Attractive Quantum Subsystems and Feedback-Stabilization Problems

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***Abstract*— We propose a general theoretical framework that is suitable to study a wide class of stabilization problems for quantum Markovian dynamical systems. Building on system-theoretic ideas, we propose definitions of invariant and attractive quantum subsystem, characterize Markovian invariance properties, and provide sufficient conditions for attraction. The general framework and results are illustrated by addressing the potential of output-feedback Markovian control strategies for quantum pure state-stabilization. In particular, constructive results for the synthesis of stabilizing semigroups in arbitrary finite-dimensional Markovian systems are established.**

I. Background and motivations

Stabilization problems are of central relevance for many quantum control applications, ranging from state preparation of quantum-optical and nano-mechanical systems to generation of noise-protected realizations of quantum information in realistic devices [1]. Dynamical systems undergoing Markovian evolution [2], [3] are both widely relevant from a physical standpoint and present distinctive control challenges – preventing, in particular, open-loop quantum-engineering and stabilization methods based on dynamical decoupling to be viable [4], [5]. However, we show here how a wide class of stabilization problems can be effectively treated in a general framework, provided by *attractive quantum subsystems*. After introducing the main ideas and definitions along with some general results, we shall explore their application to pure-state preparation problems for Markovian output-feedback control. We refer to the forthcoming journal version of the present paper [6] for detailed proofs we shall omit or merely sketch in the following sections.

Consider a separable Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$. Let $\mathfrak{B}(\mathcal{H})$ represent the set of linear bounded operators on $\mathcal{H}$, $\mathfrak{H}(\mathcal{H})$ denoting the real subspace of Hermitian operators, with $\mathbb{I}$, $\mathbb{O}$ being the identity and the zero operator, respectively. In the standard statistical formulation of quantum mechanics [7], [8], the dimension of the Hilbert space $\mathcal{H}$ associated with the quantum system of interest, $\mathcal{Q}$, is determined by the physics of the problem. In what follows, we consider *finite-dimensional* systems, i.e. $\dim(\mathcal{H}) < -\infty$. Our (possibly uncertain) knowledge of the state of $\mathcal{Q}$ is condensed in a *density operator* ρ, with $\rho \geq 0$ and $\mathrm{trace}(\rho) = 1$. Density operators form a convex set $\mathfrak{D}(\mathcal{H}) \subset \mathfrak{H}(\mathcal{H})$, with one-dimensional projectors corresponding to extreme points (*pure states*, $\rho_{|\psi\rangle} = |\psi\rangle\langle\psi|$).

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If $\mathcal{Q}$ is the composite system obtained from two other quantum systems $\mathcal{Q}_1$, $\mathcal{Q}_2$, the corresponding mathematical description is carried out in the tensor product space, $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ [7], observables and density operators being associated with Hermitian and positive-semidefinite, normalized operators on $\mathcal{H}_{12}$, respectively. The *partial trace* over $\mathcal{H}_2$ is the unique linear operator $\mathrm{trace}_2(\cdot) : \mathfrak{B}(\mathcal{H}_{12}) \to \mathfrak{B}(\mathcal{H}_1)$, ensuring that for every $X_1 \in \mathfrak{B}(\mathcal{H}_1), X_2 \in \mathfrak{B}(\mathcal{H}_2)$, $\mathrm{trace}_2(X_1 \otimes X_2) = X_1 \mathrm{trace}(X_2)$.

In general, in the presence of internal couplings, quantum measurements, or interaction with a surrounding environment, the evolution of a subsystem of interest is no longer unitary and reversible, and the general formalism of *open quantum systems* is required [9], [2], [3]. A wide class of open quantum systems obeys Markovian dynamics [2], [10], [11]. Let $\mathcal{I}$ denote the physical quantum system of interest, with associated Hilbert space $\mathcal{H}_I$, $\dim(\mathcal{H}_I) = d$. Assume that we have no access to the quantum environment surrounding $\mathcal{I}$, and that the dynamics in $\mathfrak{D}(\mathcal{H}_I)$ is continuous in time, the state change at each $t > 0$ being described by a Trace-Preserving, Completely-Positive (TPCP) map $\mathcal{T}_t(\cdot)$ [12], [1]. A differential equation for the density operator of $\mathcal{I}$ may be derived provided that a forward composition law holds:

Definition 1 (QDS): A *quantum dynamical semigroup* is a one-parameter family of TPCP maps $\{\mathcal{T}_t(\cdot),\, t \geq 0\}$ that satisfies:

(i) $\mathcal{T}_0 = \mathbb{I}$,
(ii) $\mathcal{T}_t \circ \mathcal{T}_s = \mathcal{T}_{t+s},\ \forall t, s > 0$,
(iii) $\mathrm{trace}(\mathcal{T}_t(\rho)X)$ is a continuous function of t, $\forall \rho \in \mathfrak{D}(\mathcal{H}_I)$, $\forall X \in \mathcal{B}(\mathcal{H}_I)$.

