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COOPERATIVE PATH PLANNING IN THE PRESENCE OF ADVERSARIAL BEHAVIOR

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Abstract
A collaborative control scenario is modeled in the framework of hybrid systems. The considered problems include a minimum time problem with state constraints and under adversarial behaviour. The problem is modeled as a differential game. Traditional methods are extended to handle the hybrid nature of the problem. Optimal feedback strategies are derived and the system behaviour is discussed with the aid of Krasovkii's u-stable bridge.

Key words
Optimal Control, Hybrid Systems, Differential Games, Collaborative Control

1 Introduction

Problems of collaborative multi-vehicle control are posing new challenges to control. In some problems, cooperation concerns distributing similar vehicles over an area to optimize, for example, the rate of area coverage for surveillance missions. In other problems, heterogeneous vehicles with complementary capabilities are used more advantageously when other forms of cooperation take place. One such example arises when planing operations of unmanned air vehicles (UAV) in hostile air spaces. The probability of survival of an UAV is directly proportional to the value of the path integral taken with respect to some risk function (de Sousa *et al.*, 2004); the level of risk is significantly reduced when the UAV flies under the protection of an UAV carrying a jamming device. This is an example of a collaborative control problem where vehicles interact to improve individual or group performance.

In previous work (see (de Sousa and da Silva, 2008) and (de Sousa *et al.*, 2009)) we discussed aspects of our research on using dynamic optimization for solving collaborative control problems with the help of a simple two-vehicle optimal path coordination control problem. This problem is representative of more general optimal coordination problems. The problem is modeled in the framework of hybrid systems. Here we extend that formulation and consider the case of optimization under adversarial behaviour. The adversarial behaviour models the worst case effect of disturbances.

In the case of deterministic optimal controls problems, it was shown in (de Sousa and da Silva, 2008) that it may be worthwhile for v_1 to deviate from the optimal path (of isolated operation) to join other vehicle that will contribute to improved conditions of operation (and to a reduced overall cost to go). In the present case, we consider that when v_1 operates with other vehicles the effect of the adversarial action is reduced.

We consider the problem of finding the minimum time for v_1 to reach a given region S and the respective optimal control. For independent operation of v_1, this is a classic differential game (see, for instance, (Bardi *et al.*, 1999)). However, the problem becomes highly discontinuous and gains a combinatorial flavour when collaboration between vehicles is considered. Our approach, based on Dynamic Programing (Bellman, 1957), reduces the complexity of such problem. We also consider the problem of reaching a given target set $T := \{(x,t) : x \in S, t \in [t_1, t_2]\}$. The system is studied with the aid of Krasovkii's u-stable bridge (Krasovskii, 1995). The discrete component of the system dynamics raises interesting questions: 1) if the initial condition for v_1 does not belong to the u-stable bridge corresponding to independent operation of v_1 it may happen that it may be worthwhile for v_1 to meet other vehicles before reaching the target set as required; 2) if the initial condition for v_1 belongs to the u-stable bridge, then it may happen that we can impose stricter timing constraints for T.

2 Problem formulation

2.1 The system

We consider two types of vehicles: *simple* and *jammer*. A vehicle is characterized by its type and its state. The system is composed of N vehicles and it is thus described by a set of the form

$$\{v_1 = (type_1, (x_1)), \cdots, v_N = (type_N, (x_N))\} \quad (1)$$

The *jammer* vehicles are subject to fuel constraints whereas it is assumed for *simple* vehicles that fuel consumption is of no concern. The *simple* vehicles are subject to adversarial actions. However, the adversarial action against a *simple* vehicle is eliminated when its position coincides with the position of a *jammer* vehicle. This joint operation is a case of collaboration. Notice that this is a simplification, since in practice no two vehicles may be in the same position at the same time; in practice, the *simple* vehicle would be required to be in a given neighbourhood of the *jammer* vehicle. In any case, this means that the dynamics of a *simple* vehicle is a discontinuous function of its position relative to the *jammer* vehicles. The motion model for the *simple* vehicle i, when operating alone, is given by

$$\dot{x_i}(t) = f_i(x_i, u_i) + g_i(x_i, p_i) \quad (2)$$

where $x_i \in \mathbb{R}^n$ is the position of vehicle i, $u_i \in U_i$ is the respective control input, $p_i \in P_i$ is the adversarial input and U_i and P_i are closed sets.

