lead-type PWM described, e. g., in [Andeen, 1960a; Kadota and Bourne, 1961].

Obviously, system (1)–(6) has a zero equilibrium. We will be interested in the case when without a control, i. e. when $u(t) \equiv 0$, this equilibrium is unstable. Further we will provide conditions for its local stability and give some ellipsoid estimates for its region of attraction. This paper extends our previous work [Churilov, 2019c] that addressed the case, when the system's operation was restricted to unsaturated widths, namely $\tau_n < T$. Here we admit that some of the widths can be saturated, i. e., $\tau_n = T$. Another improvement is that we consider solutions' behaviors not only at sampling times, but also between them.

3 Averaging Method

For brevity we will use notation $t_n = nT$.

Our analysis will be based on the Gelig's version of the averaging method [Gelig, 1982; Gelig and Churilov, 1998] and some additional mathematical technique from [Churilov, 2018; Churilov, 2019a; Churilov, 2019c]. The square of the nth pulse (taking the sign into account) can be calculated by the formula

$$v_n = \tfrac{1}{T}\, u_n \tau_n\,. \tag{7}$$

Introduce a nonlinear function called *equivalent nonlinearity*

$$\varphi(\sigma) = \begin{cases} \frac{1}{\sigma_*}\,\sigma, & |\sigma| \leqslant \sigma_*, \\ \operatorname{sgn}\sigma, & |\sigma| \geqslant \sigma_*. \end{cases} \tag{8}$$

Nonlinearity (8) presents a saturation function (see [Tarbouriech et al., 2011]). From (6) the following statement is valid (see [Gelig, 1982]):

$$v_n = \varphi(\sigma(nT)) \tag{9}$$

for all $n \geqslant 0$.

Let θ_* be a number, $\theta_* \geqslant \sigma_*$. Then

$$\tfrac{1}{\theta_*} \leqslant \frac{\varphi(\sigma)}{\sigma} \leqslant \tfrac{1}{\sigma_*}, \quad \forall \sigma,\ 0 < |\sigma| \leqslant \theta_* \tag{10}$$

(see Fig. 1). Thus if $|\sigma(nT)| \leqslant \theta_*$, we have an instant quadratic constraint

$$(\sigma(nT) - \sigma_* v_n)(\theta_* v_n - \sigma(nT)) \geqslant 0. \tag{11}$$

(The term "instant quadratic constraint" was introduced by A. Gelig, it means a quadratic constraint taken at a discrete time instant.) In [Churilov, 2019c] the local stabilization problem was addressed for the special case $\theta_* = \sigma_*$ (that is the analysis was limited to the linearity region of $\varphi(\sigma)$). In this paper the general case is considered.

Let us define two auxiliary scalar functions. Firstly, take a piecewise constant function

$$v(t) = v_n, \quad t_n \leqslant t < t_{n+1}.$$

Secondly, consider a function $w(t)$ that is an integrated error of replacing $u(t)$ with $v(t)$:

$$w(t) = \int_0^t (u(s) - v(s))\, ds, \quad t \geqslant 0.$$

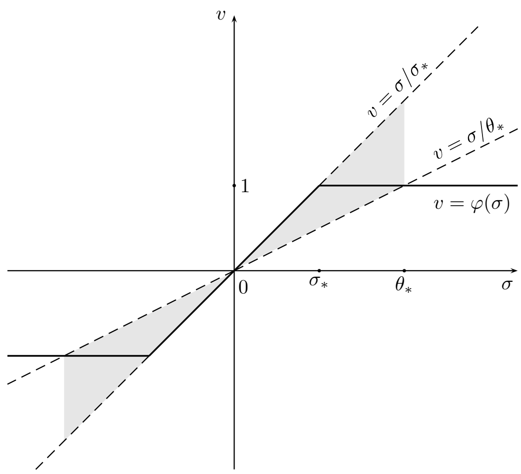

Figure 1. An illustration to the local quadratic constraint (10). The points (σ, v), $v = \varphi(\sigma)$, lie in the part of a conic sector with $-\theta_* \leqslant \sigma \leqslant \theta_*$ (filled grey).

