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HAMILTONIAN MODEL OF A WIND TURBINE TOWER STRUCTURE AS A SWITCHED SYSTEM

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Abstract
Wind turbines are being used to generate electricity as an alternative energy source to conventional fossil fuels, and it is well known that wind towers must to sustain continuous vibration forces throughout their operational life. In this paper, a stability analysis of bending deflection of a wind turbine steel tower is presented. The wind turbine is modelled as the structure of a simplified beam-column by a switched system. It is modelled by using a Hamiltonian system, which simplifies the system under study and allows to analyze the stability dynamics of the system. An eigenvalue analysis have been done in order to analyze the stability of the system; finally, also, some transient simulations of the system are presented to verify the results obtained.

Key words
Wind turbine tower, switched linear systems, reachability.

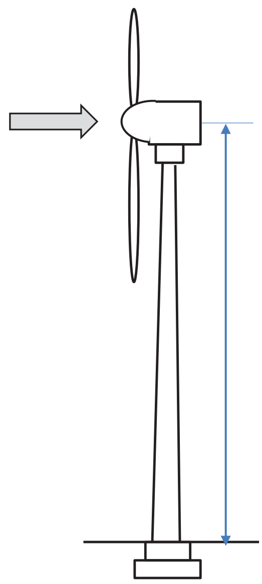

Figure 1. Structure of Wind Turbine Tower

1 Introduction

One of the major forms to produce renewable energy is to use the wind and one of the most efficient ways of converting the kinetic energy in wind into mechanical power is extracting energy from wind, using Wind turbines [Burton, Jenkins, Sharpe, and Bossanyi, 2011; Fahad, 2012].

It is known that many wind turbine damage occurs due to the structural failure of wind turbine towers. The most of these failures are caused by the strong wind striking the structure or wind induced vibrations. The aim of this paper is to analyze of bending deflection of a wind turbine steel tower

The steel tower of wind-turbine structure can be simplified as a cantilever beam-column with varying diameter involve in bending and torsion. B. Mediano in [Mediano, 2011] this kind of structures modelling them as Hamiltonian systems.Some other authors study the control of vibratory systems by means of the flatness approach as for example [Monroy-Pérez, Romero-Meléndez, 2012].

The Finite Elements theory is the most popular method used in strength and buckling analysis of wind-turbine structures, some authors as M. Wang, Zh. Wang, H. Zhao in [Wang, Wang and Zhao, 2009] analyze the structure by means a transfer matrix o a linear system. In this paper a new method to model the structure of a simplified beam-column based on switched system is presented.

Roughly speaking, a switched system is a family of continuous-time (or discrete-time) dynamical subsystems and a rule that determines the switching between them.

Given an initial time t_0, a *switching path* is a function of time $\sigma : [t_0, T) \longrightarrow M = \{1, 2, \ldots, \ell\}, T > t_0$.

Definition 1.1. *A switched system is a system which consists of several subsystems and a rule that orches-*

trates the switching between them.

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma u(t), \qquad (1)$$

where $A_\sigma \in M_n(\mathbb{R})$, $B_\sigma \in M_{n\times m}(\mathbb{R})$, *and* $\dot{x} = dx/dt$.

The pairs of matrices (A_k, B_k) for $k \in M$ are referred as the subsystems of (1) Given an initial state $x(t_0) = x_0$, input u, and a switching path $\sigma : [t_0, t_f] \longrightarrow M$, the solution of state equation (1) is given by

$$\begin{aligned} x(t) &= e^{A_{i_k}(tt_k)} \dots e^{A_{i_0}(t_1 t_0)} x_0 \\ &+ e^{A_{i_k}(tt_k)} \dots e^{A_{i_1}(t2t1)} \\ &\times \int_{t_0}^{t_1} e^{A_{i_0}(t_1\tau)} B_{i_0} u(\tau) d\tau + \dots \\ &+ \int_{t_k}^{t} e^{A_{i_k}(t\tau)} B_{i_k} u(\tau) d\tau, \end{aligned} \qquad (2)$$

for $t_k < t \leq t_{k+1}$, $1 \leq k \leq s$ where $t_0, t_1, \dots, t_s$ is the switching time sequence of σ, $t_{s+1} = t_f$, and $i_0 = \sigma(t_0), \dots i_s = \sigma(t_s)$ is the switching index sequence of σ.