The *jammer* vehicles are not affected by the adversarial behaviour. The kinematic motion model for *jammer* vehicles on independent operation is given by

$$\dot{x_i}(t) = f_i(x_i, u_i) \quad (3)$$

The motion model for joint operation of a *simple* vehicle i with a *jammer* vehicle j is given by

$$\dot{x_i}(t) = f_{ij}(x_i, u_{ij}) \quad (4)$$

where $u_{ij} \in U_{ij}$ is the control input and $f_{ij}(x, u)$ and U_{ij} are defined such that

$$f_{ij}(x_i, U_{ij}) = f_i(x_i, U_i) \bigcap f_j(t, x_i, U_j) \quad (5)$$

This means that the *jammer* vehicle will be capable of replicating the motion of the *simple* vehicle.

Remark 1. *The model can be easily extended in order to consider only a partial reduction of the disturbance. It can also be extended in order to model the reduction of the disturbance as a function of the set of jammer vehicles travelling simultaneously with the simple vehicle. This could be useful, for instance, to model a disturbance that can only be totally eliminated by the presence of two jammer vehicles.*

The amount of available fuel of *jammer* vehicle i is modeled by the state variable $c_i \in \mathbb{R}$:

$$\dot{c}_i(t) = \begin{cases} w_i(x_i, u_i) & \text{if } c_2 > 0 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

$$c_i(0) = \theta_i \quad (7)$$

where $w_i(.,.) \leq 0$.

Consider the following two value functions: $V_i^f(x)$ and $V_i^b(x)$. $V_i^f(x)$ is a map of x to the minimum amount of fuel required by *jammer* vehicle v_i to reach x after departing from its base. The value function is the solution of the following PDE (see (Kurzhanskii and Varaiya, 2001) for details on dynamic optimization techniques for reachability analysis):

$$\sup_u [\nabla V_i^f(x) f_i(x, u) - w_i(x, u)] = 0, u \in U_i \quad (8)$$

$$V_i^f(x_i(0)) = 0 \quad (9)$$

Consider $V_i^b(t, x)$. This is a map of x to the minimum amount of fuel required by *jammer* vehicle v_i to reach a destination base from x. The value function is the solution of the following PDE:

$$\sup_u [-\nabla V_i^b(x) f_i(x, u) - w_i(x, u)] = 0, x \notin B \quad (10)$$

$$V_i^b(x) = 0, x \in B \quad (11)$$

where B is the set of positions corresponding to returning bases. It can be seen that if the vehicle is not required to return to any base then $B = \mathbb{R}^n$ and $\forall x \in \mathbb{R}^n : V_i^b(x) = 0$.

Let $R_i = \{x : \theta_i - V_i^b(x) - V_i^f(x) \geq 0\}$ be the set of points that can be reached by *jammer* vehicle v_i under the above mentioned operational constraints. Joint operation between *simple* vehicles and v_i may only occur in R_i. This means that a *simple* vehicle will only benefit from reduced adversarial action when going through R_i.

The vehicles are allowed to meet once and move together while the *jammer* vehicle has enough fuel to return to a base. If the *jammer* vehicle is not required to return to a base, the vehicles may travel together until the *jammer* vehicle runs out of fuel. It is possible to devise scenarios where this policy is not optimal (e.g., heterogeneous vehicles separate in order to benefit from "corridors" that enhance their specific characteristic and then meet at a new region where joint operation is advantageous again). The advantage of this policy is that it allows us to reduce the dimension of the state space of the global system. It also simplifies the problem of coordination. In what follows, we consider a system with a single *simple* vehicle, designated by v_1. v_1 is allowed to meet and separate only once with each *jammer* vehicle.

2.1.1 Two vehicles scenario For $N = 2$ (one *simple* vehicle v_1 and one *jammer* v_2), we have three possible distinct modes:

Mode a**:** v_1 operating alone, without having met v_2. This is the initial mode.

Mode b**:** v_1 moving together with v_2 in joint operation; this mode is optional.

Mode c**:** v_1 operating alone, after meeting with v_2; this mode happens after and only after mode b.