The function $w(t)$ is continuous for $t \geqslant 0$ with $w(t_n) = 0$. Such a function was used in monographs [Gelig and Churilov, 1993a; Gelig and Churilov, 1998] and in a number of other publications on the Gelig's averaging method. Later functions with the property $w(t_n) = w(t_{n+1}) = 0$ became known under the name "looped functions" [Briat and Seuret, 2012]. Notice that the derivative of $w(t)$ has gaps at $t = t_n$.

By a direct calculation we obtain

$$w(t) = v_n\, w_n(t), \quad t_n < t < t_{n+1},$$

where (cf. [Chaudenson, 2013])

$$w_n(t) = \begin{cases} \frac{T-\tau_n}{\tau_n}(t - t_n), & t_n \leqslant t \leqslant t_n + \tau_n, \\ t_{n+1} - t, & t_n + \tau_n \leqslant t \leqslant t_{n+1}. \end{cases}$$

Obviously

$$0 \leqslant w_n(t) \leqslant T, \quad t_n \leqslant t \leqslant t_{n+1},$$

and hence we get a quadratic constraint

$$Tv(t)w(t) \geqslant w^2(t), \quad t \geqslant 0. \tag{12}$$

4 Main Statement

Let θ_* be a number, $\theta_* \geqslant \sigma_*$. Introduce the following $(m + 4)$-dimensional rows:

$$\begin{aligned} D_0 &= [\, CA \quad CB_0 \quad CB \quad 0 \quad 0], \\ D_1 &= [\, C \quad 0 \quad -\sigma_* \quad -CB \quad -1], \\ D_2 &= [\, -C \quad 0 \quad \theta_* \quad CB \quad 1]. \end{aligned} \tag{13}$$

Theorem 1. *Assume that there exist a symmetric positive definite $m \times m$ matrix H and nonnegative numbers ε_i, $i = 0, \ldots, 4$, such that linear matrix inequalities (LMI)*

$$\Pi + \varepsilon_2 \Delta^2 D_0^\top D_0 + \varepsilon_4 (D_1^\top D_2 + D_2^\top D_1) < 0, \tag{14}$$

$$\begin{bmatrix} H & C^\top \\ C & \theta_*^2 \end{bmatrix} > 0 \tag{15}$$

are satisfied. The inequalities are understood in the sense of definiteness of quadratic forms. Here $\Delta = 2T/\pi$, Π is a symmetric $(m + 4) \times (m + 4)$ matrix with the block components

$$\begin{aligned} &\Pi_{11} = HA + A^\top H - \varepsilon_0 \mu_1 \mu_2 C_0^\top C_0, \\ &\Pi_{12} = HB_0 + \tfrac{1}{2}\varepsilon_0(\mu_1 + \mu_2) C_0^\top, \\ &\Pi_{13} = HB, \quad \Pi_{14} = -A^\top HB, \\ &\Pi_{15} = \varepsilon_3 A^\top C^\top, \quad \Pi_{22} = -\varepsilon_0, \quad \Pi_{23} = 0, \\ &\Pi_{24} = -B^\top HB_0, \quad \Pi_{25} = \varepsilon_3\, CB_0, \\ &\Pi_{33} = -1, \quad \Pi_{34} = T\varepsilon_1 - B^\top HB, \\ &\Pi_{35} = \varepsilon_3\, CB, \quad \Pi_{44} = -2\varepsilon_1, \\ &\Pi_{45} = 0, \quad \Pi_{55} = -\varepsilon_2. \end{aligned}$$

Here $\Pi_{ij} = \Pi_{ji}$ ($1 \leqslant i < j \leqslant 5$), $\top$ denotes matrix transpose. Consider an ellipsoid bound space

$$\mathcal{E} = \{x \in \mathbb{R}^m \mid x^\top Hx \leqslant 1\}.$$

Then any solution of system (1)–(6) *with the initial data $x(0) \in \mathcal{E}$ satisfies $x(t) \to 0$ as $t \to +\infty$. Moreover,*

$$x(t) - Bw(t) \in \mathcal{E}, \quad \forall\, t \geqslant 0,$$

and

$$\begin{aligned} &|\sigma(nT)| < \theta_*, \quad \forall\, n \geqslant 0, \\ &|\sigma(t)| < \theta_* + |CB|\, T, \quad \forall\, t \geqslant 0. \end{aligned}$$

The blocks Π_{ij} contain parameters (A, B_0, C_0, μ_1, μ_2) of the system to be controlled, as well as parameters (B, C, T, σ_*) related to the impulsive control. The conditions of Theorem 1 present a feasibility problem of semi-definite programming (with variables H, ε_i, $i = 0, \ldots, 4$) that can be explored by standard software packages. If inequalities (14), (15) are soluble, they usually have an infinite number of feasible solutions. To make a specific choice of $\mathcal{E}$ inequalities (14), (15) can be augmented by a target condition (preserving convexity), e. g. we can extremize some convex function of H.

