DYNAMICS OF NON-STATIONARY PROCESSES THAT FOLLOW THE MAXIMUM OF CONTINUOUS TSALLIS ENTROPY

Dmitry S. Shalymov
Mathematics and Mechanics
Saint-Petersburg State University
Russian Federation
dmitry.shalymov@gmail.com

Abstract

In this paper a non-stationary processes that tend to maximize the Tsallis entropy are considered. Systems with discrete probability distribution for the Tsallis entropy have already been investigated on the basis of the Speed-Gradient principle. The evolution of probability density function and continuous form of the Tsallis entropy are considered. A set of equations describing dynamics of a system under the mass conservation and the energy conservation constraints is derived. The uniqueness of the limit distribution and asymptotic convergence of probability density function is examined for both constraints.

Key words

Nonlinear dynamics and control, Control in thermodynamics, Tsallis entropy, MaxEnt, Speed-Gradient principle

1 Introduction

Entropy is actively used by many fields of science including physics, chemistry, computer science, biology etc. [Martyushev and Seleznev, 2006]

Today there are many different types of entropy in use which often becomes the center of discussions both in statistical physics and thermodynamics. The most famous is the Shannon entropy [Shannon, 1948]:

$$H(X) = -\sum_i P(x_i)\log P(x_i), \qquad (1)$$

where X is a discrete random variable with possible values $\{x_1, ..., x_n\}$ and P is a probability mass function.

In 1988, Constantino Tsallis [Tsallis, 1988] introduced a generalized Shannon entropy.

$$H(X,q) = \frac{1}{q-1}\left(1 - \sum_i P(x_i)^q\right), \qquad (2)$$

where q is any real number. It was shown that the Tsallis entropy tends to the Shannon entropy when $q \to 1$.

Extension of the Tsallis entropy for continuous case can be defined as

$$S(X,q) = \frac{1}{q-1}\left(1 - \int p^q(x)dx\right), \qquad (3)$$

where X is an absolutely continuous random variable having probability density function (PDF) p.

The Tsallis entropy has become very popular in statistical mechanics and thermodynamics nowadays. It has also found many applications in various scientific fields such as chemistry, biology, medicine, economics, geophysics, *etc.* There is a plenty of works that use and analyze the Tsallis entropy [Tsallis, 2016].

A variety of physical systems obey the famous maximum entropy (MaxEnt) principle: their entropy achieves maximum under constraints caused by other physical laws. Since 1957, when seminal works of E.T.Jaynes were published [Jaynes, 1957], and until now [Martyushev, 2013] the MaxEnt principle has whetted a lively interest of researchers.

Although the states of maximum entropy are widely discussed in scientific articles and studies, the dynamics of evolution and transient behavior of systems are still not well investigated.

In this paper, we propose a set of equations that describe the dynamics of PDFs for non-stationary processes that follow the maximum of the Tsallis entropy principle.

The speed-gradient (SG) principle [Fradkov, 2008; Fradkov, Miroshnik and Nikiforov, 1999; Fradkov, 2005; Fradkov, 1979] used here is originated from the control theory. This principle has been already applied in [Fradkov, 2007; Fradkov, 2008] to derive the equations of dynamics for the systems with a finite number of particles for the case of maximum Shannon entropy principle. The dynamics of discrete systems for the Tsallis entropy is discussed in [Fradkov and Shalymov, 2015]. Rényi entropy is investigated from the SG-principle perspective in [Shalymov and Fradkov, 2016]. Continuous probability distributions are considered in [Fradkov and Shalymov, 2015].

We use a similar approach proposed in [Fradkov, 2008; Fradkov and Shalymov, 2015; Fradkov and Shalymov, 2015; Shalymov and Fradkov, 2016] for continuous form of the Tsallis entropy. The derived equations describe the dynamics of non-stationary (transient) states and show the way and trajectory of a system tending to the state with maximum of the Tsallis entropy.

The well-known Fokker-Planck (FP) equations [Frank, 2005] describe the time evolution of PDF. Jaynes's MaxEnt approach can also be applied to these equations, see [Hick and Stevens, 1987]. Another general form of time-evolution equations for non-equilibrium systems is known as GENERIC (general equation for the non-equilibrium reversible-irreversible coupling) [Grmela and Öttinger, 1997]. The relation between GENERIC and FP equations is established in [Grmela and Öttinger, 1997]. It states that FP is a particular case of GENERIC when a noise term is added into the GENERIC. Thus, if the FP equation is represented as a stochastic differential equation, from which fluctuations are eliminated, this equation matches the GENERIC equation [Öttinger, 1998].

