STUDY ON FRACTIONALIZED OSCILLATORY SYSTEMS IN POLAR COORDINATES

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Abstract
Planar oscillators in polar coordinates have been previously studied in integer order systems for many years. In this paper, some properties of fractionalized dynamic systems in polar coordinates which have oscillatory behavior in integer order polar coordinates have been investigated via numerical and analytical methods. It has been shown that there is angular acceleration appeared in such systems, not seen in integer order counterparts. Also, the convergence rate toward the limit cycle depends on the order of fractional derivative.

Key words
Fractional order, polar coordinates, oscillatory system

1 Introduction

The concept of limit cycle first appeared in papers by Poincare during early 1890s. In particular, the qualitative theories have been widely developed for the planar systems. From a topological viewpoint, the main result may be the Poincare-Bendixson theorem, which has no generalization to higher dimensions. This theorem states that any bounded solution not tending to a singular point, must necessarily be a periodic orbit, or tends to a limit cycle [Wiggins, 2003]. Let us recall that a limit cycle is an isolated periodic solution of a given planar system. The most difficult and important problem for planar differential systems is the determination of their limit cycles. In planar systems, it has been shown that employment of polar coordinates simplifies the use of some theorems to study the limit cycles [Chavarriga, Giacomini and Gine, 1999].

Concept of the fractional order operators is an old concept in mathematical area, but its application in various branches of physics, biology and engineering has been recently investigated. Some examples from fractional order dynamics can be found in [Podlubny, 1999; Hifler, 2001] and references therein. Studying the dynamical behavior of the fractional order systems is one of the latest issues in the recent studies. It has been found that the fractional order systems like as their integer order counterpart can generate oscillations. The oscillations may be regular or non-regular (chaotic). For instance, in [Ahmad, El-Khazali and El-Wakil, 2001] it has been pointed out that limit cycle can be generated in the fractional order Wien bridge oscillator. Existence of limit cycle for fractional Brusselator has been numerically shown in [Wang and Li, 2007]. Dynamics of the fractional order Van der Pol oscillator has been investigated in [Barbosa, Machado, Ferreira and Tar, 2004; Barbosa, Machado, Vingare and Calderon, 2007]. There are some attempts to find analytical solution for linear fractional order oscillators ([Narahari Achar, Hanneken and Clarke, 2004]). Also, investigation of chaotic behaviors in the fractional order systems has been reported in some papers such as [Hartley, Lorenzo and Qammer, 1995; Li and Chen, 2004; Tavazoei and Haeri, 2007].

In this paper, we investigate the fractionalized dynamic systems in polar coordinates via studying some special samples. The paper is organized as follows. In Section 2, some preliminaries of the study including the employed numerical method for simulations and the definitions in the fractional calculus are provided. Section 3 includes three different examples of the fractional order systems in the polar coordinates and investigations of their properties via numerical and analytical methods. The paper is concluded in Section 4.

2 Preliminaries

Fractional calculus: there are several definitions for fractional derivative. In this paper we employ the definition which was introduced by Caputo in 1969.

The Caputo fractional derivative D^{α} is defined as follows,

$$D^{\alpha} y(x) = J^{m-\alpha} y^{(m)}(x) \tag{1}$$

where $\alpha > 0$ (but not necessarily $\alpha \in N$), $m := \lceil \alpha \rceil$ is the smallest integer larger than α, $y^{(m)}$ is the ordinary m^{th} derivative of y, and

$$J^{\beta} z(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} z(t) dt \tag{2}$$

is the Riemann-Liouville integral operator of order $\beta > 0$.

A fractional differential equation of order α_n is an equation such as

$$F(x, y(x), D^{\alpha_1} y(x), \cdots, D^{\alpha_n} y(x)) = g(x) \tag{3}$$

with $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, $F(x, y_1, \cdots, y_n)$, $g(x)$ known real function, $D^{\alpha_k} (k = 1, 2, \cdots, n)$ fractional operators, and $y(x)$ is the unknown function.

When one works with Caputo's definition of the fractional derivatives, only integer order derivatives of the function i.e. $y(0), y'(0), \cdots, y^{(m-1)}(0)$ are appeared in the solution. These data have typically a well understood physical meaning and can be measured.