Due to the trace and positivity preserving assumptions, a QDS is a semigroup of contractions. It has been proved [10], [13] that the Hille-Yoshida generator for a QDS exists and can be cast in the following canonical form:

$$
\begin{aligned}
&-i[H, \rho(t)] + \sum_{k=1}^{p} \gamma_k \mathcal{D}(L_k, \rho(t)) \qquad (1)\\
&= -i[H, \rho(t)] + \sum_{k=1}^{p} \gamma_k \Big(L_k \rho(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k,\, \rho(t)\}\Big),
\end{aligned}
$$

with $\{\gamma_k\}$ denoting the spectrum of A. The *effective Hamiltonian* H and the *Lindblad operators* L_k specify the net effect of the Markovian environment on the dynamics. In general, H is equal to the isolated system Hamiltonian, H_0, plus a correction, H_L, induced by the coupling to the environment (so-called Lamb shift). The non-Hamiltonian terms $\mathcal{D}(L_k, \rho(t))$ in (1) account for non-unitary dynamics induced by L_k.

In principle, the exact form of the generator of a QDS may be rigorously derived from the underlying Hamiltonian model for the joint system-environment dynamics under appropriate limiting conditions (the so-called "singular coupling limit" or the "weak coupling limit," respectively [2], [3]). In most physical situations, however, carrying out such a procedure is unfeasible, typically due to lack of complete knowledge of the full microscopic Hamiltonian. A Markovian generator of the form (1) is then usually assumed on a phenomenological basis. In practice, it is often the case that a direct knowledge of the noise effect may be assumed, allowing to specify the Markovian generator by giving a set of noise strengths γ_k and Lindblad operators L_k (not necessarily orthogonal or complete) in (1). Each of the noise operators L_k may be thought of as corresponding to a distinct *noise channel* $\mathcal{D}(L_k, \rho(t))$, by which information irreversibly leaks from the system to the environment.

II. Quantum subsystems

Quantum subsystems are the basic building block for describing composite systems in quantum mechanics [7]. From both a conceptual and practical standpoint, renewed interest toward characterizing quantum subsystems in a variety of control-theoretic settings is motivated by Quantum Information Processing (QIP) applications [1]. A definition suitable to our scopes is the following:

Definition 2 (Quantum subsystem): A *quantum subsystem* $\mathcal{S}$ of a system $\mathcal{I}$ defined on $\mathcal{H}_I$ is a quantum system whose state space is a tensor factor $\mathcal{H}_S$ of a subspace $\mathcal{H}_{SF}$ of $\mathcal{H}_I$,

$$\mathcal{H}_I = \mathcal{H}_{SF} \oplus \mathcal{H}_R = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R, \qquad (2)$$

for some factor $\mathcal{H}_F$ and remainder space $\mathcal{H}_R$. The set of linear operators on $\mathcal{S}$, $\mathcal{B}(\mathcal{H}_S)$, is isomorphic to the (associative) algebra on $\mathcal{H}_I$ of the form $X_I = X_S \otimes \mathbb{I}_F \oplus \mathbb{O}_R$.

Let $n = \dim(\mathcal{H}_S)$, $f = \dim(\mathcal{H}_F)$, $r = \dim(\mathcal{H}_R)$, and let $\{|\phi_j^S\rangle\}_{j=1}^n$, $\{|\phi_k^F\rangle\}_{k=1}^f$, $\{|\phi_l^R\rangle\}_{l=1}^r$ denote orthonormal bases for $\mathcal{H}_S$, $\mathcal{H}_F$, $\mathcal{H}_R$, respectively. Decomposition (2) is then naturally associated with the following basis for $\mathcal{H}_I$:

$$\{|\varphi_m\rangle\} = \{|\phi_j^S\rangle \otimes |\phi_k^F\rangle\}_{j,k=1}^{n,f} \cup \{|\phi_l^R\rangle\}_{l=1}^r.$$

This basis induces a block structure for matrices acting on $\mathcal{H}_I$:

$$X = \left(\begin{array}{c|c} X_{SF} & X_P \\ \hline X_Q & X_R \end{array}\right), \qquad (3)$$

where, in general, $X_{SF} \neq X_S \otimes X_F$. Let Π_{SF} be the projection operator onto $\mathcal{H}_S \otimes \mathcal{H}_F$, that is, $\Pi_{SF} = \left(\begin{array}{c|c} \mathbb{I}_{SF} & 0 \end{array}\right)$.

