IMPROVEMENT OF NUMERICAL DESCRIPTION OF NON-LINEAR SHOCK PROFILES BY USE OF ANALYTICAL SOLUTIONS OF DIFFERENTIAL APPROXIMATIONS

Alexey Porubov
Institute of Problems in Mechanical Engineering, Russia
porubov.math@mail.ioffe.ru

Daniel Bouche
CMLA, ENS de Cachan, Cachan, France
daniel.bouche@cea.fr

Guy Bonnaud
CEA, INSTN, Centre de Saclay France
guy.bonnaud@cea.fr

Abstract

An analysis of dispersive/dissipative features of the difference schemes is developed based on particular asymptotic and exact travelling wave solutions of the differential approximation (DA) of the equation under study. It is shown on the example of the non-linear Burgers' equation, that its asymptotic travelling wave solution allows us to describe deviations in the shock wave caused by a scheme dispersion/dissipation. These analytical predictions may be used to diminish bad deviations by suitable choice of the parameters of the scheme. Then it is shown, that exact travelling wave solution of the DA for the non-linear Burgers' equation helps us to suggest artificial non-linear additions to the schemes to suppress the influence of the scheme dispersion and/or dissipation. Application of the analytical solutions is demonstrated using the Lax-Wendroff scheme.

Key words

Scheme dispersion, non-linear wave, analytical and numerical solutions

1 Introduction

A discrete model described by a scheme often possesses internal dispersive and/or dissipative properties caused by a method of discretization. It gives rise to non-physical deviations in the numerical solution. One can decrease the influence of these bad factors by varying time and space steps and modifying the method of approximation. A possibility to know how to do it is the application of the method of differential approximation (DA) [Lerat and Peyret, 1975; Shokin, 1983; Mukhin et al, 1983; Fletcher, 1991]. This method allows us to study dispersive and dissipative features of a scheme by an analysis of the *differential* equation called a differential approximation (DA). It is obtained using a substitution of the Taylor expansions of the discrete functions into a *difference* scheme. An analysis of the resulting partial differential equation (PDE) is possible if the expansion is truncated at some order. However, the DA for a dicretization of a *non-linear* equation is also non-linear and nonitegrable equation. Then, only *particular* analytical travelling wave solutions existing at specific initial conditions may be obtained. A natural question arises: may the particular asymptotic and exact solutions be used to analyze the features of the DA, thus the features of difference schemes?

In this paper, we demonstrate the efficiency of the use of analytical solutions for understanding the deviations in the shock caused by the scheme features on an example of the non-linear Burgers' equation,

$$v_t + (v^2)_x - b\, v_{xx} = 0. \tag{1}$$

In general, its DA may be written as [Lerat and Peyret, 1975; Shokin, 1983; Engelberg, 1999]

$$u_t+(u^2)_x-b\, u_{xx} \;=\; -s(u)\, u_{xxx}+\alpha(u, u_x)-q(u)\, u_{xxxx}, \tag{2}$$

In particular, we obtain for the Lax-Wendroff (LW) scheme [Lerat and Peyret, 1975; Shokin, 1983; Fletcher, 1991]

$$u_t + (u^2)_x \; - b u_{xx} = \; \frac{\triangle t^2}{24}(u^4)_{xxx} \; - \; \frac{\triangle x^2}{12}(u^2)_{xxx}, \tag{3}$$

One can try to find an exact travelling wave solution of Eq.(3). Another way is to consider the r.h.s. of it as a small perturbation and find an asymptotic solution. In this last case it was suggested in [Mukhin et al, 1983] to linearize the r.h.s. around a constant, say, the value of $u_{-\infty} \; = \; u(x \to -\infty)$. Then Eq.(3) is simplified towards the linearly perturbed Burgers equation,

$$u_t + u\, u_x \; - b u_{xx} = \; - \frac{\left(\triangle x^2 - \triangle t^2 u_{-\infty}^2\right) u_{-\infty}}{6} u_{xxx}, \tag{4}$$