Consider now

$$\Psi(t, \sigma, x_0) = e^{A_{i_k}(t-t_k)} e^{A_{i_{k-1}}(t-t_{k-1})} \dots e^{A_{i_{t_0}}(t-t_0)},$$

$t \in [t_k, t_{k+1}]$, so the state transition matrix is given by $\Phi(t_1, t_2, \sigma, x_0) = \psi(t_1, \sigma, x_0)(\psi(t_2, \sigma, x_0))^{-1}$ and the solution can be rewritten as

$$\begin{aligned} &\phi(t, t_0, x_0, u, \sigma) = \\ &\Phi(t, t_0, \sigma, x_0) x_0 + \int_{t_0}^{t} \Phi(t, \tau, \sigma, x_0) u(\tau) d\tau. \end{aligned}$$

(See [Sun and Ge, 2005], [Gurvits, Shorten, and Mason, (2007)] for more information about switched systems).

The structure of the paper is as follows. In section 2 modeling assumptions are presented, in section 3 the structure of wind turbine tower is modelled as a switched system, in section 4 the reachability of the system is analyzed. In section 5 some simulations have been realized and eigenvalues have been calculated for a particular steel tower in order to analyze the stability of the system. Finally, some conclusions are summarized in section 6.

2 Modeling Assumptions

For static analysis of a wind turbine tower, take into account the following series of forces. The first one due to gravity consist of a concentrated force at the top of the tower, representing the weight nacelle, gear box, runner generator, etc., and the weight of the tower itself distributed along of its height. The second series of forces due to the wind resistance, distributed wind pressure along the height of the tower $q(z)$ and the aerodynamics loads at the elevation of the engine axis. The wind forces on the rotors are calculated and transferred to the top of the tower by equivalent static method consisting of the following three forces and moments

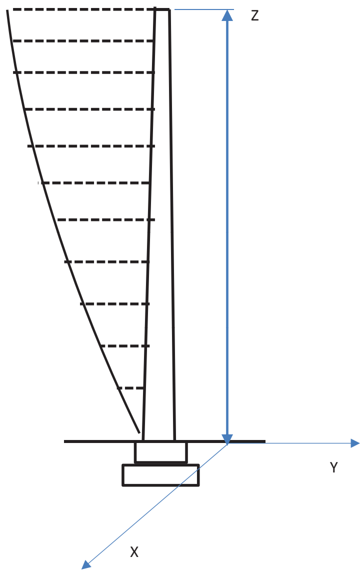

Figure 2. Wind Turbine tower seen as beam-column

1. F_y wind force
2. F_x yaw force
3. F_z vertical force
4. M_y blade bending moment
5. M_x tower bending moment
6. M_z yaw moment

The direction of the y axis is horizontal in the wind direction, the z axis is vertical with upward direction.

The structure of wind turbine tower can be simplified as a beam column with linearly varying diameter and stepwise varying thickness subject to an axial force P and it can be modeled by means the following differential equation

$$\frac{\partial^2}{\partial z^2}\left(EI(z)\frac{\partial^2 y}{\partial z^2}\right) + \frac{\partial}{\partial z}\left(P(z)\frac{\partial y}{\partial z}\right) = q(z,t) \qquad (3)$$

Where EI is the flexural rigidity, $y(z)$ is the deflection in the yz-plane and q is the transverse load.

The force q can be written as:

$$q = m_f \frac{d^2 y}{dt^2}\bigg|_{z=Ut} \qquad (4)$$

and the dynamics if the system is described in the following manner

$$EI\frac{\partial^4 y}{\partial z^4} + (P - m_q U)^2 \frac{\partial^2 y}{\partial x^2} - 2m_q U \frac{\partial^2 y}{\partial z \partial t} + -m_q \frac{\partial^2 y}{\partial t^2} = 0 \tag{5}$$

that can be approximated by a Hamilton equation.