Numerical method: Numerical simulations of this paper have been done based on Adams-type predictor-corrector method introduced in [Diethelm, Ford and Freed, 2002]. This method is a generalization of the classical Adams-Bashforth-Moulton algorithm.

3 Analysis and Simulation of Polar Fractionalized Oscillatory Systems

In this section, three fractional order systems in polar coordinates are studied. First we investigate a simple system both numerically and analytically in Example 1. Systems investigated in Examples 2 and 3 are more complex. In Example 2 we study the given system based on its numerical simulations results. Moreover, the asymptotic response of the system is determined analytically. The given system in Example 3 is studied only through numerical simulations.

Example 1: The following equations represent a nonlinear system in Cartesian and Polar coordinates.

$$\begin{cases} \dot{x} = -y - x(\sqrt{x^2 + y^2} - 1) \\ \dot{y} = x - y(\sqrt{x^2 + y^2} - 1) \end{cases} \tag{4}$$

$$\begin{cases} \dot{r} = -r(r-1) \\ \dot{\theta} = 1 \end{cases} \tag{5}$$

The first state of the polar system has two equilibrium points at $r = 0$ and $r = 1$. The first point corresponds to an unstable focus at the origin. The second one represents the polar form of a circle with origin as its center and radius one. $\dot{\theta} = 1$ implies that the circle is traversed in an anti clockwise direction with constant angular velocity of one. Thus one of the system solutions is a closed orbit. If $r < 1$ then $\dot{r} > 0$ and the trajectories spiral outwards towards the closed orbit. If $r > 1$ then $\dot{r} < 0$ and thus the trajectories spiral inwards towards the closed orbit (Fig. 1). The closed orbit is called a stable limit cycle.

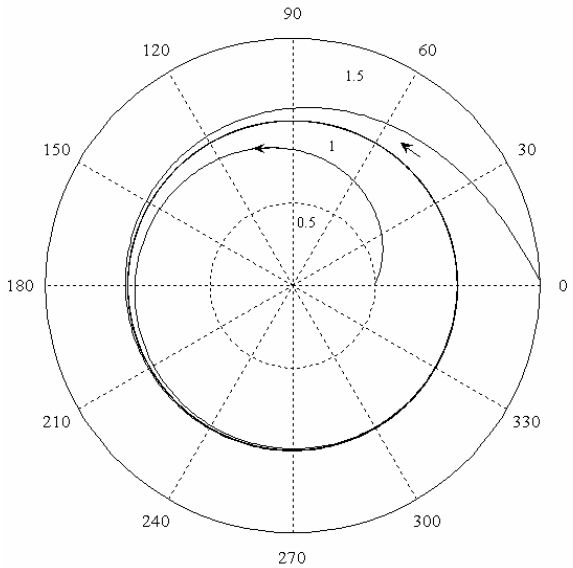

Figure 1. Time evolution of two trajectories for system in (5).

The fractionalized counterpart of the system (5) is as follows.

$$\begin{cases} D^q r = -r(r-1) \\ D^q \theta = 1 \end{cases}, \quad \theta(0) = \theta_0, r(0) = r_0 \tag{6}$$

where q is a real number such that $0 < q < 1$, r_0 and θ_0 are initial values of the states. The equilibrium points of the first state of system (6) are $r^* = 0$ and $r^* = 1$. The first state of the system can be linearized as follows [Ahmed, El-Sayed and El-Saka, 2007]

$$D^q \delta r = (-2r^* + 1)\delta r \tag{7}$$

According to Matignon theorem [Matignon, 1996], $r^* = 0$ is an unstable and $r^* = 1$ is an asymptotic stable equilibrium point for any value of parameter $0 < q \le 1$ (Fig. 2). This implies that $r^* = 1$ is a stable limit cycle of the system for any q, i.e. when a trajectory starts from any point on this orbit it remains on the orbit for ever.