5 Proof of the Main Statement

For brevity, introduce notation $t_n = nT$ for $n \geqslant 0$.

Inequality (15) and the Schur complement formula [Boyd et al., 1994] imply

$$\theta_*^2 > CH^{-1}C^\top. \tag{16}$$

Since

$$\max_{x \in \mathcal{E}} |Cx| = \sqrt{CH^{-1}C^\top},$$

from inequality (16) we obtain

$$\mathcal{E} \subset \{x \in \mathbb{R}^m \mid |Cx| < \theta_*\}. \tag{17}$$

Thus if $x(t_n) \in \mathcal{E}$ then $|\sigma(t_n)| < \theta_*$.

Let us prove that if $x(0) \in \mathcal{E}$, then $x(t_n) \in \mathcal{E}$ for all $n \geqslant 0$. This implies that $|\sigma(t_n)| < \theta_*$ for all $n \geqslant 0$.

Introduce an auxiliary function

$$\xi(t) = \sigma(t) - \sigma(t_n) - CB\, w(t), \tag{18}$$

for $t_n \leqslant t < t_{n+1}$. This implies

$$\dot{\xi}(t) = CAx(t) + CB_0\, f(t) + CB\, v(t) \tag{19}$$

for $t \neq t_n$, $n \geqslant 0$. Define an $(m+4)$-dimensional vector column

$$X(t) = \mathrm{col}\{x(t),\, f(t),\, v(t),\, w(t),\, \xi(t)\}.$$

From (13), (19) it is seen that

$$\dot{\xi}(t) = D_0 X(t). \tag{20}$$

Since the right-sided limit $\xi(t_n^+) = 0$, The Wirtinger inequality implies an integral quadratic constraint (see [Gelig and Churilov, 1993b; Gelig and Churilov, 1998])

$$\int_{t_n}^{t} \xi^2(s)\, ds \leqslant \Delta^2 \int_{t_n}^{t} \dot{\xi}^2(s)\, ds \tag{21}$$

for any t, $t_n \leqslant t \leqslant t_{n+1}$, here $\Delta = 2T/\pi$. With the help of (20) inequality (21) can be rewritten as

$$\int_{t_n}^{t} \xi^2(s)\, ds \leqslant \Delta^2 \int_{t_n}^{t} (D_0 X(s))^2\, ds. \tag{22}$$

Additionally, we have an obvious constraint

$$\int_{t_n}^{t} \xi(s) D_0 X(s)\, ds = \int_{t_n}^{t} \xi(s)\dot{\xi}(s)\, ds = \tfrac{1}{2}\, \xi^2(t^-) \geqslant 0 \tag{23}$$

for any t, $t_n \leqslant t \leqslant t_{n+1}$.

Further, we will rearrange instant quadratic constraint (11) with the help of the vectors $X(t)$, D_1, D_2.

Lemma 1. *Let* $|\sigma(t_n)| \leqslant \theta_*$. *Then* (11) *implies*

$$D_1 X(t)\, D_2 X(t) \geqslant 0, \quad t_n \leqslant t < t_{n+1}. \tag{24}$$

Proof. From (18) we obtain

$$\sigma(t_n) = \sigma(t) - \xi(t) - CB\, w(t). \tag{25}$$

Substituting (25) into inequality (11) we come to

$$\begin{aligned}(\sigma(t) - \xi(t) - CB\, w(t) - \sigma_* v(t))& \\ \times (\theta_* v(t) - \sigma(t) + \xi(t) + CB\, w(t)) \geqslant 0&,\end{aligned} \tag{26}$$

which is equivalent to (24). □

Notice that for the special case $\theta_* = \sigma_*$ considered in [Churilov, 2019c] one gets $D_1 = -D_2$ and thus (24) reduces to the equality $D_1 X(t) \equiv 0$. This allows to decrease the number of variables by expressing $\xi(t)$ through $x(t)$, $v(t)$, $w(t)$.

