OPTIMAL CONTROL OF HYBRID SYSTEMS WITH POLYNOMIAL IMPULSES

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Abstract
We address an optimal control problem for a measure driven hybrid dynamical system. The impulsive dynamics is due to the BV-relaxation (the compactification of the trajectory tube in the weak* topology of the space BV of functions of bounded variation) of a dynamical system with polynomial dependence on a control variable in the right-hand side under the constraint on the norm of a control in the Lebesgue space L_p. The relaxed system is described by a certain measure differential equation. The hybrid feature is expressed in the presence of "nonstandard mixed constraints". The latter term is used to name asymptotic constraints relating the state and the measure, and these conditions are formulated as constraints on one-sided limits of a solution to the measure differential equation. The main result is an equivalent transformation of the considered model to a usual optimal control problem. To this end we propose a special space-time transformation technique.

Key words
Hybrid systems, optimal control, impulsive control, polynomial impulses, mixed constraints, discontinuous time reparameterization technique

1 Introduction

The note trails the paper [Goncharova and Staritsyn, 2015a] on optimal polynomially-impulsive control. Now, following [Goncharova and Staritsyn, 2012, Goncharova and Staritsyn, 2011a, Goncharova and Staritsyn, 2011b, Goncharova and Staritsyn, 2015b], we put the phenomenon of polynomial impulses within the framework of hybrid systems, and therefore extend our previous results on impulsive hybrid control.

The first motivation of this extension stems from the quadratic case. Systems with square impulses and respective problems of dynamic optimization were comprehensively treated by A. Bressan and F. Rampazzo. Such models are Lagrangian mechanical systems, where time-dependent constraints are regarded as controls (see e.g. [Aldo Bressan and Motta, 1993, Bressan and Rampazzo, 1994, Bressan, 2008, Bressan and Rampazzo, 2010]).

Close relationships between the theory of hybrid dynamical systems and the paradigm of impulsive control were noticed by many researchers, see, e.g., [Aubin, 2000, Branicky, Borkar and Mitter, 1998, Haddad, Chellaboina, and Nersesov, 2006, Kurzhanski and Tochilin, 2009, Miller and Rubinovich, 2001, Miller and Bentsman, 2006, Pereira, Silva and Oliveira, 2008, Van der Schaft and Schumacher, 2000, Fraga, Gomes and Pereira, 2007]. We address a class of impulsive dynamical systems whose right-hand side is an algebraic polynomial of an impulsive control variable with coefficients depending on state and usual (bounded) control variables. Such (less general) models were studied in [Pedregal and Tiago, 2009, Rampazzo and Sartori, 2000].

Our prototypical system is a conventional one with two types of controls subject to geometrical and L_p-constraints (a uniform bound on the norm in the Lebesgue space), respectively, and the right-hand side of a polynomial structure with respect to the control of the latter type. In [Goncharova and Staritsyn, 2015a] we introduced a notion of its generalized solution, and designed a certain system's relaxation in the weak* topology of the space of functions of bounded variation. The relaxation was shown to be constructively defined by means of a discontinuous time reparameterization. Under some natural assumptions, we gave an explicit representation of generalized solutions by a measure differential equation.

General nonlinear dynamical systems with unbounded control sets were investigated by [Warga, 1962, Warga, 1972, Warga, 1987, Gurman, 1972, Miller and Rubinovich, 2001]. Systems with affine dependence on impulsive control and corresponding singular problems are studied most comprehensively, see, e.g. [Bressan and Rampazzo, 1994, Dykhta, 1990, Dykhta and Samsonyuk, 2009, Filippova, 2005, Gurman, 1972, Miller, 1996, Miller and Rubinovich, 2001, Rishel, 1965, Warga, 1962, Warga, 1972, Warga, 1987, Zavalischin and Sesekin, 1997].

2 Model description

Consider the following problem (P) of optimal impulsive control:

$$\text{Minimize } I = F(x(T))$$

subject to the dynamical system

$$dx = f_0(x,u)dt + \sum_{q\in Q\setminus\{p\}} f_q(x,u)\, l^q\, dt + \\ + f_p(x,u)\,\vartheta(dt), \quad x(0-) = x_0, \qquad (1)$$

$$\varrho = (u, \vartheta) \in \mathcal{P}, \qquad (2)$$

and the conditions

$$x(t-) \in \mathcal{Z}_-,\ x(t) \in \mathcal{Z}_+\ |\vartheta|\text{-a.e. on } [0,T]. \qquad (3)$$

The measure differential equation of the form (1) denotes a certain dynamical system specified in Section 4 together with a solution concept.