A. Invariant subsystems

We start by investigating in which sense, and under which conditions, a quantum subsystem may be defined as invariant.

Definition 3 (State initialization): The system $\mathcal{I}$ with state $\rho \in \mathfrak{D}(\mathcal{H}_I)$ is *initialized in $\mathcal{H}_S$ with state $\rho_S \in \mathfrak{D}(\mathcal{H}_S)$* if the blocks of ρ satisfy:

(i) $\rho_{SF} = \rho_S \otimes \rho_F$ for some $\rho_F \in \mathfrak{D}(\mathcal{H}_F)$;
(ii) $\rho_P = 0, \rho_R = 0$.

Condition (ii) in the above Definition guarantees that $\bar{\rho}_S = \mathrm{trace}_F(\Pi_{SF}\rho\Pi_{SF}^\dagger)$ is a valid state of $\mathcal{S}$, while condition (i) ensures that measurements or dynamics affecting the factor $\mathcal{H}_F$ have no effect on the state in $\mathcal{H}_S$. We shall denote by $\mathfrak{I}_S(\mathcal{H}_I)$ the set of states initialized in this way.

Definition 4 (Invariance): Let $\mathcal{I}$ evolve under TPCP maps $\mathcal{T}_t$. $\mathcal{S}$ is an *invariant subsystem* if $\forall\, \rho_S \in \mathfrak{D}(\mathcal{H}_S)$, $\rho_F \in \mathfrak{D}(\mathcal{H}_F)$, the state of $\mathcal{I}$ obeys

$$\mathcal{T}_t\left(\begin{array}{c|c} \rho_S \otimes \rho_F & 0 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathcal{T}_t^S(\rho_S) \otimes \mathcal{T}_t^F(\rho_F) & 0 \\ \hline 0 & 0 \end{array}\right),\ t \geq 0, \qquad (4)$$

where, for every $t \geq 0$, $\mathcal{T}_t^S(\cdot)$ and $\mathcal{T}_t^F(\cdot)$ are TPCP maps on $\mathcal{H}_S$ and $\mathcal{H}_F$, respectively, not depending on the initial states.

Thus, a subsystem is invariant if time evolution preserves the initialization of the state, that is, the dynamics is confined within $\mathfrak{I}_S(\mathcal{H}_I)$. For Markovian evolution of $\mathcal{I}$, Definition 1 requires both $\{\mathcal{T}_t^S\}$ and $\{\mathcal{T}_t^F\}$ to be QDSs on their respective domain.

We now provide two characterizations of dynamical models able to ensure invariance for a fixed subsystem – the second explicitly constraining the block-structure of the matrix representation of the operators specifying the Markovian generator.

Theorem 1 (Markovian invariance): $\mathcal{H}_S$ supports an invariant subsystem under Markovian evolution on $\mathcal{H}_I$ iff $\forall\, \rho_S \in \mathfrak{D}(\mathcal{H}_S)$, $\rho_F \in \mathfrak{D}(\mathcal{H}_F)$ the following conditions hold:

$$\frac{d}{dt}\rho(t) = \left(\begin{array}{c|c} \mathcal{L}_{SF}(\rho_S(t) \otimes \rho_F(t)) & 0 \\ \hline 0 & 0 \end{array}\right), \quad \forall t \geq 0, \qquad (5)$$

$$\mathrm{trace}_F\left[\mathcal{L}_{SF}(\rho_S(t) \otimes \rho_F(t))\right] = \mathcal{L}_S(\rho_S(t)), \quad \forall t \geq 0, \qquad (6)$$

where $\mathcal{L}_{SF}$ and $\mathcal{L}_S$ are QDS generators on $\mathcal{H}_S \otimes \mathcal{H}_F$ and $\mathcal{H}_S$, respectively.

Corollary 1 (Markovian invariance): Assume that $\mathcal{H}_I = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$, and let H, $\{L_k\}$ be the Hamiltonian and the error generators of a Markovian QDS as in (1). Then $\mathcal{H}_S$ supports an invariant subsystem iff $\forall\, k$:

$$\begin{aligned} L_k &= \left(\begin{array}{c|c} L_{S,k} \otimes L_{F,k} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array}\right), \\ iH_P &- \frac{1}{2}\sum_k (L_{S,k}^\dagger \otimes L_{F,k}^\dagger) L_{P,k} = 0, \qquad (7) \\ H_{SF} &= H_S \otimes \mathbb{I}_F + \mathbb{I}_S \otimes H_F, \end{aligned}$$

where for each k either $L_{S,k} = \mathbb{I}_S$ or $L_{F,k} = \mathbb{I}_F$ (or both).