Following this, we can also claim that the SG-principle matches GENERIC (and thus it matches FP) if a goal function is set as entropy and constraints are specified by energy [Fradkov and Shalymov, 2015]. Moreover, the SG-principle is a more general case of GENERIC because almost every smooth function can be taken as a goal function, not only entropy.

We propose evolution law of the system in the following general form:

$$\dot{P}(t) = \Gamma \frac{q}{q-1}(I - \Psi)P(t)^{q-1}, \tag{4}$$

where I is an identity operator, Ψ is a linear integral operator that does not depend on p, $\Gamma > 0$ is a constant gain. All the solutions of the equation (4) converge to the distribution that corresponds to the maximum value of the Tsallis entropy.

Along with the Tsallis entropy more general forms of relative entropies and divergences such as CR entropy or Csiszár–Morimoto conditional entropies (f-divergencies) [Morimoto, 1963] can also be considered from the SG-principle perspective.

The main goal of this paper is extension of results described in the [Fradkov, 2008; Fradkov and Shalymov, 2015; Fradkov and Shalymov, 2015; Shalymov and Fradkov, 2016] to the continuous form of the Tsallis entropy. The SG principle is used to derive equations of the dynamics of transient states of the systems that follow the MaxEnt principle in a steady state.

2 The Speed-Gradient Principle

There is a connection between the laws of control in technical systems and laws of the dynamics in physical systems. It is known that the methods for synthesis of control algorithms allow to derive the laws of dynamics for physical systems. In particular, the model of the dynamics for a number of physical systems can be derived based on the SG-method with an appropriate choice of the goal function.

Consider the class of open physical systems which dynamics can be described by the system of differential equations

$$\dot{x} = f(x, u, t), \tag{5}$$

where $x \in \mathbb{C}^n$ is the system state vector, u is the vector of input (free) variables, $t \geq 0$. The problem of system modeling can be formulated as finding the law of change (evolution) $u(t)$ which meets a certain criteria of "naturalness" for its behavior and grants a set of properties observed in real physical systems to a generated model.

Such formulations are well known in physics. Variational principles of systems models has long been recognized. They usually involve the task of an integral functional that characterizes the behavior of the system [Lanczos, 1962]. Minimization of the functional defines the real possible trajectories of the system $\{x(t), u(t)\}$ as points in the corresponding functional space. To explicitly specify the dynamics of the system a developed apparatus of the variations calculus is used.

Variational approach formed the basis of the whole direction in the control theory - the theory of optimal control, where minimization of functional is used to find an appropriate control law for a given system.

Methods of optimal control (e.g. Bellman dynamic programming, Pontryagin maximum principle etc.) are the result of development of classical variational calculus methods. Such optimal control methods can be used to build dynamic models of mechanical systems in nature and society.

Together with integral principles a differential local time principles have also been proposed such as Gauss principle of least constraint, the principle of minimum energy dissipation etc. As noted by M. Planck [Planck, 1909], local principles have some advantages over integral ones, because they do not make the current state and the movement of the system to be dependent on its later states and movements. Let us formulate another local variational principle based on the method of speed-gradient [Fradkov, Miroshnik and Nikiforov, 1999; Fradkov, 2007]:

The speed-gradient principle: *only those possible movements of the system are realized (among all possible movements) for which the input variables change proportionally to the speed-gradient of a "goal" functional.*

The speed-gradient principle offers to researcher the choice of two types of systems dynamics models:

A) models which follow the algorithm of the speed-gradient in differential form:

$$\dot{u} = -\Gamma \nabla_u \dot{Q}_t. \tag{6}$$

B) models following the algorithm of the speed-gradient in finite form:

$$u = -\Gamma \nabla_u \dot{Q}_t. \tag{7}$$

where $\dot{Q}_t$ is the rate of change of the target functional along the trajectories of the system (5). We describe the application of the SG-principle in the simplest (and most important) case when the class of models of dynamics is given by the relation:

$$\dot{x} = u. \tag{8}$$

Relation (8) means only that we are looking for a law of variation rates of the state variables of the system.