Beforehand, we discuss the model's input data and its main features.

Input data. Suppose we are given

- positive real numbers T and M,
- a rational number $p \geq 1$, and a finite set Q of distinct positive rational numbers such that $\max Q = p$, and the maps $v \mapsto v^q$, $q \in Q$, are defined for negative values v,
- a compact set $U \subset \mathbb{R}^m$,
- functions $f_q : \mathbb{R}^n \times U \to \mathbb{R}^n$, $q \in Q \cup \{0\}$,
- a vector $x_0 \in \mathbb{R}^n$,
- closed sets $\mathcal{Z}_\pm \subseteq \mathbb{R}^n$.

Controls. The dynamical system is driven in two ways: The "usual" part of a control input ϱ is played by a Borel measurable ($\mathcal{B}$-measurable) function $u : [0,T] \to \mathbb{R}^m$ such that (s.t.)

$$u(t) \in U \text{ for } \lambda\text{-almost all (a.a.) } t \in [0,T], \qquad (4)$$

and we denote the set of all such functions by $\mathcal{U}_T$. Here, λ, $\lambda(dt) := dt$ stands for the Lebesgue measure on reals. The remainder of ϱ is a collection

$$\vartheta := \left(\nu, \mu, l, \{e_\tau, u_\tau\}_{\tau\in\Delta_\nu(T)}\right),$$

referred to as an *impulsive control*. Here,

- $\nu, \mu \in C^*([0,T], \mathbb{R})$ are Lebesgue-Stieltjes measures (Lebesgue extensions of the Borel measures, induced by functions of bounded variation) with

$$|\mu| \leq \nu,\ |\mu|_c = \nu_c, \text{ and } \nu([0,T]) \leq M \qquad (5)$$

(in respect of a measure, $|\cdot|$ denotes its total variation, and $|\vartheta| = \nu$ by definition; $\nu_c := \nu_{ac} + \nu_{sc}$, where the summands are, respectively, absolutely continuous and singular continuous parts in the Lebesgue decomposition of a measure ν).
- $l : [0,T] \to \mathbb{R}$ is a $\mathcal{B}$-measurable function with

$$\int_0^t l^p(\theta)d\theta = \mu_{ac}([0,t]) \text{ for all } t \in [0,T]. \quad (6)$$

- $\{e_\tau, u_\tau\}_{\tau\in\Delta_\nu(T)}$ is a family of $\mathcal{B}$-measurable functions

$$e_\tau : [0,T_\tau] \to \mathbb{R},\ \ u_\tau : [0,T_\tau] \to \mathbb{R}^m$$

parameterized by atoms of the measure ν and meeting the constraints

$$|e_\tau(\theta)| = 1,\ u_\tau(\theta) \in U\ \lambda\text{-a.e. on } [0,T_\tau], \quad (7)$$

$$\int_0^{T_\tau} e_\tau(\theta)d\theta = \mu(\{\tau\}), \qquad (8)$$

with $\Delta_\nu(t) := \{\tau \in [0,t]|\ \nu(\{\tau\}) > 0\}$, and $T_\tau := \nu(\{\tau\})$ ("a.e." abbreviates "almost everywhere").

The used definition of impulsive control is similar to [Arutyunov, Karamzin and Pereira, 2010, Karamzin, 2006].

Notice that if $v \mapsto v^p$ is an even function, then $\mu = \nu$, and $e_\tau(\theta) = 1$ on $[0,T_\tau]$ for any atom τ of μ; and if the mapping $v \mapsto v^p$ is odd, then $l = \left(\dot{F}_{\mu_{ac}}\right)^{1/p}$, where $F_{\mu_{ac}}$ is the distribution function of μ_{ac}.

$\mathcal{P}$ denotes the set of admissible controls, i.e. all ϱ satisfying (4)–(8).