3 Switched System Model

When the column is divided into a number of elements it can be assumed that the axial force, wind loads, section modules are constant with respect z in each element.

As a consequence the equation is reduced to

$$EI\frac{\partial^4 y}{\partial z^4} + P\frac{\partial^2 y}{\partial z^2} = q \tag{6}$$

corresponding to the linear dynamical system

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{P}{EI_i} & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{q}{EI_i} \end{pmatrix} = \begin{pmatrix} y' \\ y'' \\ y''' \\ y^{iv} \end{pmatrix} \tag{7}$$

Notice that y is the deflection, $y' = \theta$ the slope angle, $EIy'' = M$ the moment and $EIy''' = Q$ the shear force at the corresponding section.

Remark 3.1. *A solution of the equation* (7) *is* $X(z) = X_p(z) + X_h(z)$ *where* $X_p(z)$ *is a particular solution and* $X_h(z)$ *is the general solution of the homogeneous associated linear system*

Proposing a second order polynomial vector as a particular solution, we obtain $X_p(z) = \begin{pmatrix} \frac{q}{2P}z^2 \\ \frac{q}{P}z \\ \frac{q}{P} \\ 0 \end{pmatrix}$.

The characteristic polynomial of the homogeneous system is $t^4 + \frac{P}{EI}t^2$ *and the roots are* $t = 0$ *double and* $t = \pm\sqrt{-\frac{P}{EI}} = \pm\sqrt{\frac{P}{EI}}i$. *So*

$$X_h(z) = \begin{pmatrix} \alpha z + \beta + \gamma \sin\sqrt{\frac{P}{EI}}z + \delta\cos\sqrt{\frac{P}{EI}}z \\ \alpha + \gamma\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}z - \delta\sqrt{\frac{P}{EI}}\sin\sqrt{\frac{P}{EI}}z \\ -\gamma\frac{P}{EI}\sin\sqrt{\frac{P}{EI}}z - \delta\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}z \\ -\gamma\frac{P}{EI}\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}z + \delta\frac{P}{EI}\sqrt{\frac{P}{EI}}\sin\sqrt{\frac{P}{EI}}z \end{pmatrix}$$

and the general solution is

$$X(z) = \begin{pmatrix} \alpha z + \beta + \gamma \sin\sqrt{\frac{P}{EI}}z + \delta\cos\sqrt{\frac{P}{EI}}z + \frac{q}{2p}z^2 \\ \alpha + \gamma\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}z - \delta\sqrt{\frac{P}{EI}}\sin\sqrt{\frac{P}{EI}}z + \frac{q}{P}z \\ -\gamma\frac{P}{EI}\sin\sqrt{\frac{P}{EI}}z - \delta\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}z + \frac{q}{P} \\ -\gamma\frac{P}{EI}\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}z + \delta\frac{P}{EI}\sqrt{\frac{P}{EI}}\sin\sqrt{\frac{P}{EI}}z \end{pmatrix}. \tag{8}$$

The term $\frac{q}{2p}z^2$ *in the solution is the particular solution related to the uniform load q, Whereas inner bending forces can be expressed in the following manner*

$$M(z) = EI\frac{\partial^2 y}{\partial z^2}$$
$$Q(z) = \frac{\partial M}{\partial z}$$