In accordance with the SG-principle the goal functional $Q(x)$ has to be determined first. Selection of $Q(x)$ should be based on the physics of the real system and reflect the presence of a tendency to decrease the current value of $Q(x(t))$. After that, the law of dynamics can be written in the form (6) or (7).

The use of a law of dynamics in the form (6) generates differential equations of motion of the second order. These equations are invariant with respect to the replacement of time t by $(-t)$. It corresponds to reversible processes. On the contrary, the choice of the final form (7) corresponds to irreversible processes in general.

3 Jaynes Maximum Entropy Principle

The approach proposed by Jaynes [Jaynes, 1957] became the foundation for statistical physics nowadays. Its main ideas are described below.

Let $p(x)$ be a PDF of a multidimensional random variable x. This is an unknown function that needs to be defined on the basis of certain system information. Let us suppose that there is the information about some average values $\overline{H}_m$ which are known a priori:

$$\overline{H}_m = \int H_m(x)p(x)dx, \; m = 1, ..., M. \tag{9}$$

The next equality is also true for the density function

$$\int p(x)dx = 1. \tag{10}$$

Conditions (9) and (10) in general can be insufficient to derive $p(x)$. In this case, according to Jaynes, applying maximization of information entropy S_I is the most objective method to define the density function.

$$S_I = -\int p(x)\log p(x)dx.$$

Maximum search with additional conditions (9) and (10) is performed by using Lagrange multipliers; it leads to

$$p(x) = \frac{1}{Z} exp\left(-\sum_{m=1}^{M} \lambda_m H_m\right), \tag{11}$$

$$Z = \int exp\left(-\sum_{m=1}^{M} \lambda_m H_m\right) dx, \tag{12}$$

where λ_m can be derived from conditions (9).

In case of equilibrium these formulas show that the maximum information entropy coincides with the Boltzmann-Gibbs entropy and can be identified with the thermodynamic entropy.

4 The Speed-Gradient Dynamics of the Continuous Tsallis Entropy Maximization Process

Consider a system with a continuous distribution of possible states that evolves on a compact carrier. Distribution over states is characterized by PDF $p(t, x)$ which is continuous everywhere except of a set with zero measure. It is true that

$$\int_\Omega p(t, x)dx = 1, \forall t \tag{13}$$

where Ω is a compact carrier.

The Tsallis entropy for continuous PDF is defined as

$$S(X, q) = \frac{1}{q-1}\left(1 - \int_\Omega p^q(t, x)dx\right), \tag{14}$$

where q is any real number.

From the constraint (13) it follows that

$$\int_\Omega u(t, x)dx = 0, \tag{15}$$

where $u(t,x) = \dot{p}(t,x)$.

According to the SG-principle we calculate $\dot{S}$:

$$\dot{S}(X,q) = \frac{q}{1-q}\int_{\Omega} p^{q-1}(t,x)\dot{p}(t,x)dx$$

Gradient of $\dot{S}$ by u is equal to

$$\nabla_u \dot{S}(X,q) = \frac{q}{1-q}\nabla_u \langle p^{q-1}(t,x), u\rangle = \frac{q}{1-q}p^{q-1}(t,x) \quad (16)$$

Speed-gradient principle of motion forms the evolution law:

$$u = -\Gamma\frac{q}{1-q}p^{q-1}(t,x) + \lambda'$$

where Γ can be taken as a scalar value and Lagrange multiplier λ' is selected to satisfy the constraint (15).

$$\int_{\Omega}\left(-\Gamma\frac{q}{1-q}p^{q-1}(t,x) + \lambda'\right)dx = 0 \Rightarrow$$
$$\lambda' = \frac{q\Gamma}{(1-q)\mathrm{mes}(\Omega)}\int_{\Omega} p^{q-1}(t,x), \quad (17)$$

where

$$\mathrm{mes}(\Omega) = \int_{\Omega} 1 d\Omega. \quad (18)$$

Final system dynamics equation has the following form

$$\dot{p} = -\Gamma\frac{q}{1-q}p^{q-1}(t,x) + \frac{q\Gamma}{(1-q)\mathrm{mes}(\Omega)}\int_{\Omega} p^{q-1}(t,x) \quad (19)$$

Eq. (19) can be represented in the more general form

$$\dot{p} = \Gamma\frac{q}{q-1}(I-\Psi)p^{q-1}, \quad (20)$$

where $\Psi = \frac{\int_{\Omega}(\cdot)dx}{\mathrm{mes}(\Omega)}$ is a linear operator which is invariant for p and I is an identity operator.