Hybrid feature: Mixed constraints. The main system's peculiarity is due to the mixed constraints (3) relating a state trajectory and an impulsive control. The presence of such conditions imparts hybrid features to the impulsive dynamical system, and this relation has been discussed in [Goncharova and Staritsyn, 2012]. In terms of hybrid systems, the sets $\mathcal{Z}_-$ and $\mathcal{Z}_+$ are called "jump permitting" and "jump destination" sets, respectively. The notions come from a formalization of switching rules of hybrid automata [Branicky, Borkar and Mitter, 1998,Van der Schaft and Schumacher, 2000]. This formalism is mostly typical for robot motion planning [Brogliato, 2000].

Notice that definition (6) implies that the inclusions (3) hold also $l\lambda$-a.e. on [0,T], where $l\lambda$ is the measure absolutely continuous w.r.t. the Lebesgue measure with density l.

3 A toy example: Hybrid system with square impulses

In the linearly-impulsive case, constraints of the type (3) naturally arise when treating mechanical systems with so-called blockable degrees of freedom [Goncharova and Staritsyn, 2012]. In such systems control actions result in switching the number of the system's degrees of freedom. This process is instantaneous and can be correctly formalized by introducing a measure term (impulsive input) related with a trajectory by a common condition. Below we address a simplest model with quadratic impulses.

Consider a hybrid version of the example from [Bressan and Rampazzo, 1993]. The original system with one degree of freedom is as follows: a bead of the unit mass is moving frictionless along a rotating rod in the horizontal plane,

$$\ddot{l} = l\dot{\phi}^2.$$

The linear position $l \in BV$ (henceforth, BV denotes the Banach space of right continuous functions of bounded variation) of the bead is controlled by switching the rod's angle $\phi \in BV$, and the derivatives are considered in the generalized sense.

We now demand that each switching should steer the bead to one of the prescribed positions $l_1, l_2, \ldots, l_N$, $N \leq \infty$. This can be formalized by introducing the constraint

$$l(t) \in \{l_1, \ldots, l_N\} \ \nu\text{-a.e.},$$

where ν is a measure majorating the total variation of differential measure $d(\phi^2)$.

4 Solution to the measure differential equation

Hypotheses $(\mathbf{H})$: The functions f_q, $q \in Q \cup \{0\}$, are continuous in all variables, uniformly Lipschitz continuous in x, and satisfy the linear growth condition w.r.t. x.

Solution concept. By a solution to the measure differential equation (1) we mean a right-continuous function of bounded variation satisfying the following integral relation

$$\begin{aligned} x(t) = x_0 + &\int_0^t f_0(x(\theta), u(\theta))\, d\theta + \\ + &\sum_{q \in Q \setminus \{p\}} \int_0^t f_q(x(\theta), u(\theta))\, l^q(\theta)\, d\theta + \\ + &\int_0^t f_p(x(\theta), u(\theta)) \mu_c(d\theta) + \\ + &\sum_{\tau \in \Delta_\nu(t)} \left[\kappa_\tau(T_\tau) - x(\tau-)\right]. \end{aligned} \quad (9)$$

Here, for each $\tau \in \Delta_\nu(T)$, κ_τ is a solution to the auxiliary "limit" system [Gurman, 1972]

$$\frac{d}{d\theta}\kappa = f_p(\kappa, u_\tau) e_\tau, \quad \kappa(0) = x(\tau-). \quad (10)$$

The existence and uniqueness of a solution $x[\varrho]$ to the measure differential equation (1) under a control $\varrho \in \mathcal{P}$ follows from the general result [Miller and Rubinovich, 2001, Theorem 8.22] thanks to hypotheses $(\mathbf{H})$ and boundedness of L_1-norms of the maps $t \mapsto l^q(t)$, $q \in Q \setminus \{p\}$.

Given a control ϱ, the family

$$\mathcal{X} = \mathcal{X}[\varrho] := \{\kappa_\tau\}_{\tau \in \Delta_\nu(T)}$$

of respective trajectories of the limit system is called *graph completion* associated with a trajectory $x[\varrho]$. Clearly, a discontinuous solution of the measure differential equation can admit a plenty of its graph's completions.

A couple $\sigma = (x, \varrho)$ with $\varrho \in \mathcal{P}$ and $x = x[\varrho]$ is referred to as an *admissible control process*, and $\Sigma(P)$ denotes the set of all admissible processes for (P). We are to assume $\Sigma(P) \neq \emptyset$.