The exact solution of the differential equation $D^q \theta(t) = 1$ is

$$\theta(t) = \frac{1}{\Gamma(q+1)} t^q + \theta_0 \tag{8}$$

where $\Gamma(.)$ is Euler's Gamma function [Podlubny, 1999]. This solution declares that the system has a time varying angular velocity, which is a new property not seen in its integer order counterpart. The angular velocity is calculated as follows

$$\omega(t) = \frac{1}{\Gamma(q)} t^{q-1} \tag{9}$$

Eq. (9) implies that the angular frequency of the solution is decreased, as time progresses. Time varying angular velocity results in non zero (and also time varying) angular acceleration which is also not seen in the integer order counterpart system. The asymptotic behavior of the Cartesian components of the solution is as follows.

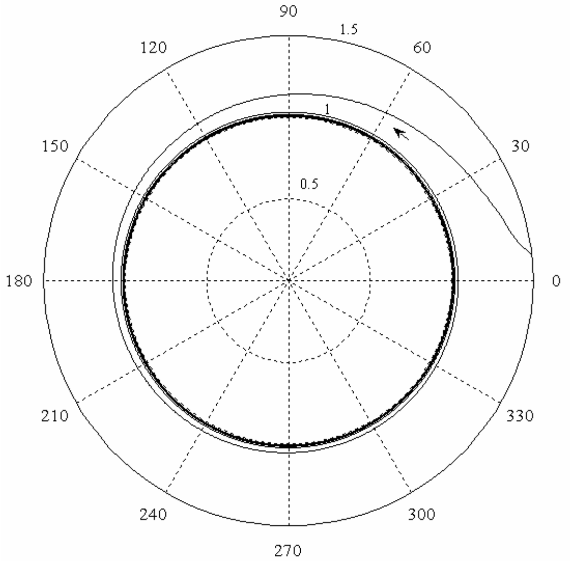

Figure 2. Time evolution of a trajectory for system in (6) ($q = 0.6$).

$$
\begin{cases}
x(t) = \cos(\dfrac{1}{\Gamma(q+1)} t^q + \theta_0) \\
y(t) = \sin(\dfrac{1}{\Gamma(q+1)} t^q + \theta_0)
\end{cases}
\tag{10}
$$

Fig. 3 shows the simulation results of system (6). It is obvious from the figure that the oscillation periods are increased as time progresses. Furthermore, the solutions get faster as q increases.

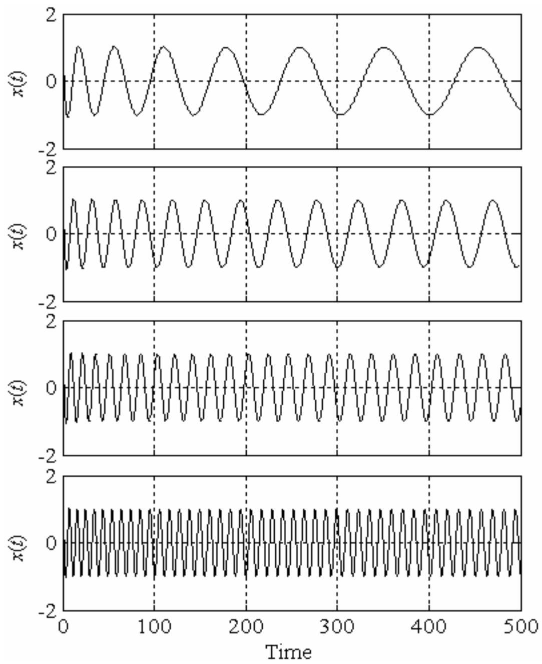

Figure 3a. Cartesian variable $x(t)$ in simulation of system (6) (From up to down q equals 0.6, 0.7, 0.8 and 0.9).

Another interesting point here is the form of effects that q has on the oscillating characteristics of the fractional order polar system. Fig. 4 illustrates the time trajectory of r for different values of q and similar initial value $r_0 = 1.5$. Apart from the q value, the asymptotic response is always located at $r = 1$. However, enough close to the asymptotic response, the convergence rate of the system toward its limit cycle is increased as q increases but for sufficiently far values of r from value $r = 1$, the convergence rate of the system toward the limit cycle is increased as q decreases.