B. Attractive subsystems

If we require a subsystem to have dynamics independent from the rest also in the case it is not "perfectly initialized" as in Definition 3, it turns out [6], [14] that an additional contraint on the Lindblad operators is required. That is, it must be $L_{P,k} = 0$ for every k, which basically decouples the evolution of the SF-block of the state from the rest. However, this imposes tighter conditions on the noise operators, which may be demanding to ensure and, from a control perspective, leave less room for Hamiltonian compensation of the noise action (see Section III-A). In order to both address situations where such extra constraints cannot be met, as well as a question which is interesting on its own, we explore conditions for a subsystem to be attractive:

Definition 5 (Attractive Subsystem): Assume that $\mathcal{H}_I = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$. Then $\mathcal{H}_S$ supports an attractive subsystem with respect to a family $\{\mathcal{T}_t\}_{t\geq 0}$ of TPCP maps if $\forall \rho \in \mathfrak{D}(\mathcal{H}_I)$ the following condition is asymptotically obeyed:

$$\lim_{t\to\infty} \left(\mathcal{T}_t(\rho) - \left(\begin{array}{c|c} \bar{\rho}_S(t) \otimes \bar{\rho}_F(t) & 0 \\ \hline 0 & 0 \end{array} \right) \right) = 0, \qquad (8)$$

where $\bar{\rho}_S(t) = \text{trace}_F[\Pi_{SF}\mathcal{T}_t(\rho)\Pi_{SF}^\dagger]$, $\bar{\rho}_F(t) = \text{trace}_S[\Pi_{SF}\mathcal{T}_t(\rho)\Pi_{SF}^\dagger]$.

An attractive subsystem may be thought of as a subsystem that “self-initializes” in the long-time limit, by somehow reabsorbing initialization errors. Although such a desirable behavior only emerges asymptotically, for QDSs one can see that convergence is exponential, as long as some eigenvalues of $\mathcal{L}$ have strictly negative real part. We begin with a negative result which, in particular, shows how the initialization-free and attractive characterizations are mutually exclusive.

Proposition 1: Assume $\mathcal{H}_I = (\mathcal{H}_{NS} \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$, $\mathcal{H}_R \neq 0$, and let H, $\{L_k\}$ be the Hamiltonian and the error generators as in (1), respectively. Let $\mathcal{H}_{NS}$ support a NS. If $L_{P,k} = L_{Q,k}^\dagger = 0$ for every k, then $\mathcal{H}_{NS}$ is not attractive.

Remark: The conditions of the above Proposition are obeyed, in particular, for NSs in the presence of purely *Hermitian* noise operators, that is, $L_k = L_k^\dagger$, $\forall k$. As a consequence, attractivity is never possible for *unital* Markovian noise, as defined by the requirement of preserving the fully mixed state. Still, even if the condition $L_{P,k} = L_{Q,k}^\dagger = 0$ condition holds, attractive subsystems may exist in the pure-factor case, where $\mathcal{H}_R = 0$. Sufficient conditions are provided by the following:

Proposition 2: Assume $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F$ ($\mathcal{H}_R = 0$), and let $\mathcal{H}_S$ be invariant under a QDS, hence of the form

$$\mathcal{L} = \mathcal{L}_S \otimes \mathbb{I}_F + \mathbb{I}_S \otimes \mathcal{L}_F.$$

If $\mathcal{L}_F(\cdot)$ has a unique attractive state $\hat{\rho}_F$, then $\mathcal{H}_S$ is attractive.

Interesting linear-algebraic conditions for determining whether a generator $\mathcal{L}_F(\cdot)$ is has a unique attractive state, but not necessarily pure, are presented in [15], [16]. Since the main application for the present paper will be *pure*-state stabilization problems, our main emphasis is on attractivity in the subspace case.

Theorem 2 (Attractive Subspace): Assume $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$ ($\mathcal{H}_F = \mathbb{C}$), and let $\mathcal{H}_S$ be an invariant subspace under $\mathcal{L}$. Assume that there exist a functional $V(\rho) \geq 0$ on $\mathfrak{D}(\mathcal{H}_I)$, continuous with continuous derivative, such that $\dot{V}(\rho) \leq 0$ in $\mathfrak{D}(\mathcal{H}_I) \setminus \mathfrak{I}_S(\mathcal{H}_I)$. Let

$$\begin{aligned} \mathcal{W} &= \{\rho \in \mathfrak{D}(\mathcal{H}_I) |\, \dot{V}(\rho) = 0\}, \\ \mathcal{Z} &= \{\rho \in \mathfrak{D}(\mathcal{H}_I) |\, \text{trace}[\Pi_R \mathcal{L}(\rho)] = 0\}, \end{aligned}$$

where Π_R is the orthogonal projector on $\mathcal{H}_R$. If $\mathcal{W} \cap \mathcal{Z} \subseteq \mathfrak{I}_S(\mathcal{H}_I)$, then $\mathcal{H}_S$ is attractive.