5 Prototypical dynamics

The prototype of the measure driven system (1) is the following dynamical system:

$$\dot{x} = f_0(x, u) + \sum_{q \in Q} f_q(x, u) v^q, \quad (11)$$

$$x(0) = x_0, \quad (12)$$

$$u \in \mathcal{U}_T, \quad v \in \mathcal{V}_T. \quad (13)$$

Here, $\mathcal{V}_T$ denotes the set of functions $v : [0, T] \to \mathbb{R}$ with $\|v\|_{L_p} \leq M^{1/p}$.

Consider the set $\mathcal{F}(x)$ of vectors $(a^p, a^p f_0(x, w) + \sum_{q \in Q \setminus \{p\}} a^{p-q} b^q f_q(x, w) + b^p f_p(x, w), |b|^p) \in \mathbb{R}^{n+2}$ such that $(a, b) \in K$ and $w \in U$. Here, $K \triangleq \operatorname{co}\{(a, b) \in \mathbb{R}^2 |\ a \geq 0,\ a^p + |b|^p = 1\}$, and $\operatorname{co} A$ denotes the convex hull of a set A.

Assumed that $\mathcal{F}(x)$ is convex for any $x \in \mathbb{R}^n$, (1) describes the closure of the trajectory tube of system (11)–(13) in the weak* topology of BV, i.e., solutions to the measure differential equation are generalized solutions of (11)–(13). The adapted concept of generalized solution is similar to [Miller and Rubinovich, 2001] and employs Warga's approach based on metric compactification of the set of solutions [Warga, 1962, Warga, 1972, Warga, 1987].

Definition 5.1. *A function $x \in BV$ is said to be a generalized solution to system (11)–(13) iff there exists a sequence $\{(u_k, v_k) |\ k \in \mathbb{N}\}$ of controls $u_k \in \mathcal{U}_T$, $v_k \in \mathcal{V}_T$ such that the respective sequence of Carathéodory solutions $x_k = x[u_k, v_k]$ of (11) converges to x in the weak* topology of BV (i.e., at all points of continuity, and at t = T).*

Denote $\widetilde{\mathbb{X}}$ the set of generalized solutions to (11)–(13). In [Goncharova and Staritsyn, 2015a] we prove that $\{x[\varrho] |\ \varrho \in \mathcal{P}\} = \widetilde{\mathbb{X}}$.

6 Main result: Problem transformation

On a time interval $[0,S]$, $S \le T+2M$, consider the following reduced problem (RP):

$$\text{Minimize } J = F(y_+(S))$$

subject to the constraints

$$\frac{d}{ds}y_\pm = \alpha^p f_0(y_\pm,\omega) + \sum_{q\in Q\setminus\{p\}} \alpha^{p-q}\beta^q f_q(y_\pm,\omega) + \\ + \gamma_\pm \beta^p f_p(y_\pm,\omega), \quad y_\pm(0) = x_0, \tag{14}$$

$$\frac{d}{ds}\xi = \alpha^p, \ \frac{d}{ds}(\eta,\zeta)_\pm = \gamma_\pm(\beta^p, |\beta|^p), \tag{15}$$

$$\xi(0) = \eta_\pm(0) = \zeta_\pm(0) = 0, \tag{16}$$

$$y_+(S) = y_-(S), \quad \eta_+(S) = \eta_-(S), \tag{17}$$

$$\xi(S) = T, \quad \zeta_+(S) = \zeta_-(S) \le M, \tag{18}$$

$$\zeta_- - \zeta_+ \le 0, \tag{19}$$

$$\int_0^S \Psi(\alpha,\beta,\gamma,y,\eta,\zeta)ds = 0, \tag{20}$$

$$\omega \in \mathcal{U}_S, \quad (\alpha,\beta,\gamma) \in \mathcal{A}_S, \ \gamma = (\gamma_+,\gamma_-). \tag{21}$$

Here, $\mathcal{A}_S$ denotes the set of control functions (α,β,γ) with $\mathcal{B}$-measurable components $\alpha,\beta,\gamma_\pm : [0,S] \to \mathbb{R}$, satisfying the constraints:

- $(\alpha,\beta)(s) \in K$ for λ-a.a. $s \in [0,S]$, while $|\beta|^p$ is the pth power of the absolute value $|\beta|$ of β;
- $\gamma_\pm(s) \ge 0$, and $\gamma_+(s) + \gamma_-(s) = 1$ λ-a.e. on $[0,S]$.