4.1 Equilibrium stability

Let us investigate a stability of obtained equilibrium equation (19). Consider function $V(p) = S_{max} - S(X,q) \geq 0$. Derivative of this function is

$$\dot{V}(p) = -\dot{S}(X,q) = -\frac{q}{1-q}\int_{\Omega} u(t,x)p^{q-1}(t,x)dx \quad (21)$$

After substitution of expression for u from (19) to (21) we obtain:

$$\dot{V}(p) = \frac{\Gamma q^2}{\mathrm{mes}(\Omega)(q-1)^2}\times$$
$$\times\left(\left(\int_{\Omega} p^{q-1}(t,x)dx\right)^2 - \mathrm{mes}(\Omega)\int_{\Omega}\left(p^{q-1}(t,x)\right)^2 dx\right) \quad (22)$$

Then we use the CBS inequality in integral form

$$\left|\int_{\Omega} f(x)g(x)dx\right|^2 \leq \left(\int_{\Omega}|f(x)|^2\right)\left(\int_{\Omega}|g(x)|^2\right) \quad (23)$$

for functions $f = p^{q-1}$ and $g = 1$. Taking into account that a scalar value Γ is positive we get that $\dot{V}(p) \leq 0$. It is known that equality in the CBS inequality is achieved when multiplicity occurs, i.e. $f(x) = \alpha g(x)$. In our case $\dot{V}(p) = 0$ is true when $p^{q-1} = \alpha$. It is possible only when $p(t,x) = C = const$. Using the constraint (13) we get that $C = \mathrm{mes}^{-1}(\Omega)$. It means that there is the only one unique limit PDF $p^* = \mathrm{mes}^{-1}(\Omega)$ for equilibrium state of the system which evolves by evolution law (19).

4.2 Asymptotic convergence

To show an asymptotic convergence of all solutions to p^* we use the Barbalat's lemma.

Theorem 4.1 (Barbalat's lemma). *If differentiable function $f(t)$ has a finite limit for $t \to \infty$ and its derivative $\dot{f}(t)$ is uniformly continuous then $\dot{f}(t) \to 0$ for $t \to 0$.*

Theorem 4.2. *For all PDFs defined by equation (19) it is true that $p(t,x) \to p^*$ for $t \to \infty$.*

Proof. For the sake of simplicity we define a notation for V in (22) as $v(t) = V(p(t))$. We will use Barbalat's lemma to show that $\dot{v}(t) \to 0$. We use $v(t)$ as a function $f(t)$ in Barbalat's lemma. Because of $v(t) \geq 0$ and $\dot{v} \leq 0$ the function $v(t)$ has a finite limit for $t \to \infty$.

It can be shown that $\dot{v}$ is uniformly continuous. Consider expression for $|\ddot{v}(t)|$.

$$\ddot{v}(t) =$$
$$\frac{q^2}{1-q}\int_{\Omega} u(t,x)p^{q-1}(t,x)dx\int_{\Omega} p^{q-1}(t,x)u(t,x)dx +$$
$$\frac{q}{q-1}\int_{\Omega}\dot{u}(t,x)p^{q-1}(t,x)dx +$$
$$\frac{q}{q-1}\int_{\Omega}(q-1)u^2(t,x)p^{q-2}(t,x)dx \quad (24)$$

Due to the constraint (13) and compactness of the carrier Ω it can be shown that function $|\ddot{v}(t)|$ is bounded. That lead us to the fact that $\dot{v}$ is uniformly continuous.

As all necessary conditions of the Barbalat's lemma for differentiable function $v(t)$ are satisfied, we have that $\dot{v}(t) \to 0$ for $t \to 0$.