In our particular setup the boundary conditions of the i *section are*

$$y_{z=0} = y_{i-1}, \quad \theta_{z=0} = \theta_{i-1},$$
$$M_{z=0} = M_{i-1}, \quad Q_{z=0} = Q_{i-1}$$

so, substituting the above conditions in the general solution of the linear system we obtain the values of the parameters for this case:

$$\alpha = y'_{i-1} + \frac{EI_{i-1}}{P}y'''_{i-1},$$
$$\beta = y_{i-1} + \frac{EI_{i-1}}{P}y''_{i-1} - \frac{EI_{i-1}}{P^2}q,$$
$$\gamma = -\frac{EI_{i-1}}{P}\frac{1}{\sqrt{\frac{P}{EI_{i-1}}}}y'''_{i-1},$$
$$\delta = \frac{q}{P^2}EI_{i-1} - \frac{EI_{i-1}}{P}y''_{i-1}.$$

Substituting the value of these parameters to (8) *we can obtain the deflection, slope angle, moment and shear force status of the ends of the* i*the beam section.*

In order to obtain the solution for any section from initial conditions we consider the switched linear system

$$A_\sigma X + B_\sigma = \dot{X}, \; \sigma = \{0, 1, \ldots, n\}$$

where

$$A_i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{P}{EI_i} & 0 \end{pmatrix}, \; B_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{q}{EI_i} \end{pmatrix}$$

for each i, $z \in [z_1, z_{i+1}]$ $[0, L) = [0, z_1) \cup [z_1, z_2) \cup \ldots \cup [z_n, L)$.

It is possible to obtain the state vector at any section of the wind turbine tower from initial state vector in the following manner:

$$X_i = A_i \ldots A_1 X_0 + A_i \ldots A_2 B_1 + \ldots + A_i B_{i-1} + B_i.$$

4 Reachability of the System

Consider the switched linear control system defined in (1). An important requirement for the design of feedback control systems is the knowledge of the structural properties of the switched systems under consideration, a fundamental structural property is the reachability.

The reachability character of a system is an important condition for solution of the control, and the optimization problems among others.

Definition 4.1. *A state $x_f \in \mathbb{R}^n$ is reachable at t_0 from x_0, if there exist a time instant $t_f > t_0$, a switching path $\sigma : [t_0, t_f] \longrightarrow M$, and inputs $u_k : [t_0, t_f] \longrightarrow \mathbb{R}^m$, $k \in M$, such that $\phi(t_f, t_0, x_0, u, \sigma) = x_f$.*

Definition 4.2. *The reachable set of the given switched linear system at t_0 is the set of states which are reachable at t_0.*

Definition 4.3. *A switched linear system is said to be (completely) reachable at t_0, if its reachable set at t_0 is $\mathbb{R}^n$.*

Let us consider the following matrix sequence.
$\mathcal{N}_0 = \left(B_1 \ \dots \ B_\ell \right)$
$\mathcal{N}_k = \left(I_n \ A_1 \ \dots \ A_1^{n-1} \ \dots \ A_\ell \ \dots \ A_\ell^{n-1} \right) \cdot$
$\operatorname{diag}(\mathcal{N}_{k-1}, \dots, \mathcal{N}_{k-1}),$
for $k > 0$

Remark 4.1. *Note that*

$$\operatorname{rk}\mathcal{N}_0 \leq \operatorname{rk}\mathcal{N}_1 \leq \dots \leq \operatorname{rk}\mathcal{N}_\mathrm{n} = \operatorname{rk}\mathcal{N}_\mathrm{n+1} = \dots$$

Proposition 4.1. *A necessary and sufficient condition for reachability of a switched linear system* (1) *is*

$$\operatorname{rk}\mathcal{N}_n = n$$

Proposition 4.2. *If there exists $j \in \{1, \dots, n\}$ such that $\mathcal{N}_j$ has full rank, then the switched linear system* (1) *is reachable.*

Remark 4.2. *Obviously, when one of the subsystems (A_k, B_k) of system* (1)*, is reachable, the system is also reachable.*

In our particular setup we analyze the reachability matrix of any subsystem (A_i, B_i) corresponding to the i-section of the beam

$$\begin{pmatrix} 0 & 0 & 0 & \frac{q}{EI_i} \\ 0 & 0 & \frac{q}{EI_i} & 0 \\ 0 & \frac{q}{EI_i} & 0 & 0 \\ \frac{q}{EI_i} & 0 & 0 & 0 \end{pmatrix}$$

which rank is 4. So the subsystem (A_i, B_i) is reachable, then the switched system is reachable.