Proof. Consider $V_1(\rho) = \text{trace}(\Pi_R\rho) + \text{trace}(\Pi_R\rho)V(\rho)$. It is zero iff $\rho_S = 0$, i.e. for perfectly initialized states. By computing $\mathcal{L}(\rho)$, we get;

$$\text{trace}[\Pi_R \mathcal{L}(\rho)] = -\text{trace}\Big(\sum_k L_{P,k}^\dagger L_{P,k} \rho_R \Big), \qquad (9)$$

that is always negative or zero. Hence

$$\dot{V}_1(\rho) = \text{trace}(\Pi_R \mathcal{L}(\rho))(1 + V(\rho)) + \text{trace}(\Pi_R\rho)\dot{V}(\rho) \leq 0,$$

for every $\mathfrak{D}(\mathcal{H}_I)$, and it is zero only in $\mathcal{W} \cap \mathcal{Z} \cup \mathfrak{I}_S(\mathcal{H}_I)$. If $\mathcal{W} \cap \mathcal{Z} \subseteq \mathfrak{I}_S(\mathcal{H}_I)$, by applying Krasowskii-LaSalle invariance theorem, we conclude. ■

The following result immediately follows:

Corollary 2: Assume $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$ ($\mathcal{H}_F = \mathbb{C}$), and let $\mathcal{H}_S$ support an invariant subspace under $\mathcal{L}$. Assume that

$$\sum_k L_{P,k}^\dagger L_{P,k} > 0, \qquad (10)$$

where $>$ means strictly positive. Then $\mathcal{H}_S$ is attractive.

Proof. It suffices to note that (10) guarantees that (9) in the proof of the Theorem above is zero iff $\rho_R = 0$. The conclusion follows by taking a $V(\rho)$ constant and positive on $\mathfrak{D}(\mathcal{H}_I)$. ■

Remark: From considerations on the rank of the l.h.s. of (10) and the $n \times r$ dimension of $L_{P,k}$, the condition of Corollary 2 may be obeyed only if $n \geq r$, i.e. $\dim(\mathcal{H}_S) \geq \dim(\mathcal{H}_R)$. An application of this result will be given in Section III-A.

III. Markovian feedback control

Building on pioneering work by Belavkin [17], it has been long acknowledged for a diverse class of controlled quantum system that intercepting and feeding back the information leaking out of the system allow to better accomplish a number of desired control tasks (see [18], [19], [20], [21], [22], [23] for representative contributions). This requires the ability to both effectively monitor the environment and condition the target evolution upon the measurement record. The basic setting we consider is a measurement scheme which mimicks optical homo-dyne detection for field-quadrature measurements, whereby the target system (e.g. an atomic cloud trapped in an optical cavity) is indirectly monitored via measurements of the outgoing laser field quadrature [18], [24]. The conditional dynamics od the state is stochastic, driven by the fluctuation one observes in the measurement. Considering a suitable feedback infinitesimal operator determined by a *feedback Hamiltonian* F, and taking the expectation with respect to the noise trajectories, this leads to the Wiseman-Milburn *Markovian Feedback Master equation* (FME) [18], [19]:

$$\frac{d}{dt}\rho_t = \mathcal{F}\Big(H + \frac{1}{2}(FM + M^\dagger F), \rho_t\Big) + \mathcal{D}(M - iF, \rho_t). \quad (11)$$

In the following sections, we will tackle state-stabilization and NS-synthesis problems for controlled Markovian dynamics described by FMEs.

A. Control assumptions

The feedback state-stabilization problem for Markovian dynamics has been extensively studied for the single-qubit case [25], [26]. In the existing literature, however, the standard approach to design a Markovian feedback strategy is to assign both the measurement and feedback operators M, F, and to treat the measurement strength and the feedback gain as the relevant control parameters accordingly.

Throughout the following sections, we will pretend to have more freedom, considering, for a fixed measurement operator M, *both* F and H as tunable control Hamiltonians.

Definition 6 (CHC): A controlled FME of the form (11) supports *complete Hamiltonian control* (CHC) if (i) arbitrary feedback Hamiltonians $F \in \mathfrak{H}(\mathcal{H}_I)$ may be enacted; (ii) arbitrary *constant* control perturbations $H_c \in \mathfrak{H}(\mathcal{H}_I)$ may be added to the free Hamiltonian H.