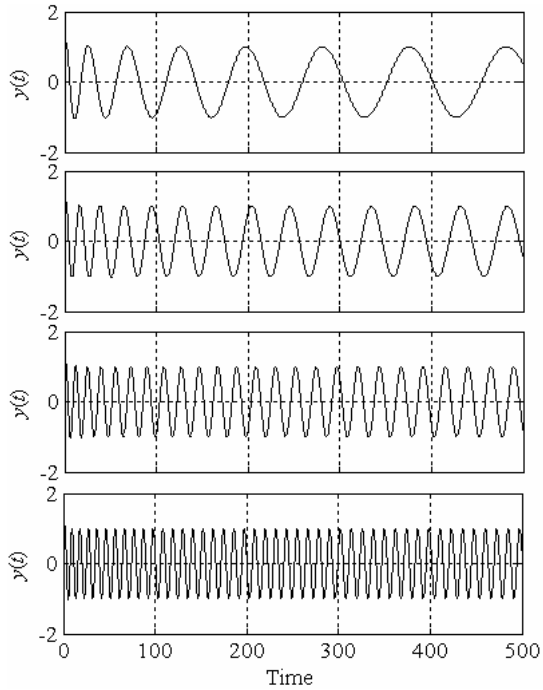

Figure 3b. Cartesian variable $y(t)$ in simulation of system (6) (From up to down q equals 0.6, 0.7, 0.8 and 0.9).

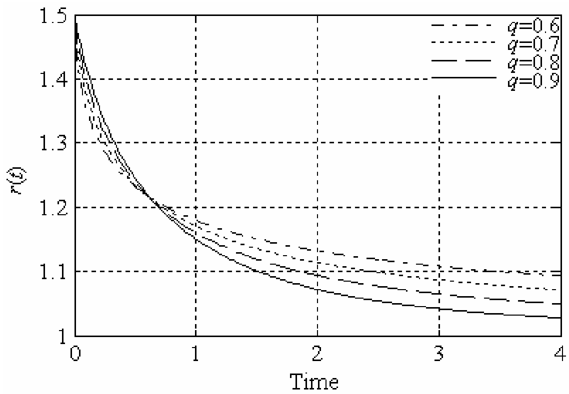

Figure 4. Variation of r for similar initial value $r_0 = 1.5$ and different values of q (Example 1).

In the system of Example 1, the fractional derivative of variable θ does not relate to variable r or θ and thus we have a simple dynamical system. A more complex system, containing variable r in the fractional derivative of θ, is studied in Example 2.

Example 2: Let us consider a dynamical system with the following relations in the Cartesian and polar coordinates respectively.

$$
\begin{cases}
\dot{x} = x - y - (x + 1.5y)(x^2 + y^2) \\
\dot{y} = x + y + (1.5x - y)(x^2 + y^2)
\end{cases}
\tag{11}
$$

$$\begin{cases} \dot{r} = r - r^3 \\ \dot{\theta} = 1 + 1.5r^2 \end{cases} \tag{12}$$

The fractionalized equations in the polar coordinates are as follows.

$$\begin{cases} D^q r = r - r^3 \\ D^q \theta = 1 + 1.5r^2 \end{cases}, \quad \theta(0) = \theta_0, r(0) = r_0 \tag{13}$$

Similar to the reasoning given in Example 1, $r = 0$ is an unstable and $r = 1$ is an asymptotic stable equilibrium point for any value of q in (0, 1]. In other words, a stable limit cycle exists at $r = 1$ for any value of q in (0, 1]. Fig. 5 illustrates time evolution of a trajectory of the system (13) for $q = 0.6$.

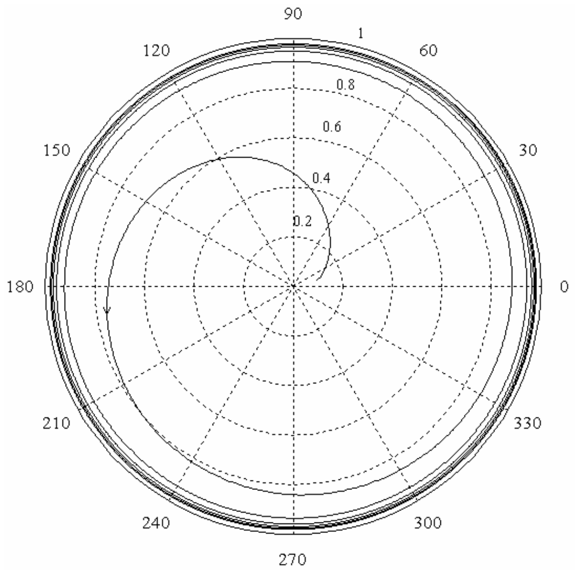

Figure 5. Time evolution of a trajectory for system in (13) ($q = 0.6$).