In mode a, only the state of v_1 is tracked. v_1 must decide whether it meets with v_2 or not. If it decides to meet v_2, it will have to define the meeting point where the system will switch from mode a to mode b. From the perspective of v_1, all that really matters in what concerns v_2 is: 1) the point where the meeting takes place, which must be inside R_i; and 2) the amount of the fuel remaining in the fuel tank of v_2 at the meeting position, given by $\theta_2 - V_2^f(x)$. When the vehicles reach the meeting position, the system switches to mode b. In mode b, the state variable corresponding to the fuel of v_2 is tracked along with the state of v_1. Given (4) and (5), this is enough to describe the global system's state. v_1 may decide to abandon v_2 still inside R_2. On the other hand, as soon as v_1 leaves R_2 v_2 must head back to its returning base. On both cases, the system switches to mode c. On mode c, like on mode a, only the state of v_1 is tracked. This model of operation can be represented by the hybrid automaton on Figure 1, where each mode of operation corresponds to a discrete state. We use the notation of (Branicky, 1995), where "?" and "!" stand for controlled and autonomous transitions, respectively; the expressions between square brackets are *guards* for the respective transition (a transition may only occur if the guard condition is verified); finally, resets of the state variables are defined after the slash symbol.

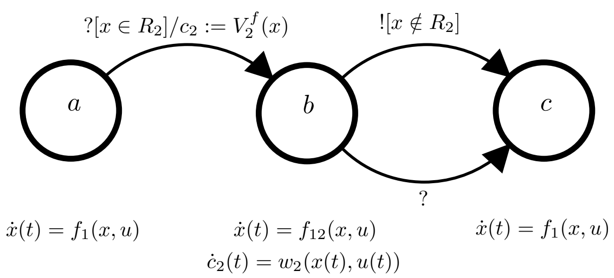

Figure 1. Hybrid automaton modeling the system. The continuous state space on mode b has one additional dimension to model available fuel in v_2.

2.1.2 General case In a scenario with multiple *jammer* vehicles, there are $3^{(N-1)}$ possible modes of operation if all possible interactions between the *simple* vehicle and the remaining vehicles are considered. For instance, for $N = 5$, we could have, at a certain time, v_2 already used, v_3 and v_4 being used and v_5 still not used. We denote the set of all possible discrete modes as Q. The full continuous state for each discrete mode is $x_v, v \in Q$. The hybrid state is defined by the tuple (x_v, v). For $N = 2$, we have $Q := \{a, b, c\}$, $x_a = x_b = x$ and $x_b = (x, c_2)$.

2.1.3 Computational requirements Consider that we want to store some specific data for each point of the state space (e.g., the optimal-time to the origin). If no parametric description exists, some type of grid must be defined. For simplicity, let us assume a regular grid, with the same number of nodes D along each dimension. Assume that each node requires K units of memory. In the case $N = 2$, it can be seen that the required number of continuous state variables is $n + 1$ for the proposed formulation, as opposed to the $2n + 1$ variables that would be required if explicit tracking of v_2's state was to be performed at all times. However, we must take in account that the continuous state space must be described on each mode. Under these assumptions, the memory requirements for our approach would be $K2D^n + D^{(n+1)}$ against $KD^{(2n+1)}$.

In the general case, the memory requirements for a model with full state tracking of each vehicle would be $KD^{(Nn+N-1)}$. In the proposed approach, the maximum number of continuous state variables is $n+N-1$. Let us assume that all $N-1$ fuel variables are tracked on each mode. This is an over-approximation, since it is only necessary to track the fuel of the *jammer* vehicles being used. Then the memory requirements are $KD^{(n+N-1)}3^M$ leading to a decrease ratio of memory requirements of more than $(\frac{D^n}{3})^{(N-1)}$

2.2 Problem 1

Let $I(x_v, v) \to ([0, t_f] \to U_1 \times Q)$ denote an admissible control for v_1 with respect to the starting hybrid state (x_v, v) and $\Lambda(x_v, v)$ denote the set of all admissible controls with respect to the same state. The cost of a trajectory from (x_v, v) to a predefined target region $S \in \mathbb{R}^n$ under control $I(x_v, v)$ is

$$\tilde{J}_1(I(x_v, v), p(.), S) = t_f \tag{12}$$

where $x(t_f) \in S$.

The problem consists of finding the minimum time to reach S from x along with the respective optimal hybrid control:

$$t_f^* = \inf_{I(x,a)} \sup_{p(.)} \tilde{J}_1(I(x, a), p(.), S) \tag{13}$$

with $I(x, a) \in \Lambda(x, a)$.