We will apply the S-procedure with multiple quadratic forms (see [Gusev and Likhtarnikov, 2006]). For an $(m+4)$-dimensional column vector

$$X = \mathrm{col}\{x,\, f,\, v,\, w,\, \xi\}, \tag{27}$$

where x is an m-dimensional vector, f, v, w, ξ are scalars, consider a quadratic form

$$\begin{aligned} W(X) = {}& \varepsilon_0(\mu_2 C_0 x - f)(f - \mu_1 C_0 x) \\ &+ 2\varepsilon_1\big(Tvw - w^2\big) \\ &+ \varepsilon_2\big[\Delta^2 (D_0 X)^2 - \xi^2\big] \\ &+ 2\varepsilon_3\, \xi\, D_0 X + 2\varepsilon_4\, D_1 X\, D_2 X. \end{aligned} \tag{28}$$

Here ε_i, $i = 0, \ldots, 4$, are the same as in the formulation of Theorem 1. It can be easily verified that inequality (14) implies negative definiteness of the quadratic form

$$2(x - Bw)^\top H(Ax + B_0 f + Bv) + W(X)\,.$$

Hence there exists a sufficiently small number $\delta_0 > 0$ such that

$$2(x - Bw)^\top H(Ax + B_0 f + Bv) + W(X) \leqslant -\delta_0 \|X\|^2 \tag{29}$$

for all vector columns X with coordinates (27).

Let us take a quadratic Lyapunov function

$$V(x, w) = (x - Bw)^\top H(x - Bw).$$

Then along the solutions of system (1)–(6) inequality (29) implies

$$\dot{V}(x(t), w(t)) + W(X(t)) \leqslant -\delta_0 \|X(t)\|^2 \tag{30}$$

for any sampling interval $t_n < t < t_{n+1}$.

From quadratic bounds (3), (12), (24) and integral quadratic constraints (22), (23) we obtain:

$$\int_{t_n}^{t} W(X(s))\, ds \geqslant 0,\ \forall t,\ t_n \leqslant t \leqslant t_{n+1},\ \forall n \geqslant 0. \tag{31}$$

Notice that (24) needs an additional supposition $|\sigma(t_n)| \leqslant \theta_*$ to hold.

Assume that $x(t_n) \in \mathcal{E}$ for some n, and hence $|\sigma(t_n)| \leqslant \theta_*$. Recall that $w(t_n) = w(t_{n+1}) = 0$. Integrating (30) over the interval $[t_n,\, t_{n+1}]$ and taking (31) into account, we get

$$\begin{aligned} V(x(t_{n+1}), 0) - V(x(t_n), 0)& \\ \leqslant -\delta_0 \int_{t_n}^{t_{n+1}} \|X(t)\|^2 dt&. \end{aligned} \tag{32}$$

From (32) it follows that $x(t_{n+1}) \in \mathcal{E}$ and hence $|\sigma(t_{n+1})| \leqslant \theta_*$. Thus if an initial value is taken so that $x(0) \in \mathcal{E}$, then $x(t_n) \in \mathcal{E}$ and $|\sigma(t_n)| \leqslant \theta_*$ for all $n \geqslant 0$. Hence we arrive at the statement: if $x(0) \in \mathcal{E}$ then (32) is valid for all $n > 0$. Since $H > 0$, for any $n > 0$ we have

$$\int_0^{t_n} \|X(t)\|^2 dt \leqslant \tfrac{1}{\delta_0}\, V(x(0), 0),$$

so the function $\|X(t)\|$ is square integrable, that is $\|X(\cdot)\| \in L^2([0, +\infty))$. In particular, this implies $\|x(\cdot)\| \in L^2([0, +\infty))$.