For $r = 1$, $\theta(t)$ has the following solution

$$\theta(t) = \frac{2.5}{\Gamma(q+1)} t^q + \theta_0 \tag{14}$$

This indicates that systems in Examples 1 and 2 have similar asymptotic response for the second variable. The angular velocity of the system in steady state is

$$\omega(t) = \frac{2.5}{\Gamma(q)} t^{q-1} \tag{15}$$

Similar to Example 1, the angular velocity of the system is time varying. For $r = 1$, The Cartesian components $x(t)$ and $y(t)$ are

$$\begin{cases} x(t) = \cos(\dfrac{2.5}{\Gamma(q+1)} t^q + \theta_0) \\ y(t) = \sin(\dfrac{2.5}{\Gamma(q+1)} t^q + \theta_0) \end{cases} \tag{16}$$

Fig. 6 shows the simulation results of system (13). These figures confirm that the frequency of Cartesian variables is decreased as time progresses. Furthermore, the solutions get faster as q increases. Fig. 7 illustrates variation of r for different values of q. As in Example 1, enough close to the limit cycle, the system converges toward its limit cycle faster as q increases; but for sufficiently far values of r from $r = 1$, the convergence rate of the system toward the limit cycle is increased as q decreases.

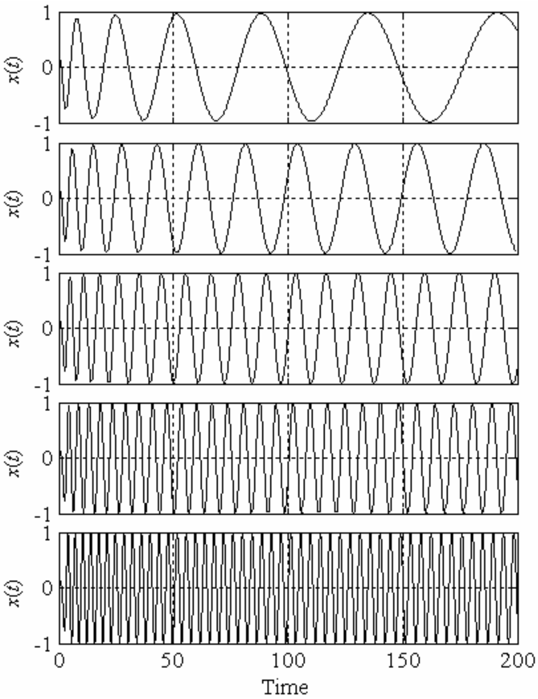

Figure 6a. Cartesian variable $x(t)$ in simulation of system (13) (From up to down q equals 0.5, 0.6, 0.7, 0.8 and 0.9).

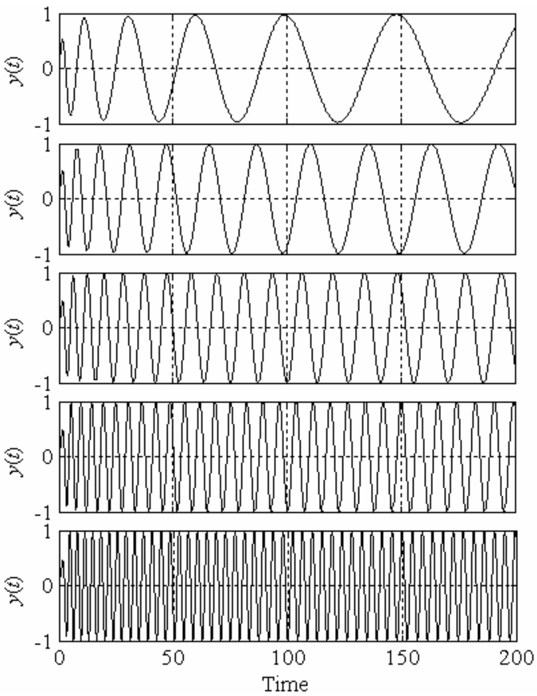

Figure 6b. Cartesian variable $y(t)$ in simulation of system (13)(From up to down q equals 0.5, 0.6, 0.7, 0.8 and 0.9).