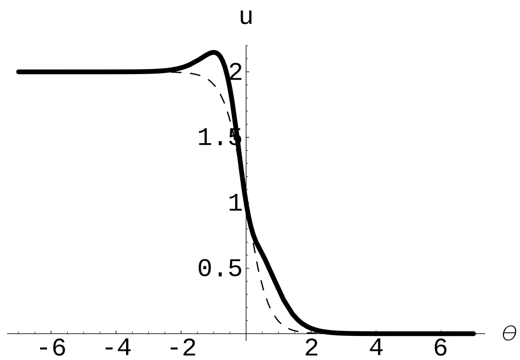

Figure 1. Influence of the weak dispersion on the Burgers shock wave solution described by the asymptotic solution (9)

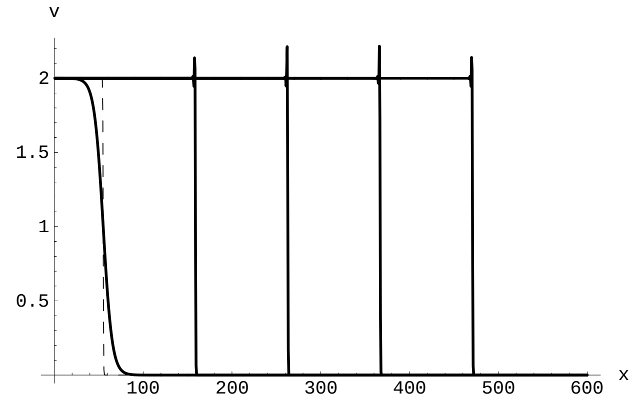

Figure 2. Numerical simulation of a shock wave described by Eq. (1) using the LW scheme. Shown by dashed line is the Burgers' kink.

where the r.h.s. accounts for the linear scheme dispersion. Similar equations may be written for the DA of another schemes with dispersion, and with dissipative features.

We suggest two kinds of the analysis of the DA. The first one is to obtain a travelling wave asymptotic solution. It accounts for the deviations in the shock profile of the solution of the Burgers equation caused by the scheme dispersion. The analytical relationship is obtained to find how these deviations depend on the temporal and spatial steps of the scheme. The deviations in the shock caused by the scheme dispersion may be diminished but not suppressed by variations in the space and time step according to the asymptotic solution. Then the exact solutions are employed to find the conditions where the smooth profile is achieved as a result of the compensation of dispersion by artificial additional nonlinear modification of the scheme.All analytical finding are confirmed by numerical simulations of the Burgers equation using the classic Lax-Wendroff scheme and this scheme modified by adding the artificial nonlinearity prescribed by the exact solution.

2 Use of asymptotic solution

Let us recall that the Burgers equation,

$$u_t + (u^2)_x - b\, u_{xx} = 0, \qquad (5)$$

possesses the shock-wave solution (or a kink), see,e.g., [Whitham, 1974],

$$u_0 = \frac{1}{2}\left(2\, b\, p\, \tanh(p(X - V\, t)) + V\right). \qquad (6)$$

where p and V are free parameters to be defined by the boundary conditions, i.e., for given values of u at infinities. For simplicity we assume $u_\infty = 0$. Then we have

$$V = u_{-\infty},\ p = -u_{-\infty}/(2b). \qquad (7)$$

The solution has familiar shape of the smooth shock wave. However, presence of dispersion in the LW scheme results in parasitic oscillations in the shape of the shock wave. To describe an influence of dispersion, a travelling wave asymptotic solution to Eq. (4) is sought in the form

$$u(\theta) = u_0(\theta) + \delta u_1(\theta) + ... \qquad (8)$$

where $\theta = x - Vt$, and $u_1 \to 0$ at $\theta \to \pm\infty$. Substituting this series into Eq.(4) we obtain the Burgers shock (6) in the leading order, while the first correction is obtained by solving the linear inhomogeneous equation for u_1. As a result the first two terms in (8) give rise to the solution

$$u = \frac{1}{2}\left(2\, b\, p\, \tanh(p(X - V\, t)) + V\right) +$$

$$\frac{p^2\left(\triangle x^2 - \triangle t^2 u_{-\infty}^2\right) u_{-\infty}}{6} \cosh(p\theta)^{-2} \log(\cosh(p\,\theta)) \qquad (9)$$