It must be remarked that $t = 0$ corresponds to the instant when v_1 starts moving. In certain scenarios, the *jammer* vehicles must start moving before v_1 in order to reach the meeting point at the optimal instant;

in these scenarios, v_1 must wait after the optimal coordination has been decided. In order to consider the time of coordination decision as $t = 0$, V_i^b, V_i^f and R would have to be defined as time dependent. We do not explore that perspective in the current work.

2.3 Problem 2

Find the set of initial states for which there is an admissible trajectory of v_1 to the target set

$$T := \{(x, t) : x \in S, t \in [t_1, t_2]\} \tag{14}$$

3 Solution approach

Our approach is inspired by (Sethian and Vladimirsky, 2002) and (Zhang and James, 2006). The former do not consider state constraints. The later discusses the conditions under which the value function is the solution of the Hamilton-Jacobi-Bellman equation.

3.1 Problem 1

Without loss of generality, only the $N = 2$ case is considered. Define

$$C_2 = \{(x, c_2) : V_2^b(x) \leq c_2 \leq \theta_2 - V_2^f(x)\} \tag{15}$$

as the set states that can be reached by v_i from its initial state. Consider the following value function:

$$V(x_v, v) = \inf_{I(x_v,v)} \sup_{p(.)} \tilde{J}_1(I(x_v, v), p(.), S) \tag{16}$$

with $V(S, a) = V(S, c) = 0$ and the state constraint $\forall x_b \notin C_2 : V(x_b, b) = +\infty$. The minimum time to reach S when starting from x is $\overline{V}(x) = V(x, a)$.

Assume that f_i, g_i and w_i fulfil the standard assumptions for uniqueness and existence of a solution to Problem 1 (Bardi *et al.*, 1999). Then the value function $V(x_v, v)$ satisfies the principle of optimality for every discrete state $v \in Q$. Moreover, $V(x_v, v)$ is the viscosity solution of the Hamilton-Jacobi-Isaacs Partial Differential Equation (HJI PDE):

$$\sup_{u \in U_v} \inf_{p \in P_v} [-\nabla V(x_v, v) \cdot f_v(x_v, u, p)] = 1 \tag{17}$$

where f_v describes the continuous flow on each discrete state. In the case $N = 2$, $f_a = f_c = f_1 - g_1$ and $f_b = (f_{12}, w_2)$. Additionally, the following conditions must be enforced when $x \in R_2$:

$$V(x_b, b) > V(x, c) \Rightarrow V(x_b, b) = V(x, c) \tag{18}$$

$$V(x, a) > V(x_b, b) \Rightarrow V(x, a) = V(x_b, b) \tag{19}$$

Notice that (18) is valid only if $x_b \in C_2$.

The formulation can be extended in order to deal with time dependent systems (e.g., $g_i(t, x, u)$). This would imply the definition of $V_i^f(t, x)$, $V_i^b(t, x)$ and $R_i(t)$ in order to model the switching behaviour. The static HJI would have to be replaced by the time-dependent HJI PDE

$$\frac{\partial V}{\partial t} + H(t, x, \nabla V) = 0 \tag{20}$$

with $V(t, S, v) = 0$, $t \in [0, t_{\max}]$ and $t_{\max}$ some acceptable finite time horizon.

3.1.1 Optimal Strategies

The value function (16) allows us to implement optimal feedback strategies. The optimal continuous control u^* is given by

$$\text{argmax}_{u \in U_v} \inf_{p \in P_v} [-\nabla V(x_v^*, v^*) \cdot f_v(x_v^*, u, p)] \tag{21}$$

with ∇V as some form of quasi-gradient of V. However, it can be seen that the computation of u^* requires the knowledge of the optimal mode v^*. This is done by parts. Let us define $\overline{V}_u(x)$ as the minimum time for v_1 to reach S on independent operation. If $\overline{V}_u(x) \leq \overline{V}(x)$ then $x_v^* = a$. If $\overline{V}_u(x) > \overline{V}(x)$ then the optimal trajectory must be computed up to the point where $\exists c_2 : V(x, a) \geq V((x, c_2), b)$. The corresponding position will mark the meeting point between v_1 and v_2. In a practical implementation this must be done before v_1 starts moving. The remaining computation can be done in real-time. The transition from mode b to mode c occurs when $V((x, c_2), b) \geq V(x, c)$.