The rest of this section is devoted to a more complex fractional system in polar coordinates that has not simple analytical solution. We will study this example system via numerical simulations.

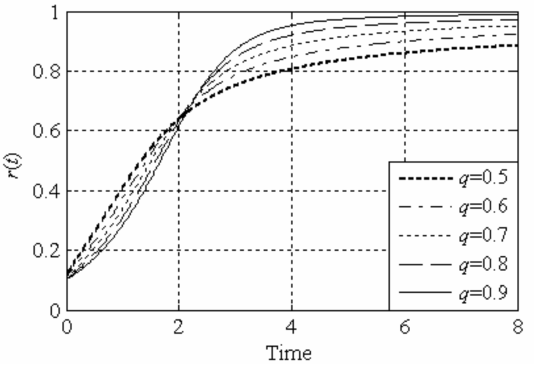

Figure 7. Variation of r for similar initial value $r_0 = 1.5$ and different values of q (Example 2).

Example 3: Let us consider the following nonlinear system which is represented both in Cartesian and polar coordinates.

$$\begin{cases} \dot{x} = x - y - x(x^2 + 2y^2) \\ \dot{y} = x + y - y(x^2 + 2y^2) \end{cases} \tag{17}$$

$$\begin{cases} \dot{r} = r - r^3(1 + 0.25\sin^2 2\theta) \\ \dot{\theta} = 1 + 0.5(r^2 \sin^2\theta \sin 2\theta) \end{cases} \tag{18}$$

This system has a limit cycle located in $0.4\sqrt{5} < r < 1$ as shown in Fig. 8.

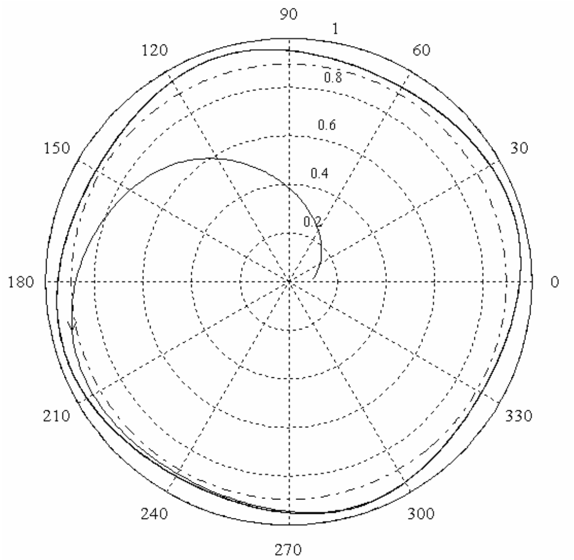

Figure 8. Time evolution of a trajectory for system (18).

The corresponding fractionalized system in polar coordinates is as follows.

$$\begin{cases} D^q r = r - r^3(1 + 0.25\sin^2 2\theta) \\ D^q \theta = 1 + 0.5(r^2 \sin^2\theta \sin 2\theta) \end{cases} \tag{19}$$

$$\theta(0) = \theta_0,\ r(0) = r_0$$

The limit cycle of this more complex system is depicted in Fig. 9 for $q = 0.7$. Simulation results of the system are shown in Fig. 10. Similar to the previous examples, the angular velocity of the system is time varying. The frequency of Cartesian variables is decreased as time progresses and the solutions get faster as q increases.