The states of the reduced system are $(y,\xi,\eta,\zeta)(s)$, $y(s) = (y_+,y_-)(s)$, $\eta(s) = (\eta_+,\eta_-)(s)$, $\zeta(s) = (\zeta_+,\zeta_-)(s)$, $\xi(s), \eta_\pm(s), \zeta_\pm(s) \in \mathbb{R}_+$, $y_\pm(s) \in \mathbb{R}^n$, where $\mathbb{R}_+$ is the nonnegative half-line.

The function Ψ in (20) is of the form

$$\Psi = \alpha\{\zeta_+ - \zeta_- + W^{\mathbb{R}}_{\{0\}}(\eta_+ - \eta_-)\} + \\ W^{\mathbb{R}^n}_{\{0\}}(y_+ - y_-) + |\beta|\{\gamma_+ W^{\mathbb{R}^n}_{Z_-}(y_-) + \gamma_- W^{\mathbb{R}^n}_{Z_+}(y_+)\}.$$

Here $W^X_Y : X \to \mathbb{R}_+$ is a continuous function vanishing only on a subset Y of a finite-dimensional space X (as is well known, such a function, even a smooth one, does exist for any closed Y).

Notice that (RP) is a conventional variational problem with absolutely continuous trajectories weighted by state, terminal and functional constraints.

A collection $\varsigma = (y,\xi,\eta,\zeta,\alpha,\beta,\gamma,\omega;S)$ is said to be an admissible process for (RP) iff it satisfies all the conditions (14)–(21). By $\Sigma(RP)$ we denote the set of all admissible processes for (RP).

For problem (P), given a control $\varrho = (u,\vartheta) \in \mathcal{P}$, $\vartheta = \big(\nu,\mu,l,\{e_\tau,u_\tau\}_{\tau\in\Delta_\nu(T)}\big)$, we introduce a function $\Upsilon : [0,T] \to [0,T+2\nu([0,T])]$ as follows

$$\Upsilon(t) = t + 2\nu([0,t]), \ t \in [0,T), \\ \Upsilon(T) = T + 2\nu([0,T]),$$

and let $\upsilon : [0,T+2\nu([0,T])] \to [0,T]$ be its inverse.

Given $S \in [T,T+2M]$, $(\alpha,\beta,\gamma) \in \mathcal{A}_S$ such that the respective solution ξ of (15), (16) satisfies (18), and $\omega \in \mathcal{U}_S$, define a function $\Xi : [0,T] \to [0,S]$ by the formulas

$$\Xi(t) = \inf\{s \in [0,T] |\ \xi(s) > t\}, \ t \in [0,T), \\ \Xi(T) = S.$$

Problems (P) and (RP) are equivalent to each other according to the following

Theorem 6.1. *1) For any control process $\sigma \in \Sigma(P)$, there exists a process $\varsigma = (y,\xi,\eta,\zeta,\alpha,\beta,\gamma,\omega;S) \in \Sigma(RP)$, $y = (y_+,y_-)$, $\eta = (\eta_+,\eta_-)$, $\zeta = (\zeta_+,\zeta_-)$, $\gamma = (\gamma_+,\gamma_-)$ such that*

$$\upsilon = \xi \ \text{ on } [0,S]; \\ x = y_\pm \circ \Upsilon, \ F_\mu = \eta_\pm \circ \Upsilon, \ F_\nu = \zeta_\pm \circ \Upsilon \text{ on } [0,T].$$

Here, F_μ, F_ν denote the distribution functions of measures, and the symbol $\circ$ stands for the composition of functions.

2) For any process $\varsigma \in \Sigma(RP)$, there exists a process $\sigma = (x,\varrho) \in \Sigma(P)$, $\varrho = (u,\vartheta) \in \mathcal{P}$, $\vartheta = \big(\nu,\mu,l,\{e_\tau,u_\tau\}_{\tau\in\Delta_\nu(T)}\big)$, such that

$$y_\pm \circ \Xi = x, \ \eta_\pm \circ \Xi = F_\mu, \ \zeta_\pm \circ \Xi = F_\nu \text{ on } [0,T].$$

3) Optimal solutions for problems (P) and (RP) can exist only simultaneously. For optimal processes $\sigma^* \in \sigma(P)$ and $\varsigma^* \in \Sigma(RP)$ one has

$$I(\sigma^*) = J(\varsigma^*). \tag{22}$$

The proof is rather technical and similar to [Goncharova and Staritsyn, 2012]. It is based on the below formulas for the direct and inverse transforms.