This leads to both new insights and constructive control protocols for systems where the noise operator is a generalized angular momentum-type observable, for generic finite-dimensional systems. Physically, the CHC assumption must be carefully scrutinized on a case by case basis, since constraints on the form of the Hamiltonian with respect to the Lindblad operator may emerge, notably in so-called weak-coupling limit derivations of Markovian models [2]. A first, interesting consequence of assuming CHC emerges directly from the following observation:

Lemma 1: The Markovian generator

$$\frac{d}{dt}\rho_t = -i[H,\,\rho_t] + \sum_k \mathcal{D}(L_k,\rho_t) \tag{12}$$

is equivalent to

$$\frac{d}{dt}\rho_t = -i[H + H_{corr},\,\rho_t] + \sum_k \mathcal{D}(\tilde{L}_k,\rho_t), \tag{13}$$

where for all k:

$$\tilde{L}_k = L_k + c_k\mathbb{I}, \quad c_k \in \mathbb{C}, \tag{14}$$

$$H_{corr} = -i\sum_k (c_k^* L_k - c_k L_k^\dagger). \tag{15}$$

Note that for Hermitian L and real c, $H_{corr} = 0$. In general, by exploiting CHC, we may vary the trace of the Lindblad operators through transformations of the form (14), and, if needed or useful, appropriately counteract the Hamiltonian correction H_{corr} with a constant control Hamiltonian. This may allow to stabilize subsystems that were not invariant for the uncontrolled equation, *without directly modifying the non-unitary part*.

Example 1. Consider a generator of the form:

$$\frac{d}{dt}\rho(t) = -i[\sigma_z,\rho(t)] + \Big(L\rho(t)L^\dagger - \frac{1}{2}\{L^\dagger L,\,\rho(t)\}\Big),$$

where $L = \sigma_z + \sigma_+$. Suppose that the task is to make $\rho_d = \mathrm{diag}(1,0)$ invariant. Since $H_P = 0, L_S = 1, L_p = 1$, invariance is not ensured by the uncontrolled dynamics. Using the above result, it suffices to apply a constant Hamiltonian $H_{corr} = -i(L - L^\dagger) = \sigma_y$. The desired state turns out to be also attractive, see Proposition 3 below.

B. Pure-state preparation with Markovian feedback: Paradigmatic examples

Let us first consider a two-dimensional system. Our perspective differs from the one presented in [25] not only because we mainly focus on continuous measurement of *Hermitian spin observables* but, more importantly, because we start from identifying what constraints must be imposed to a two-dimensional Lindblad equation as in (1) for ensuring that one of the system's pure states is a stable attractor. Without loss of generality, let such a state be written as $\rho_d = \mathrm{diag}(1,0)$, and write, accordingly,

$$L_k = \begin{pmatrix} l_{k,S} & l_{k,P} \\ l_{k,Q} & l_{k,R} \end{pmatrix}, \quad H = \begin{pmatrix} h_S & h_P \\ h_P^* & h_R \end{pmatrix}.$$

As a straightforward application of the results of the previous Section, we have the following.

Proposition 3: The pure state $\rho_d = \mathrm{diag}(1,0)$ is a stable attractor for a two-dimensional quantum system evolving according to (1) iff:

$$ih_P - \frac{1}{2}\sum_k l_{k,S}^* l_{k,P} = 0, \tag{16}$$

$$l_{k,Q} = 0, \quad \forall k, \tag{17}$$

and there exists a $\bar{k}$ such that $l_{\bar{k},P} \neq 0$.

We provide next a characterization of the stabilizable manifold.

Proposition 4: Assume CHC. For any measurement operator M, there exist a feedback Hamiltonian F and a Hamiltonian compensation H_c able to stabilize an arbitrary desired pure state ρ_d for the FME (11) iff

$$[\rho_d, (M + M^\dagger)] \neq 0. \tag{18}$$

Proof. Consider as before a basis where $\rho_d = \mathrm{diag}(1,0)$, and let M^H and M^A denote the Hermitian and anti-Hermitian part of M, respectively. By (18), M^H cannot be diagonal in the chosen basis. In fact, assume M^H to be diagonal, then, by Proposition 3, $M^S - F$ must be brought to diagonal form to ensure invariance of ρ_d. Hence, by the same result it follows that ρ_d cannot be made attractive. On the other hand, if M^H is not diagonal, we can always find an appropriate F in order to get an upper diagonal $L = M^H + i(M^S - F)$, and $H' = H + (FM + M^\dagger F)/2$. To conclude, it suffices to devise a compensation Hamiltonian H_c such that the condition $i(H' + H_c)_P - \frac{1}{2}l_S^* l_P = 0$ is satisfied. ∎

The above proof naturally suggests a constructive algorithm for designing the feedback and correction Hamiltonian needed for stabilizing the intended pure state. From our analysis, we also recover the results of [25] recalled before. For example, the states that are never stabilizable within the control assumptions of [25] are the ones commuting with the Hermitian part of $M = \sigma_+$, that is, $M^H = \sigma_x$. On the xz plane in the Bloch's representation, the latter correspond precisely to the equatorial points. The following examples serve to illustrate the basic ideas we shall extend to the d-level case.