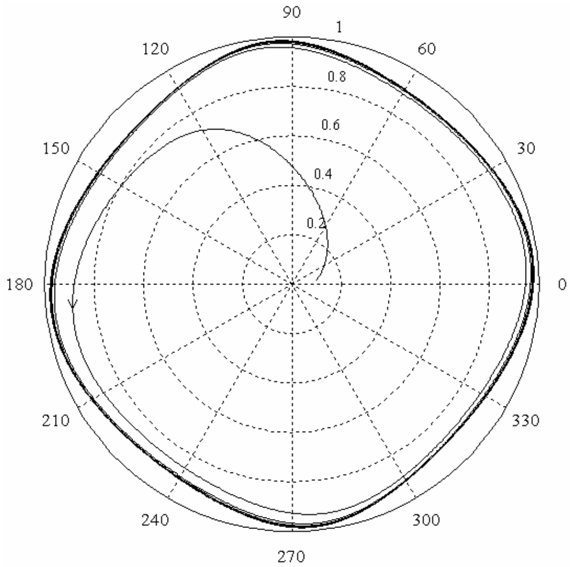

Figure 9. Time evolution of a trajectory for system (19) for $q = 0.7$.

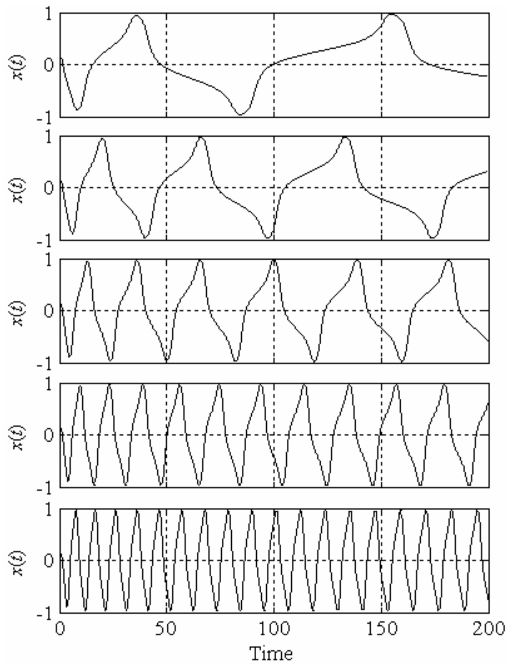

Figure 10a. Cartesian variable $x(t)$ in simulation of system (19) (From up to down q equals 0.5, 0.6, 0.7, 0.8 and 0.9).

4 Conclusion

In this paper, the fractionalized systems in polar coordinates were investigated through three different examples. We showed that in such systems, the angular velocity is time varying which in turn results in time varying angular acceleration. This property does not show up in the integer order counterpart in polar coordinates. Also the limit cycle of fractional polar systems were investigated and concluded that any changes of fractional order q affects on the convergence rate of system trajectories toward the limit cycle. Analytical investigation of the properties for the complex fractional polar systems, such as the one in Example 3, is an ongoing research topic in our group. Furthermore, analysis of fractionalized oscillatory systems in Cartesian coordinates is another topic for future works. Simulations of the paper have been performed in MATLAB writing appropriate m-files required for each example.

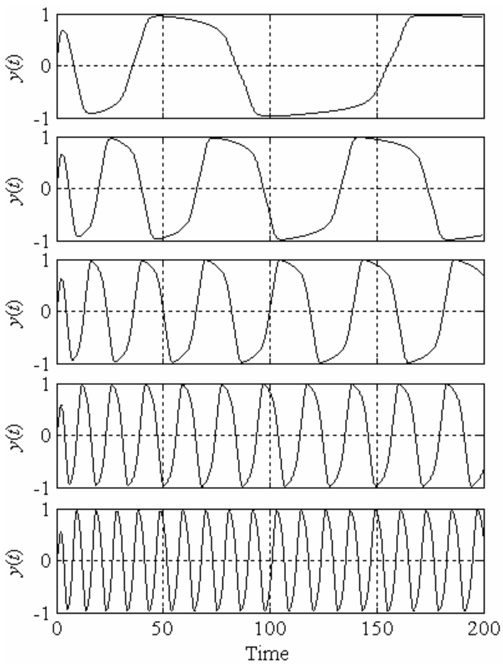

Figure 10b. Cartesian variable $y(t)$ in simulation of system (19) (From up to down q equals 0.5, 0.6, 0.7, 0.8 and 0.9).

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