Inequality (32) also implies that the sequence $V(x_n, 0)$ is bounded for $n \neq 0$, and hence the sequence $x(t_n)$, $n \geqslant 0$, is also bounded. From (30) we also get

$$V(x(t), w(t)) \leqslant V(x(t_n), 0), \quad t_n \leqslant t \leqslant t_{n+1}. \tag{33}$$

Thus the function $\|x(t) - Bw(t)\|$ is bounded for $t \geqslant 0$. Since $|v_n| \leqslant 1$, we have $|w(t)| \leqslant T$ for $t \geqslant 0$, so the function $\|x(t)\|$ is bounded for $t \geqslant 0$. We have $|u(t)| \leqslant 1$ for $t \geqslant 0$, and

$$|f(t)| = |\varphi_0(\sigma_0(t), t)| \leqslant \max\{|\mu_1|, |\mu_2|\}|\sigma_0(t)|,$$

so the right-hand side of (1) is bounded for $t \geqslant 0$, and the function $\|x(t)\|^2$ is uniformly continuous. Applying the Barbălat's lemma [Popov, 1973], we conclude that $x(t) \to 0$ as $t \to +\infty$.

From (33) and (17) we also obtain that if $x_0 \in \mathcal{E}$, then $x(t) - Bw(t) \in \mathcal{E}$ and

$$|\sigma(t)| = |Cx(t)| \leqslant \theta_* + |CB|T$$

for $t \geqslant 0$. □

Remark. It was shown previously that under the conditions of Theorem 1 inequality (29) is valid for all x, f, v, w, ξ. Let us put $w = \xi = 0$ in (29). Then (29) implies

$$\begin{aligned} &2x^\top H(Ax + B_0 f + Bv) \\ &\quad + \varepsilon_0(\mu_2 C_0 x - f)(f - \mu_1 C_0 x) \\ &\quad + 2\varepsilon_4(Cx - \sigma_* v)(\theta_* v - Cx) \\ &\quad \leqslant -\delta_0(\|x\|^2 + f^2 + v^2) \end{aligned} \tag{34}$$

for all x, f, v. Let us take $x = \mu C_0 x$, $v = \nu Cx$ in (34), where μ, ν are any numbers satisfying

$$\mu_1 \leqslant \mu \leqslant \mu_2, \quad \tfrac{1}{\theta_*} \leqslant \nu \leqslant \tfrac{1}{\sigma_*}. \tag{35}$$

Since $H > 0$, from (34) it follows that the matrix

$$A_{\mu,\nu} = A + \mu B_0 C_0 + \nu BC \tag{36}$$

is Hurwitz stable for any μ, ν from intervals (35). This requirement is necessary to satisfy the conditions of Theorem 1.

6 Illustrative Example

Consider a system (with a continuous part from [Seifullaev and Fradkov, 2015])

$$\begin{aligned} \dot{x}_1 &= -2x_1 + \sin x_2, \\ \dot{x}_2 &= x_1 - x_2 + 2\sin x_2 - u(t). \end{aligned} \tag{37}$$

Here $\eta(t) = x_2(t)$, $\varphi_0(\eta) = \sin\eta$. Thus we can take $\mu_1 = -0.2173$, $\mu_2 = 1$ in (3).

Originally, the stabilization problems for (37) was adressed in [Seifullaev and Fradkov, 2015] with the help of the zero-order hold control. In our recent research (37) was taken as a benchmark system for different schemes of stabilization. In [Churilov, 2019a] a system with a nonuniform sampling and a bounded duty ratio was analyzed. A sawtooth stabilizing signal was considered in [Churilov, 2019b]. In all these cases the control signal was supposed to be unbounded, so a global stabilization (for any initial data) was achieved. In [Churilov, 2019c] a local stabilization of system (37) was studied under an unsaturated PWM control, and here we treat a general case of PWM, where saturation is allowed.

The phase portrait of system (37) without a control ($u(t) \equiv 0$) is shown in Fig. 2. The zero solution of the system is unstable (a saddle point). At the same time, the system has two stable equilibria (foci) with the coordinates $(0.425,\ 2.125)$ and $(-0.425,\ -2.125)$.

Let us consider the PWM controlled system (37) with $\sigma(t) = x_2(t)$. Its continuous-time part can be rewritten in the form of (1)–(2) with matrix coefficients

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$C_0 = [0 \quad 1], \quad C = C_0\,.$$

Define $A_{\mu,\nu}$ by formula (36). It is easily verified that Hurwitz stability of $A_{\mu,\nu}$ over all μ, ν satisfying (35) with $\mu_1 = -0.2173$, $\mu_2 = 1$ is ensured by the inequality $\sigma_* < \frac{2}{3} = 0.6666$.