λ_1	λ_2	λ_3	λ_4
0	0	j$\cdot$ $0.4849 \cdot 10^{-3}$	-j$\cdot$ $0.4849 \cdot 10^{-3}$

Table 1. Eigenvalues of the first section

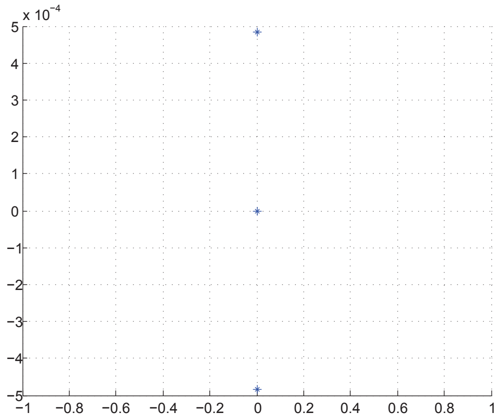

Figure 3. Eigenvalues corresponding to the first section

5 Application to a Steel Tower

As an application case of study, a steel tower for wind turbine generation is considered. For the analysis, the following parameters are considered $P = 11000 \cdot 9.81$ which represents the nacelle weight supported by the tower, $E = 210000 \frac{\mathrm{N}}{\mathrm{mm}^2}$ corresponding to the steel material, and $I_1 = 2.185 \cdot 10^6$, and $I_2 = 2.195 \cdot 10^6$ and $I_3 = 2.197 \cdot 10^6$, as the inertia value for the analyzed tower sections. They are different since the diameter of the tower is not constant.

5.1 First Section

The first section (lower inertia constant) represents a section of the wind turbine tower top part. Its inertia is lower due to the fact that its diameter is smaller.

From Figure 3 and Table 1, it can be observed that the system under study is marginally stable, since all the eigenvalues of the system exist on the negative or null real part of the complex plain. It is worth to say that in Hamiltonian systems it can be understood as a stable system.

In order to verify, the results obtained in the steady state analysis (eigenvalues), some transient simulations are presented. In Figure 4, it can be seen that all the variable states are keeping almost constant since the axis values are really low. Such results reveal that the system is keeping stable as predicted by the previous analysis.

5.2 Second Section

The second section (medium inertia constant) represents a section of the wind turbine tower middle part.

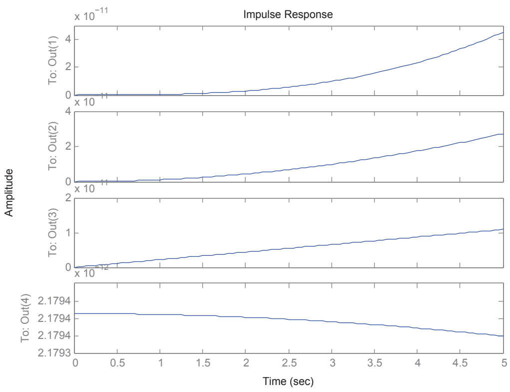

Figure 4. Impulse-response graphics corresponding to the first section

λ_1	λ_2	λ_3	λ_4
0	0	j· $0.4838 \cdot 10^{-3}$	-j· $0.4838 \cdot 10^{-3}$

Table 2. Eigenvalues of the second section

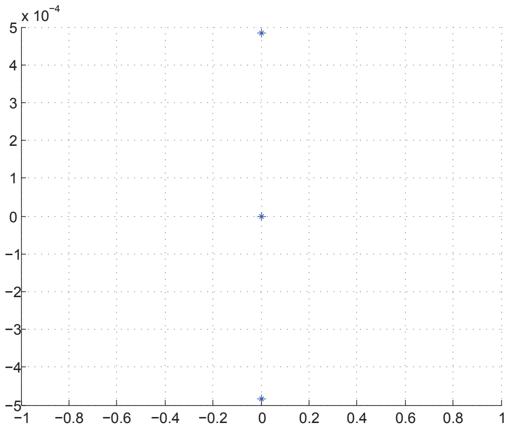

Figure 5. Eigenvalues corresponding to the second section

Its inertia is in between the maximum and minimum due to the fact that its diameter is medium.