Direct transformation. Set $S = T + 2\nu([0,T])$ and define the functions

$$\omega(s) = \begin{cases} (u_\tau \circ \theta_{\tau\pm})(s), & \text{if } \exists\, \tau \in D_\Upsilon \text{ s.t. } s \in \Upsilon_{\tau\pm}, \\ (u \circ \upsilon)(s), & \text{otherwise,} \end{cases}$$

$$\alpha(s) = \begin{cases} 0, & \text{if } \exists\, \tau \in D_\Upsilon \text{ s.t. } s \in [\Upsilon(\tau-),\Upsilon(\tau)], \\ (m_1^{1/p} \circ \upsilon)(s), & \text{otherwise,} \end{cases}$$

$$\beta(s) = \begin{cases} (e_\tau \circ \theta_{\tau\pm})(s), & \text{if } \exists\, \tau \in D_\Upsilon \text{ s.t. } s \in \Upsilon_{\tau\pm}, \\ (l \circ \upsilon)(s)\cdot\alpha(s), & \text{if } \upsilon(s) \in \operatorname{supp}\nu_{ac}, \\ (m_2 \circ \upsilon)(s), & \text{if } \upsilon(s) \in \operatorname{supp}\nu_{sc}. \end{cases}$$

$$\gamma_+(s) = \begin{cases} 1, & \text{if } \exists\, \tau \in D_\Upsilon \text{ s.t. } s \in \Upsilon_{\tau+}, \\ 0, & \text{if } \exists\, \tau \in D_\Upsilon(T) \text{ s.t. } s \in \Upsilon_{\tau-}, \\ 1/2, & \text{otherwise,} \end{cases}$$

$\gamma_-(s) = 1 - \gamma_+(s)$, $s \in [0, S]$. Here, m_1 denotes the Radon-Nikodym derivative of the Lebesgue measure λ w.r.t. the measure $(\lambda + \nu)$, and m_2 the Radon-Nikodym derivative of the singular continuous part μ_{sc} of μ w.r.t. the measure ν;

$$\theta_{\tau+}(s) = s - \Upsilon(\tau-),\ \theta_{\tau-}(s) = \theta_{\tau+}(s) - \nu(\{\tau\})$$
$$\text{for } s \in [\Upsilon(\tau-), \Upsilon(\tau)];$$
$$D_\Upsilon = \{\tau \in [0,T] |\ \Upsilon(\tau) - \Upsilon(\tau-) > 0\},$$
$$\Upsilon_{\tau+} = \Upsilon(\tau-) + [0, T_\tau],$$
$$\Upsilon_{\tau-} = \Upsilon_{\tau+} + [0, T_\tau],$$

and $\operatorname{supp} \nu$ is the support of a measure ν. As before, $T_\tau := \nu(\{\tau\})$.

Inverse transformation. Define a desired control $\varrho = (u, \vartheta) \in \mathcal{P}$, $\vartheta = \big(\nu, \mu, l, \{e_\tau, u_\tau\}_{\tau \in \Delta_\nu(T)}\big)$ through the following formulas:

- $u = \omega \circ \Xi$.
- $\mu = dF_\mu$, and $\nu = dF_\nu$, where the functions F_μ, F_ν of bounded variation are introduced by

$$F_\mu(0-) = 0,\ F_\nu(0-) = 0,$$
$$F_\mu(t) = (\eta_+ \circ \Xi)(t), \text{ and}$$
$$F_\nu(t) = (\zeta_+ \circ \Xi)(t),\ t \in [0, T].$$