One can see in Fig. 1 a non-symmetric influence on the upper and lower parts of the shock due to the addition to the Burgers kink in the solution (9). A "hat" or a bump appears at the upper part of the shock while the lower one exhibits a smoother profile. The sign of the dispersion coefficient in Eq.(4) cannot be changed without exceeding the stability criterium $u_{-\infty}\triangle t < \triangle x$. According to the asymptotic solution disturbances will be weaker for higher temporal step since the dispersion coefficient decreases with increase in $\triangle t$. Numerical simulations of the Burgers equation using the LW scheme is shown in Fig. 2. The initial condition is chosen in the form of the shock with the slope differing from that of the Burgers shock (6)), shown by dashed line. As time goes, numerical solution demonstrates transformation to the shock with the slope coinciding

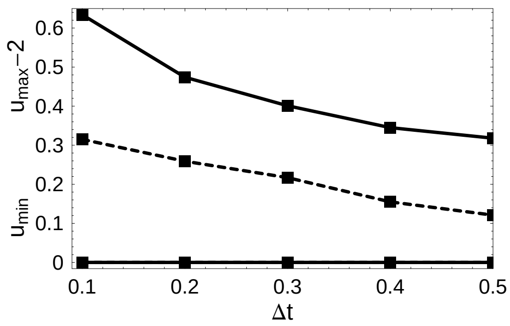

Figure 3. Variations of minimum and maximum of the shock vs temporal step for the LW scheme: $u_{max} - 2$ varies between upper solid line and dotted line, while lower solid line corresponds to u_{min}.

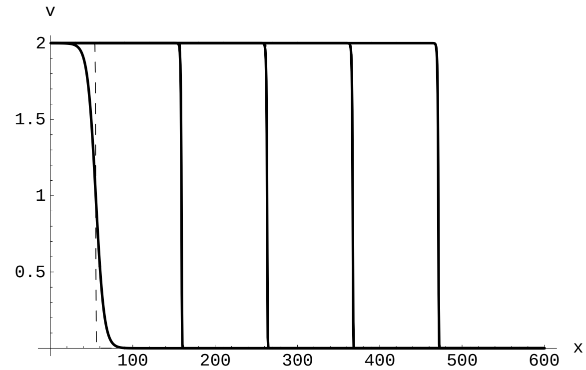

Figure 4. Temporal evolution of an initial smooth kink-shaped profile for the LW scheme with artificial nonlinear term. Shown by dashed line is the Burgers' kink.

with that of (6)) since no dashed lines are visible at all four stages shown in Fig. 2 against a background of the numerical solution. Also the bump develops at the upper side of the wave front in the agreement with the asymptotic solutions, c.f. Fig. 1. The increase in the values of the space and time steps done following the solution (9) provides a decrease in the height of the bump, see Fig. 3, but not the full disappearance of the bump. Note that the prediction remains valid even for the moderate values of $u_{-\infty} - u_{\infty} = 2$ chosen outside a formal applicability of the linearized equation (4), $|u - u_{-\infty}| << 1$.

3 Use of exact solutions

The shock wave of the Burgers equation arises as a result of a balance between nonlinearity and dissipation described by the second and the third terms in Eq.(5) respectively. The balance between nonlinearity and dissipation may be destroyed or perturbed by the presence of dispersion. It does not refuse existence of an exact travelling wave solution. In particular, Eq.(4) often called the Korteweg-de Vries-Burgers (KdVB) equation, possesses well known exact kink-shaped solution

$$u = B\,\mathrm{sech}^2(p(x - Vt)) + F\,\tanh(p(x - Vt)) + C. \qquad (10)$$

whose shape is close to that of the Burgers shock wave (6). However, it does not contain free parameters, all of them are defined by the coefficients of the KdVB equation,

$$B = 6p^2\,s,\ F = -\frac{6b\,p}{5},\ C = \pm\frac{3b^2}{25s},\ p = \mp\frac{b}{10s},$$

$$V = \pm\frac{6b^2}{25s}, s = \frac{\left(\triangle x^2 - \triangle t^2 u_{-\infty}^2\right) u_{-\infty}}{6}$$

As a result, boundary conditions cannot be satisfied for any $u_{-\infty}$, and the velocity of the wave differs from that of the Burgers shock wave (6). It does not mean that the DA cannot predict the deviations in profile- it means that not any *particular* exact solution may account for it. Also it is described by the *asymptotic* solution of the DA in the last section. The problem is to find the improvement of the scheme providing absence of the bump shown in Fig. 2.