Taking into account that $\int_\Omega f dx = (1, f)$ and $\int_\Omega f^2 dx = \|f\|^2$ the expression for $\dot{v}$ from (22) may be rewritten as

$$\dot{v} = -\frac{\Gamma q^2}{\text{mes}(\Omega)(q-1)^2}\left(1 - \frac{(1, p^{q-1})^2}{\text{mes}(\Omega)\|p^{q-1}\|^2}\right) = \\ = -\frac{\Gamma q^2}{\text{mes}(\Omega)(q-1)^2}\|p^{q-1}\|^2\left(1 - \cos^2(\alpha(t))\right), \tag{25}$$

where $\alpha(t) = \cos(\langle 1, p^{q-1}\rangle)$.

If $\|p^{q-1}\|^2 \to 0$ then $\text{mes}_t\{r : p_t(r) \neq 0\} \to 0$. But this case conflicts with the constraint (13). Given (25) we obtain that $\alpha(t) \to 0$. It means that $\widehat{p}^{q-1} \to \widehat{1}$, where $\widehat{p}^{q-1}$ and $\widehat{1}$ are normalized values for p^{q-1} and 1 respectively.

It follows that $p(t, x)$ tends to the stationary distribution. As explained earlier this distribution is unique. Thus, $p_t \to p^*$ for $t \to \infty$.

5 Total Energy Constraint

The constraint (13) can be interpreted as the mass conservation law on the space Ω. Consider a system with additional constraint for the total energy conservation, i.e. a conservative case when energy does not depend on a time. The new constraint may be described as

$$\int_\Omega p(t, x)h(x)dx = E, \tag{26}$$

where E is a common energy of a system and $h(x)$ is a density of energy.

Equation of dynamics can be defined in the form

$$u = -\Gamma\nabla_u \dot{S}(X, q) + \lambda_1 h + \lambda_2 \tag{27}$$

Based on constraints (13) and (26) we can find expressions for Lagrange multipliers λ_1 and λ_2:

$$\begin{cases} \lambda_1 &= \frac{\Gamma q}{(1-q)}\frac{\text{mes}(\Omega)\int_\Omega p^{q-1}h(x)dx - \int_\Omega p^{q-1}dx\int_\Omega h(x)dx}{\text{mes}(\Omega)\int_\Omega h(x)^2dx - (\int_\Omega h(x)dx)^2} \\ \lambda_2 &= \frac{\Gamma q}{(1-q)}\frac{\int_\Omega p^{q-1}dx\int_\Omega h^2(x)dx - \int_\Omega p^{q-1}h(x)dx\int_\Omega h(x)dx}{\text{mes}(\Omega)\int_\Omega h(x)^2dx - (\int_\Omega h(x)dx)^2} \end{cases} \tag{28}$$

Equations in (28) are valid when denominator in both fractions is not equal to zero. If we use the CBS inequality for $f = h$ and $g = 1$ then the following inequality comes true

$$\left|\int_\Omega h dx\right|^2 \leq \text{mes}(\Omega)\int_\Omega h^2 dx, \tag{29}$$

This inequality becomes equality when $h = const$. It means that all energy levels coincide. This case is supposed to be degenerate and not considered here. Thus the expression

$$\text{mes}(\Omega)\int_\Omega h^2 dx \neq \left(\int_\Omega h dx\right)^2. \tag{30}$$

is always true.

Result equation of dynamics can be obtained by substituting (16) and (28) into (27). It can be transformed to the more general form:

$$\dot{p} = \Gamma\frac{q}{q-1}(I - \Psi)p^{q-1}, \tag{31}$$

where I is an identity operator, Ψ is a linear integral operator that is independent of p :

$$\Psi = \frac{(1, \cdot)}{\text{mes}(\Omega)} + \frac{\tilde{h}(\tilde{h}, \cdot)}{\|h\|^2 - \frac{1}{\text{mes}(\Omega)}(1, h)^2}, \tag{32}$$

$\tilde{h} = h - \frac{1}{\text{mes}(\Omega)}\int_\Omega h dx$.