- $l = (\alpha^\oplus \beta) \circ \Xi$, where $^\oplus$ denotes the operation of pseudoinversion , i.e., $\alpha^\oplus = 0$, if $\alpha = 0$, and $\alpha^\oplus = \alpha^{-1}$, otherwise.
- For each $\tau \in \Delta_\nu(T)$, we put

$$e_\tau = \beta \circ s_\tau,\ u_\tau = \omega \circ s_\tau,$$

where

$$s_\tau(\theta) := \inf\{s \in [\Xi(\tau-), \Xi(\tau)] :\ \theta_\tau(s) > \theta\}$$
$$\text{for } \theta \in [0, \nu(\{\tau\}))$$

with $\theta_\tau(s) := \zeta_{\tau+}(s) - \nu([0, \tau))$, and

$$s_\tau(\nu(\{\tau\})) := \Xi(\tau).$$

The idea of the proposed space-time transformation is as follows: A discontinuous trajectory is stretched into an absolutely continuous one by putting its original dynamics together with its graph completion in a new common time scale. During jumps, a trajectory is split in two branches. The first branch corresponds to the right one-sided limit of a solution to the measure differential equation, and the second branch is due to the left one. As the interval of the extended time, associated with a jump, is over, the branches are reunited, and we have a unique extended trajectory. The inverse discontinuous reparameterization of the extended solution restores the original discontinuous trajectory, that meets (among others) the key constraint (3). State, terminal and functional constraints of the reduced problem serve a realization of this idea.

7 Conclusion

The discussed problem transformation allows one to treat the original (a rather sophisticated) model as a conventional problem of dynamical optimization. A discontinuous time reparameterization approach is known as a basic tool of the impulsive control theory. Our main theorem, as all results of this type, can be used in both qualitative and numerical problem's examination. The most typical its application is to derive optimality conditions, and develop algorithms for optimal impulsive control.

In what concerns practical relevance, here the cases $p = 1, 2$ are of the most evidence. Models of these types appear in robotics: they are a mathematically correct and fruitful formalization of manipulators, controlled by changing the number of system's degrees of freedom, or by so-called moving constraints (i.e. by a part of system's state coordinates). In this respect, the nonstandard mixed constraints that we introduce serve to design such important hybrid phenomena as state switchings' permitting and destination areas in the configuration space of a manipulator.

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References

Arutyunov, A., Karamzin, D. and Pereira, F. (2010). On constrained impulsive control problems. *J. Math. Sci.*, **165**, pp. 654–688.

Arutyunov, A., Karamzin, D. and Pereira, F. (2005). A nondegenerate Maximum Principle for the impulse control problem with state constraints. *SIAM J. Control Optim.,* **43 (5)**, pp. 1812–1843.

Aubin, J.-P. (2000). *Impulse differential equations and hybrid systems: A viability approach. Lecture Notes*. Univ. California, Berkeley.

Branicky, M., Borkar, V., and Mitter, S. (1998). A unified framework for hybrid control: Model and optimal control theory. *IEEE Trans. Automat. Control*, 43(1), pp. 31–45.

Bressan, A. (2008). Impulsive control of Lagrangian systems and locomotion in fluids. *Discr. Cont. Dynam. Syst.*, **20**, pp. 1–35.

Bressan, Aldo and Motta, M. (1993). A class of mechanical systems with some coordinates as controls. A reduction of certain optimization problems for them. Solution methods. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Mem.*, **9**, pp. 5–30.

Bressan, A. and Rampazzo, F. (2010). Moving constraints as stabilizing controls in classical mechanics. *Arch. Ration. Mech. Anal.*, **196(1)**, pp. 97–141.

Bressan, A. and Rampazzo, F. (1993). On systems with quadratic impulses and their application to Lagrangean mechanics. *SIAM J. Control Optim.*, **31**, pp. 1205–1220.

Bressan, A. and Rampazzo, F. (1994). Impulsive control systems without commutativity assumptions, *J. Optim. Theory Appl.*, **81(3)**, pp. 435–457.

Brogliato, B. (2000). *Nonsmooth Impact Mechanics. Models, Dynamics and Control*, Springer- Verlag, London.

Dykhta, V. (1990). Impulse-trajectory extension of degenerate optimal control problems, *IMACS Ann. Comput. Appl. Math.*, **8**, pp. 103–109.

Dykhta, V. and Samsonyuk, O. (2009). A maximum principle for smooth optimal impulsive control problems with multipoint state constraints, *Comput. Math. Math. Phys.*, **49(6)**, pp. 942–957.

Filippova, T. F. (2005). On the generalized solutions for uncertain systems with applications to optimal control and estimation problems. *WSEAS Transactions on Systems*, **5(4)**, pp. 481–486.