Let us take parameters

$$\sigma_* = 0.6, \quad \theta_* = 0.8, \quad T = 0.45$$

and apply Theorem 1. The YALMIP software package [Löfberg, 2004] for MATLAB was used to maximize the objective function $\mathrm{tr}(H)$ (the trace of H) over the set of variables H, ε_i, $i = 0, \ldots, 4$, subject to inequalities (14), (15). An estimate for the domain of attraction of the zero equilibrium was obtained in the form

$$\mathcal{E} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ : \ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top H \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leqslant 1 \right\} \tag{38}$$

with

$$H = \begin{bmatrix} 4.6024 & -1.4075 \\ -1.4075 & 1.9929 \end{bmatrix}. \tag{39}$$

If a solution starts in $\mathcal{E}$, then $|x_2(t)| \leqslant 1.2$ for all $t \geqslant 0$ and $|x_2(nT)| \leqslant 0.8$ for all $n \geqslant 0$.

For the same system parameters, the phase portrait obtained by direct simulation is shown in Fig. 3. The ellipsoid estimate for the region of attraction is calculated from (38), (39).

Notice that since $|\sin x_2| \leqslant 1$, $|u(t)| \leqslant 1$ for all t, and the matrix A is Hurwitz stable, all the solutions of (37) are ultimate bounded. More precisely, for any initial data the following limit relationships hold:

$$\limsup_{t\to+\infty} |x_1(t)| \leqslant 0.5, \quad \limsup_{t\to+\infty} |x_2(t)| \leqslant 3.5\,.$$

Recall that $\eta(t) = x_2(t)$. It is easily seen that if we restrict values of η to $|\eta| \leqslant 3.5$, then sectoral bound (3) is satisfied with $\mu_2 = 1$, $\mu_1 = -0.1002$. To apply Theorem 1 with the latter constraint, the initial time t_0 has to be taken not zero, but sufficiently large.

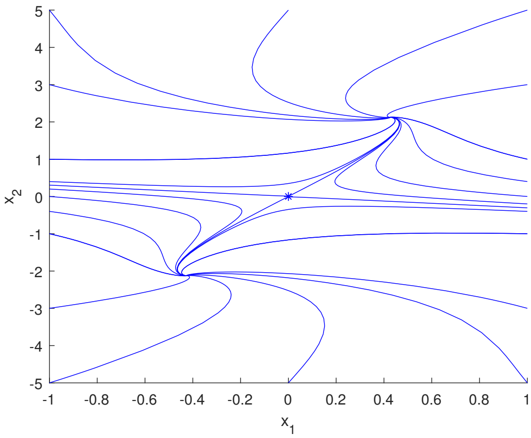

Figure 2. The phase portrait of system (37) without a control ($u(t) \equiv 0$). The system has three equilibria — a saddle (in the origin) and two stable foci. The unstable zero equilibrium is marked by asterisk.

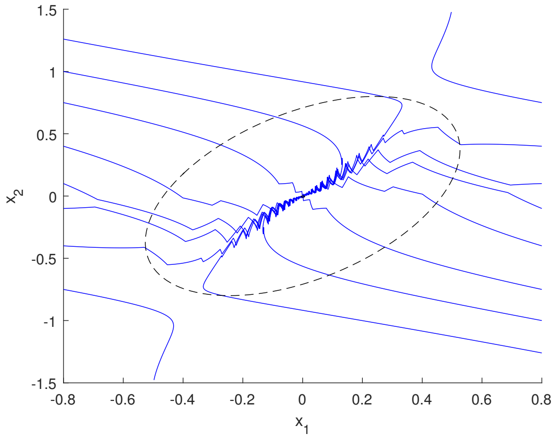

Figure 3. The phase portrait of the PWM controlled system in a vicinity of the stable zero equilibrium. The border of the set $\mathcal{E}$ is shown by the dashed line.

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7 Conclusion

Our approach relies on the Gelig's averaging method for pulse-modulated systems. For a sector bound nonlinear system under the PWM control, an LMI technique was proposed to find parameters of the stabilizing feedback. The system is stabilized locally, in some neighborhood of the origin, and the proposed Lyapunov-like method also provides an ellipsoid estimate for this neighborhood.

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