5.1 Equilibrium stability

Let us examine the equilibrium of obtained equation (27). We use the same Lyapunov function $\dot{V}$ as we have used in previous section with only one constraint. For two constraints the new expression for $\dot{V}$ is

$$\dot{V}(p) = \frac{\Gamma q^2}{\text{mes}(\Omega)(q-1)^2}(A - B), \tag{33}$$

where
$A = \frac{\left(\text{mes}(\Omega)\int_\Omega h p^{q-1}dx - \int_\Omega h dx\int_\Omega (p^{q-1})^2 dx\right)^2}{\text{mes}(\Omega)\int_\Omega h^2 dx - \left(\int_\Omega h dx\right)^2}$,
$B = \left(\text{mes}(\Omega)\int_\Omega (p^{q-1}(t, x))^2 dx - \left(\int_\Omega p^{q-1}(t, x)dx\right)^2\right)$
We will prove that

$$\dot{V}(p) \leq 0. \tag{34}$$

Let us define a functional:
$\langle\cdot,\cdot\rangle : L_2(\Omega)\times L_2(\Omega) \to \mathbb{R}, \forall f, g \in L_2(\Omega)$

$$\langle f, g\rangle = \text{mes}(\Omega)\int_\Omega fg dx - \int_\Omega f dx\int_\Omega g dx. \tag{35}$$

New functional has several useful properties (proof of each property is provided in [Fradkov and Shalymov, 2015]):

1. Linearity for the first argument
$\forall f, g, h \in L_2(\Omega),\ \forall\lambda \in \mathbb{R}$
$\langle \lambda f + g, h\rangle = \langle \lambda f, h\rangle + \langle g, h\rangle$.

2. Symmetry
$\forall f, g \in L_2(\Omega)\ \langle f, g\rangle = \langle g, f\rangle$.
3. Positiveness and the condition of zero value
$\forall f \in L_2(\Omega)\ \langle f, f\rangle \geq 0$,
$\langle f, f\rangle = 0 \Leftrightarrow f = \mu = const$.
Let's prove inequality (34) base on properties 1-3.
Obvious that for any $f, g \in L_2(\Omega)$ and $\lambda \in \mathbb{R}$ it is true that $f - \lambda g \in L_2(\Omega)$. This function has property 3: $\langle f - \lambda g, f - \lambda g\rangle \geq 0$. Using properties 1 and 2 we get the quadratic inequality with respect to λ:

$$0 \leq \langle f - \lambda g, f - \lambda g\rangle = \lambda^2\langle g, g\rangle - 2\lambda\langle f, g\rangle + \langle f, f\rangle.$$

This inequality holds for any real λ. Hence the discriminant can not be positive.

$$D = 4\langle f, g\rangle^2 - 4\langle f, f\rangle\langle g, g\rangle.$$

Thus

$$\langle f, g\rangle^2 \leq \langle f, f\rangle\langle g, g\rangle. \tag{36}$$

If the equality takes place in (36) then there exists a unique solution $\lambda \in \mathbb{R}$ of an equation $\langle f - \lambda g, \langle f - \lambda g\rangle = 0$. But then by property 3 we have $\exists \mu \in \mathbb{R} :\ f - \lambda g = \mu\mathbf{1}$.
Substituting $f = p^{q-1}, g = h$ to the inequality (36) we get

$$\left(\text{mes}(\Omega)\int_\Omega hp^{q-1}dx - \int_\Omega hdx\int_\Omega p^{q-1}dx\right)^2 \leq \left(\text{mes}(\Omega)\int_\Omega \left(p^{q-1}\right)^2 pdx - \left(\int_\Omega p^{q-1}dx\right)^2\right) \times \left(\text{mes}(\Omega)\int_\Omega h^2 pdx - \left(\int_\Omega hdx\right)^2\right).$$

Which implies the inequality (34).
Note that the equality (34) holds if and only if

$$\exists \lambda, \mu \in \mathbb{R} :\ p^{q-1} = \lambda h + \mu. \tag{37}$$

According to the speed-gradient for H (16) the expression (27) can be rewritten for the case of equilibrium as

$$p(t, x)^{q-1} = \frac{C\lambda_1 h(x)}{\Gamma} + \frac{C\lambda_2}{\Gamma}, \tag{38}$$

where $C = \frac{(1-q)}{q}$. It coincides with (37) for $\lambda = \lambda_1\frac{C}{\Gamma}$, $\mu = \lambda_2\frac{C}{\Gamma}$ where λ_1 and λ_2 are defined in (28). So there is only one PDF p^* for an equilibrium state.

5.2 Asymptotic convergence

We will prove asymptotic convergence similarly to the case with one constraint.