From Figure 5 and Table 2, it can be seen that the system under study is marginally stable, as in the previous case, since all the eigenvalues of the system do not belong to the right hand plane.

Again, some dynamic simulations are done to compare them to the results shown by the eigenvalues. In Figure 6, can be seen that the system is stable as expected.

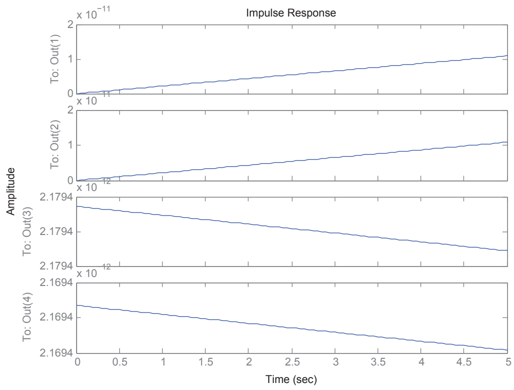

Figure 6. Impulse-response graphics corresponding to the second section

5.3 Third Section

The third section (bigger inertia constant) represents a section of the wind turbine tower lower part. Its inertia is higher since its tower diameter is bigger.

As in all the previous cases, Figure 7 and Table 3 show that the system under study is marginally stable, since all the eigenvalues of the system exist on the negative or null real part of the complex plain.

λ_1	λ_2	λ_3	λ_4
0	0	j· $0.4836 \cdot 10^{-3}$	-j· $0.4836 \cdot 10^{-3}$

Table 3. Eigenvalues of the third section

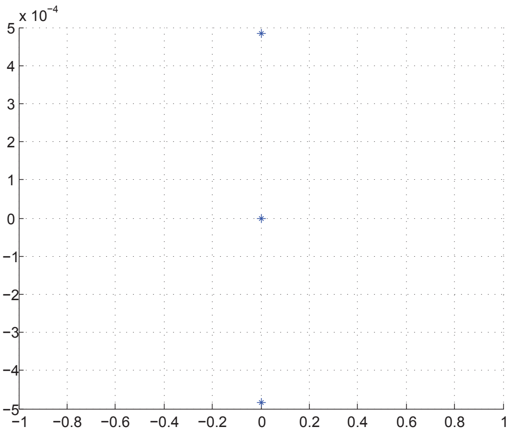

Figure 7. Eigenvalues corresponding to the third section

Finally, the simulations are drawn in Figure 8, where all the variable states are keeping almost constant, as occur in all the previous cases.

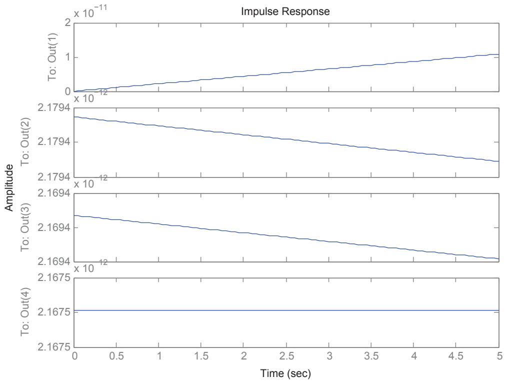

Figure 8. Impulse-response graphics corresponding to the first section

6 Conclusion

Theoretical model of a wind turbine tower has been introduced, in this paper. An approximation of such model to a hamiltonian system has been proposed. It is worth to remark that wind turbine tower can be understood as a switched system, which is an important issue to consider. The proposed model is useful to predict instability of the system depending on the parameters. Finally, in order to verify the results obtained from the model, dynamic responses and eigenvalue analysis have been developed considering real case parameters.

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