One possibility is to add artificial term in the DA (10). It was suggested in [Mukhin et al, 1983] to add artificial terms in the scheme to suppress numerical dispersion. It was found that addition of the terms produced by DA (e.g., r.h.s. of (3) with opposite sign coefficients does not solve the problem due to the influence of the higher-order terms omitted in the truncated expansions (3).

We suggest to add nonlinear term to balance the influence of dispersion and to employ exact solutions to find suitable nonlinear additions providing an exact solution with the shape and velocity close as much as possible to those of the kink solution of the Burgers equation. For this purpose, an additional non-linear term is suggested to add to Eq. (4),

$$u_t + u_x^2 - b\,u_{xx} + s\,u_{xxx} + \gamma\,u_{xx}^2 = 0, \qquad (11)$$

Exact solution of Eq.(11) is sought using the method of ansatz. It turns out that the last equation possesses the exact kink-shaped solution in the form (6) provided that the velocity V is defined by the boundary conditions as in Eq. (7) and

$$\gamma = -\frac{s}{b + 2p\,s}, p = \frac{-b \pm \sqrt{b^2 + 4s\,u_{-\infty}}}{4s}, \qquad (12)$$

Therefore, the slope of the last solution differs from that of the Burgers kink while velocity is the same. We are able to define suitable value of γ through the parameters of the scheme, and it disappears as the temporal and spatial steps tend to zero, i.e., in the continuum limit. The last means that we still model the Burgers equation even if we add in the discrete LW scheme the central difference representation of the term $\gamma\ u_{xx}^2$ choosing the coefficient γ according to Eq. (12). Numerical simulations are shown in Fig. 4 where the initial profile is chosen the same as in Fig. 2. One can see the suppression of the bump (c.f. Fig. 2) while the slope of the wave coincides with that of the Burgers' kink since again the dashed line representing exact travelling wave solution (6) is not visible against a background of the numerical solution. One can note that deviations in the value of the artificial term coefficient from that of Eq. (12) result either in an appearance of a bump or in an instability of the scheme.

4 Conclusions

The main result of the paper is that particular analytical solutions of the differential approximation of a numerical scheme for the Burgers' equation predict shock profiles arising in its numerical study. The solutions provide us with the explicit relationships between the parameters of the scheme and the coefficients in the equation responsible for the scheme dispersion. It allows us to choose these parameters so as to avoid one or another perturbation of the numerical shock profile. We are able to predict deviations in the shock in dependence of the value of the steps of a numerical scheme. Therefore, the scheme deviations of the numerical shock may be diminished by choosing suitable values according to our asymptotic analysis. Also we manage to compensate the influence of dispersion by adding artificial nonlinear term. It is exact travelling wave solution that helps us to choose suitable term to avoid unreasonable perturbations on the shock profile.

The presentation here is restricted to the LW scheme with dispersion. The same analysis may be done not only for other schemes with dispersion (Mac-Cormack, Warming-Beam) but for the schemes with dissipative features (3d order LW scheme) also. These results may be found in our paper [Porubov et al, 2008].

The non-linear Burgers' equation is chosen to demonstrate the efficiency of our approach excluding huge analytical solutions. The applicability of the method to more complicated equations, e.g., gas-dynamics equations will be the topic of future work.

A more rigorous mathematical justification of the method is improbable since the differential approximations belong to the class of so-called non-integrable equations.

Acknowledgment

This work of one of the authors (AVP) was supported by the CEA/DAM grant 0611S/DIR and by the grant of the Russian Science Support Foundation.

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