Theorem 5.1. *For all PDFs defined by equation (27) it is true that* $p(t, x) \to p^*$ *for* $t \to \infty$.

Proof. To use Barbalat's lemma we have to check conditions under which the function $\ddot{v}$ is bounded. Based on the expression for $\dot{v}$ in (33) and according to the similar logic as it was used in Proposition 1 we can conclude that $\ddot{v}$ is bounded for the compact carrier Ω. According to Barbalat's lemma it is true that

$$\dot{v} \to 0,\ t \to \infty. \tag{39}$$

We introduce a scalar product as $\langle f, f\rangle = \text{mes}(\Omega)\int_\Omega f^2(x)dx - (\int_\Omega f(x)dx)^2$. Having $\|f\|^2 = \langle f, f\rangle$ the expression (33) can be rewritten as

$$\dot{v} = \frac{\Gamma q^2}{\text{mes}(\Omega)(q-1)^2}\left(\|p^{q-1}\|^2 - \frac{(p^{q-1}, h)^2}{\|h\|^2}\right) = \frac{\Gamma q^2\|p^{q-1}\|^2}{\text{mes}(\Omega)(q-1)^2}\left(1 - \frac{(p^{q-1}, h)^2}{\|p^{q-1}\|^2\|h\|^2}\right). \tag{40}$$

As $\dot{v} \to 0$, consider the case when $\|p^{q-1}\|^2 \to 0$ first. In CBS inequality

$$\left(\int_\Omega p^{q-1}dx\right)^2 \leq \text{mes}(\Omega)\int_\Omega \left(p^{q-1}\right)^2 pdx$$

the equality takes place only when $p^{q-1} = \alpha$. We have previously demonstrated in (38) that the equality $\dot{v} = 0$ holds in the only one case when $p^{q-1} = \lambda h + \mu$, where λ and μ are constants. Then h must be a constant since the equality $\lambda h + \mu = \alpha$. This case we assume to be degenerate.
Given (39), (40) and $\|p^{q-1}\|^2 \to 0$ we have that $\frac{\left(p^{q-1}, h\right)^2}{\|p^{\beta-1}\|^2\|h\|^2} \to 1$. Which implies that $\widehat{p}^{q-1} \to \widehat{h}$, where $\widehat{p}^{q-1}$ and $\widehat{h}$ are normalized values for p^{q-1} and h respectively. Thus p tends to the only one stationary distribution p^* since h does not depend on time.

6 Correspondence to the Tsallis distribution

Let us show that distribution p^* corresponds to the Tsallis distribution which is equilibrium distribution for the Tsallis MaxEnt with a given set of constraints. According to [Tsallis, 1988; Tsallis, Mendes and Plastino, 1998] this distribution for continuous case can be defined as:

$$p_i = \frac{1}{Z_q}\left(1 - \beta(q-1)E_i\right)^{\frac{1}{q-1}} \tag{41}$$

where β is a special Lagrange multiplier and $Z_q = \int_\Omega (1-\beta(q-1)h(x))^{\frac{1}{q-1}}\,dx$ stands for the normalization constant.

Taking into account eq. (27) for the case of stationary distribution p^* it is true that $\Gamma\frac{q}{q-1}p^{q-1}+\lambda_1 h+\lambda_2 = 0$. We get that

$$p(x,t) = \left((\lambda_1 h(x)+\lambda_2)\,\frac{1-q}{\Gamma q}\right)^{\frac{1}{q-1}} \qquad (42)$$

Let us substitute $p(x,t)$ from (42) into (13). We get

$$\left(\frac{1-q}{\Gamma q}\right)^{\frac{1}{q-1}}\int_\Omega (\lambda_1 h(x)+\lambda_2)^{\frac{1}{q-1}} = 1$$
$$\Rightarrow \left(\frac{1-q}{\Gamma q}\right)^{\frac{1}{q-1}} = \frac{1}{\int_\Omega (\lambda_1 h(x)+\lambda_2)^{\frac{1}{q-1}}} \qquad (43)$$

After substitution (43) into (42) we get that

$$p(x,t) = \frac{(\lambda_1 h(x)+\lambda_2)^{\frac{1}{q-1}}}{\int_\Omega (\lambda_1 h(x)+\lambda_2)^{\frac{1}{q-1}}} \qquad (44)$$