Example 2: The simplest choice to obtain an attractive generator is to engineer a dissipative part determined by

$$L = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $H = n_0\mathbb{I}_2 + n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, with $n_0, n_x, n_y, n_z \in \mathbb{R}$. Consider e.g. $M = \frac{1}{2}\sigma_x$ and $F = -\frac{1}{2}\sigma_y$. Notice that in this case $\frac{1}{2}(FM + M^\dagger F) = 0$, thus $H' = H$. Substituting in the FME (11), one clearly obtain the desired result, provided that $H_c = -n_x\sigma_x - n_y\sigma_y$.

Assume that it is possible to continuously monitor a single observable, e.g. σ_x in the above example. Since the choice of the reference frame for the spin axis is conventional, by suitably adjusting the relative orientation of the measurement apparatus and the sample, it is then in principle possible to prepare and stabilize any desired pure state with the same control strategy.

Example 3: Consider next a three-level system undergoing continuous observation of a spin observable, for instance:

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Assume that our goal is to make the state $\rho_d = \mathrm{diag}(1,0,0)$ attractive. In analogy with the above example, one is led to consider the feedback Hamiltonian

$$F = -J_y = -\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

thereby obtaining:

$$L = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In fact, if we define $D = \mathrm{diag}(1,0,0)$ and $V(\rho) = \mathrm{Tr}(D\rho)$, we get $\dot V(\rho) = -\sqrt{2}\rho_{33}$. Therefore, $V(\rho)$ is positive, has negative semi-definite derivative, and the derivative is zero in $\mathscr{Z} = \{\rho | \rho_{33} = 0\}$. By invoking Theorem 2, $\mathcal{V} = \{\rho|\rho_{22} = 0\}$, the two sets are disjoint beside ρ_d itself, hence the state is attractive. Notice that this also provides an example of an attractive state, which does *not* satisfy the sufficient condition of Corollary 2 (in fact, Corollary 2 is interesting for state stabilization only with two-dimensional systems [6]).

The previous examples naturally extend to generic d-level systems, provided that similar structure of the observable is ensured. Let the pure state to be FME-stabilized be written as $\rho_d = \mathrm{diag}(1, 0, \ldots, 0)$.

Proposition 5: Assume that H is diagonal. The pure state ρ_d is a stable attractor for the FME (11) conditioned over the continuous measurement of the operator:

$$M = \frac{1}{2}\begin{pmatrix} 0 & m_1 & 0 & \cdots & 0 \\ m_1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & m_{d-1} \\ 0 & \cdots & 0 & m_{d-1} & 0 \end{pmatrix},$$

and a Markovian feedback Hamiltonian:

$$F = -\frac{i}{2}\begin{pmatrix} 0 & m_1 & 0 & \cdots & 0 \\ -m_1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & m_{d-1} \\ 0 & \cdots & 0 & -m_{d-1} & 0 \end{pmatrix},$$

with $m_i \neq 0$, for $i = 1, \ldots, (d-1)$.

By contruction, F and M play a role analogous to the σ_y and σ_x observables of the $d = 2$ case. Notice that their form is not different from that of standard, higher-dimensional spin observables. The main advantage of Markovian feedback techniques with respect to other design strategies, based on estimation of the underlying quantum state (so-called *Bayesian techniques*), is the simplicity of a *direct output-feedback loop* – as opposed to the task of integrating a d^2-dimensional stochastic master equation in real time, which becomes rapidly prohibiting as d grows. On the other hand, direct feedback requires strong control capabilities and perfect detection. The parameters one has to accurately tune are the feedback and measurement operators, along with both the system Hamiltonian and its control perturbation, if needed. Also notice that Bayesian feedback-stabilization problems usually aim at stabilizing a pure state that commutes with a Hermitian observable, namely one of its eigenstates. This, in the light of Proposition 4, would not be possible with Markovian feedback. In the Markovian approach, however, it is substantially simpler to stabilize states that are not equilibrium points for the uncontrolled dynamics.