Fraga, S. L., Gomes, R. and Pereira, F. L. (2007). An Impulsive Framework for the Control of Hybrid Systems, In *Proc. The 46 IEEE Conf. on Decision Cont.*, New Orleans, USA, Dec. 12–14, pp. 5444–5449

Goncharova, E., and Staritsyn, M. (2015). Optimal control of dynamical systems with polynomial impulses. *Discr. Cont. Dynam. Syst.*, **35(9)**, to appear.

Goncharova, E., and Staritsyn, M. (2015). Problem of optimal impulsive control with state and mixed constraints: The case of vector-valued measure *Autom. Re. Control*, **76(3)**, pp. 377–384.

Goncharova, E., and Staritsyn, M. (2012). Optimization of measure-driven hybrid systems. *J. Optim. Theory Appl.*, **153(1)**, pp. 139–156.

Goncharova, E., and Staritsyn, M. (2011). Optimal impulsive control problem with phase and mixed constraints. *Doklady Math.*, 84(3), pp. 882–885.

Goncharova, E., and Staritsyn, M. (2011). Optimal Control of Impulsive Hybrid Systems. In *Proc. The 18th IFAC World Congress*, pp. 10255–10260.

Gurman, V. (1972). On optimal processes with unbounded derivatives. *Autom. Remote Control*, **17**, pp. 14–21.

Haddad, W., Chellaboina, V., and Nersesov, S. (2002). Hybrid Nonnegative and Compartmental Dynamical Systems. *Mathematical Problems in Engineering*, volume 8, number 6, pp. 493-515.

Haddad, W., Chellaboina, V., and Nersesov, S. (2006). *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control,* Princeton University Press, Princeton.

Karamzin, D. (2006). Necessary conditions of the minimum in an impulse optimal control problem. *J. Math. Sci.*, **139(6)**, pp. 7087–7150.

Kurzhanski, A., and Tochilin, P. (2009). Impulse Controls in Models of Hybrid Systems. newblock *Differential Equations*, 45(5), pp. 731-742.

Miller, B. (1996). The generalized solutions of nonlinear optimization problems with impulse control. *SIAM J. Control Optim.*, **34**, pp. 1420–1440.

Miller, B. and Rubinovich, E. (2001). *Impulsive Control in Continuous and Discrete- Continuous Systems*, Kluwer Academic / Plenum Publishers, New York.

Miller, B., and Bentsman, J. (2006). Optimal Control Problems in Hybrid Systems with Active Singularities, *Nonlinear Analysis*, 65, pp. 999-1017.

Pereira, F., Silva, G. and Oliveira, V. (2008). Invariance for impulsive control systems. *Autom. Remote Control*, **69(5)**, pp. 788–800.

Pedregal, P. and Tiago, J. (2009). Existence results for optimal control problems with some special nonlinear dependence on state and control. *SIAM J. Control Optim.*, **48(2)**, pp. 415–437.

Rampazzo, F. and Sartori, C. (2000). Hamilton-Jacobi-Bbellman equations with fast gradient-dependence. *Indiana Univ. Math. J.*, **49(3)**, pp. 1043–1077.

Rishel, R. (1995). An extended Pontryagin principle for control systems whose control laws contain measures. *J. Soc. Indust. Appl. Math. Ser. A Control*, **3**, pp. 191–205.

Sanfelice, R., Goebel, R., Teel, A., Hespanha, J., and Tiwari, A (2006). A Feedback Control Motivation for Generalized Solutions to Hybrid Systems. *HSCC 2006, LNCS 3927*, pp. 522-536.

Sousa, J., and Pereira, F. (2001). Some questions about hybrid systems, In *Proc. European Control Conference*.

Van der Schaft, A., and Schumacher, H. (2000). *An introduction to hybrid dynamical systems*. Springer-Verlag, London.

Warga, J. (1962). Relaxed variational problems. *J. Math. Anal. Appl.*, **4**, pp. 111–128.

Warga, J. (1972). *Optimal control of differential and functional equations*, Academic Press, New York.

Warga, J. (1987). Variational problems with unbounded controls. *J. SIAM Control Ser.A*, **3(3)**, pp. 424–438.

Zavalischin, S., and Sesekin, A. (1997). *Dynamic impulse systems: theory and applications*, Kluwer Academic Publishers, Dorderecht.