Let us denote

$$\frac{\lambda_1}{\lambda_2} = -\beta(q-1) \qquad (45)$$

As $\lambda_1 E_i + \lambda_2 = \lambda_2\left(1+\frac{\lambda_1}{\lambda_2}E_i\right)$ and taking into account Equation (45), the Equation (43) can be transformed to

$$p(x,t) = \frac{(1-\beta(q-1))^{\frac{1}{q-1}}}{\int_\Omega (1-\beta(q-1))^{\frac{1}{q-1}}}, \qquad (46)$$

where $\beta = -\frac{1}{q-1}\frac{\lambda_1}{\lambda_2}$. We can see that (46) coincides with the Tsallis distribution (41). As mentioned in [Tsallis, Mendes and Plastino, 1998], β in Equation (41) is not the Lagrange multiplier associated to the internal energy constraint (which is λ_1 in our notation). Following by notation of C. Tsallis (see Equation (10) in [Tsallis, 1988]) we have that $\lambda_1 = -\lambda_2\beta(q-1)$. It explains the variable substitution in (45).

It is evident that (46) satisfies the mass conservation constraint (13). Let us check that the second constraint for energy (26) is also satisfied.

Let us substitute $p(x,t)$ from (42) into (26). Then we get

$$\left(\frac{1-q}{\Gamma q}\right)^{\frac{1}{q-1}}\int_\Omega (\lambda_1 h(x)+\lambda_2)^{\frac{1}{q-1}}\,h(x)dx = E$$
$$\Rightarrow \left(\frac{1-q}{\Gamma q}\right)^{\frac{1}{q-1}} = \frac{E}{\int_\Omega (\lambda_1 h(x)+\lambda_2)^{\frac{1}{q-1}}\,h(x)dx} \qquad (47)$$

After substitution (47) into (42) we get that

$$p(x,t) = (\lambda_1 h(x)+\lambda_2)^{\frac{1}{q-1}} \times$$
$$\times\left(\frac{E}{\int_\Omega (\lambda_1 h(x)+\lambda_2)^{\frac{1}{q-1}}\,h(x)dx}\right)$$
$$\Rightarrow\ h(x)p(x,t) = \frac{(\lambda_1 h(x)+\lambda_2)^{\frac{1}{q-1}} * E}{\int_\Omega (\lambda_1 h(x)+\lambda_2)^{\frac{1}{q-1}}\,h(x)dx}$$
$$\Rightarrow\ \int_\Omega h(x)p(x,t)dx = E \qquad (48)$$

Which means that internal energy constraint (26) is also true for (46).

7 Conclusion

The Tsallis entropy is widely used in communication and coding theory, signal processing, data mining and many other areas [Tsallis, 2016]. Stationary states which maximizes the Tsallis entropy are already well investigated for the discrete case [Fradkov and Shalymov, 2015]. The MaxEnt principle defines the asymptotic behavior of the system, but does not answer for the question about how the system moves to this asymptotic behavior.

In this paper a non-stationary states of processes that follow the MaxEnt principle for the continuous form of the Tsallis entropy are investigated. The equations (19), (20), (27) and (31) which describe dynamics of PDF for the system that tends to the state with maximum Tsallis entropy have been derived. Systems with discrete probability distribution and continuous PDFs are considered under mass conservation and energy conservation constraints. It is shown that the limit PDF p^* is unique and corresponds to the Tsallis distribution. The convergence of PDFs with dynamics described by equations (19) and (27) to PDF p^* which corresponds to the state with maximum value of the Tsallis entropy has been proved.

From the physics view point the new derived equations (19), (20), (27) and (31) allow us to predict the behavior of complex non-stationary systems which tend to maximize its continuous form of the Tsallis entropy. It may find further applications in statistical physics and thermodynamics, also in the field of communication and signal processing.

The key point of proposed approach is using the SG-method with the goal function chosen as a continuous form of the Tsallis entropy. The SG-principle is originated from the control theory and it generates equations for the transient (non-stationary) states of the system operation which help to track how the system evolves to the steady-state.

More general forms of relative entropies and divergences such as CR entropy or Csiszár–Morimoto conditional entropies (f-divergencies) [Morimoto, 1963] can also be considered from the SG-principle perspective. Investigation of dynamics for these entropies seems to be promising for further investigations.

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