C. Imperfect detection case: Perturbative analysis

From an experimental viewpoint, the perfect detection assumption may seriously constrain the applicability of the analysis and synthesis techniques developed so far. Nevertheless, for state stabilization problems, one may assess the role of the perfect-detection hypothesis and the possibility to relax it. If $\eta < 1$, the FME is modified as follows [24]:

$$\begin{aligned} \frac{d}{dt}\rho_t \;&=\; \mathcal{F}\Big(H + 1/2(FM + M^\dagger F), \rho_t\Big) \\ &\quad + \; + \mathcal{D}(M - iF, \rho_t) + \varepsilon\mathcal{D}(F, \rho_t), \end{aligned} \tag{19}$$

where we defined $\varepsilon = (1-\eta)/\eta$.

In [2], generators of the form (1) are rewritten in a convenient way by choosing a suitable Hermitian basis in $\mathfrak{B}(\mathcal{H}_i) \approx \mathbb{C}^{d\times d}$ [1]. In such a basis, all density operators are represented by d^2-dimensional vectors $\bar\rho = (\rho_0, \rho_1, \ldots, \rho_{d^2-1})^T$, where the first component ρ_0, relative to $\frac{1}{\sqrt{d}}\mathbb{I}_d$, is invariant and equal to $\frac{1}{\sqrt{d}}$ for TP-dynamics. Let $\rho_v = (\rho_1, \ldots, \rho_{d^2-1})^T$. Hence, any QDS generator $\mathcal{L}(\rho)$ must take the form:

$$\frac{d}{dt}\rho = \mathcal{L}(\rho) = \left(\begin{array}{c|c} 0 & 0 \\ \hline C & D \end{array}\right)\begin{pmatrix} 1/\sqrt{d} \\ \rho_v \end{pmatrix}. \tag{20}$$

[1] In fact, endowing $\mathbb{C}^{d\times d}$ with the inner product $\langle X, Y\rangle := \mathrm{trace}(X^\dagger Y)$ (Hilbert-Schmidt), we may use a basis where the first element is $\frac{1}{\sqrt{d}}\mathbb{I}_d$, and complete it with a orthonormal set of Hermitian, traceless operators. This can always be done for finite d, for example by employing the natural d-dimensional extension of the Pauli matrices [2], [27].

Assume that the dynamics has a unique attractive state $\bar{\rho}^{(0)}$. Thus D must be invertible and we get:

$$\bar{\rho}^{(0)} = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 \\ -D^{-1}C \end{pmatrix}.$$

Consider now a small perturbation of the generator depending on the continuous parameter ε, with $1-\delta < \eta < 1$, and δ sufficiently small so that $(D+\varepsilon D')$ remains invertible. The generator becomes:

$$\frac{d}{dt}\rho = \mathcal{L}(\rho) = \left[\left(\begin{array}{c|c} 0 & 0 \\ \hline C & D \end{array} \right) + \varepsilon \left(\begin{array}{c|c} 0 & 0 \\ \hline C' & D' \end{array} \right) \right] \begin{pmatrix} 1/\sqrt{d} \\ \rho_v \end{pmatrix}, \tag{21}$$

and the new attractive, unique equilibrium state is:

$$\bar{\rho}^{(\varepsilon)} = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 \\ -(D+\varepsilon D')^{-1}(C+\varepsilon C') \end{pmatrix}.$$

Being $\bar{\rho}^{(\varepsilon)}$ a continuous function of ε, we are guaranteed that for a sufficiently high detection efficiency the stable attractor will be arbitrarily close to the desired one in trace norm. Therefore, if we relax our control task to a state preparation problem with sufficiently high fidelity, this may be accomplished with a sufficiently high detection efficiency, yet strictly less than 1.

IV. Conclusion

We have revisited some fundamental concepts on Markovian dynamics for quantum systems and reformulated the notion of a general quantum subsystem in *linear-algebraic* terms. A complete characterization of invariant subsystems for Markovian quantum dynamical systems has been provided. When imperfect subsystem initialization is considered, the conditions to be imposed on the Markovian generator become more demanding, motivating the new notion of asymptotically stable *attractive subsystem*. The linear-algebraic approach we adopted, along with Lyapunov's stability techniques, provided us with explicit stabilization results that have been illustrated in simple yet paradigmatic examples.

In the second part of the work, the conditions identified for subsystem invariance and attractivity serve as the starting point for designing output-feedback Markovian strategies able to actively achieve the intended quantum stabilization. We have completely characterized the state-stabilization problem for two-level systems. While the analysis assumes perfect detection efficiency, a perturbative arguments indicates how unique attractive states depend in a continuous fashion on the model parameters.

Further work is needed in order to establish entirely general Markovian feedback stabilization results, including finite bandwidth and detection efficiency, as well as simultaneous monitoring of multiple observables. Among the most interesting perspectives, additional investigation is required to establish the full power of Hamiltonian control and Markovian feedback in synthesizing NS structures. This may point to new venues for producing protected realizations of quantum information for physical systems whose dynamics is described by quantum Markovian semigroups.

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