3.2 Problem 2

Krasovkii's u-stable bridge (Krasovskii, 1995) allows us to study the solution of this kind of problems and it may be also used to develop extremal aiming feedback strategies. The u-stable bridge $W(t)$ is the set of states at time t for which reachability to the target set is assured. The extended u-stable bridge corresponding to the hybrid system may be computed by noting that

$$W(t) = \{x : V(t, x_v, a) \neq \infty\} \tag{22}$$

where $V(t, x, v)$ is computed by (20). If the initial condition for v_1 does not belong to the u-stable bridge then there are no guarantees that v_1 reaches the target set. By comparing the extended u-stable bridge to that of the independent operation one notices that, for some intervals of t in the considered time horizon, the set of admissible states is enlarged. This answers question 1 of the introduction. The answer for question 2 is illustrated in the next section.

4 Examples

The first example was chosen just to illustrate some of the subtleties of the cooperation problem. Examples

2 and 3 illustrate Problems 1 and 2 respectively. All examples assume the case $N = 2$, one *jammer* vehicle and one *simple* vehicle.

4.1 Example 1

If the rate of fuel consumption is constant, the time-optimal trajectories on mode b will consist of segments traveled at maximum speed. Otherwise, the optimal speed profile will have to be computed along with the optimal path. The following example consider a one-dimensional motion model. The objective is to illustrate the nonlinear relation between the optimal speed and the travelling distance in a example where it is possible to find the solution in explicit form.

On modes a and c, $\dot{x}(t) = u_u$ (assuming the worst case adversarial input). On mode b, $\dot{x}(t) = u_c, 0 < u_{\min} \le u_c < \infty$; for the model of fuel consumption we have $w_2(x, u) = Ku^3$. v_2 departs from $x_{20} = \frac{\theta_2}{2Ku_{\min}^2}$ and must return to that same position. Given the fuel model, v_2 will choose the minimum speed when moving alone, in order to save fuel for joint operation. Therefore, $V_2^f(x) = V_2^b(x) = Ku_{\text{min}}^2|x - x_{20}|$. The target set is $S = \{0\}$. The problem consists of finding the minimum time to reach S from any given $x \in [0, x_{20}]$. In this scenario, v_1 will always benefit from joint operation. Therefore, the problem amounts to finding the optimal u_c and the points of transition between modes. The transition from mode a to b must occur right at the starting point of v_1. The point of transition from mode b to c, x_t, is a function of the starting point of v_1. The objective function can be defined in the following way:

$$\min\left(\frac{x_t}{u_u} + \frac{x - x_t}{u_c}\right) \tag{23}$$

In this case, it is obvious that the optimal solution implies total fuel expenditure. Thus, we have:

$$\theta_2 = K\left[(2x_{20} - x_t - x)u_{\min}^2 + (x - x_t)u_c^2\right] \tag{24}$$

Let us define $\alpha_1 = \frac{u_c}{u_u}$ and $\alpha_2 = \frac{u_{\min}}{u_u}$. Then, (24) and (23) become respectively

$$x_t = x\frac{(\alpha_1^2 - \alpha_2^2)}{(\alpha_1^2 + \alpha_2^2)} \tag{25}$$

$$\min_{\alpha_1}\left(x\frac{(\alpha_1^3 - \alpha_2^2\alpha_1 + 2\alpha_2^2)}{\alpha_1(\alpha_1^2 + \alpha_2^2)}\right) \tag{26}$$

The minimum of (26) is achieved for $\alpha_1 = \alpha_1^*$ such that $(\alpha_1^*)^3 - 1.5(\alpha_1^*)^2 - 0.5\alpha_2^2 = 0$, as long as the resulting u_c and x_t verify $u_c \ge u_{\min}$ and $x_t \in [0, x_{20}]$. For $\alpha_2 \in [0, 2]$, the exact value of α_1^* may be obtained by the explicit formula for the solution of the cubic equation (with one real root and two complex roots). For $\alpha_2 = 0$, we have $\alpha_1^* = 1.5$. This is a limit case. Physically, it means that v_2 would not spend any fuel to meet v_1 and to return home. For $\alpha_2 \ge 2$, we have $\alpha_1^* = \alpha_2$, i.e., $u_c^* = u_{\min}$. Notice that in every case the optimal coordinated speed, u_c^*, is independent of the initial position of v_1.

4.2 Example 2

The motion models are characterized as follows:

$$f_1(x, (u, \psi)) = \begin{cases} 40u\cos(\psi) \\ 40u\sin(\psi) \end{cases} \tag{27}$$

$$g_1(x, (p, \psi_p)) = \begin{cases} 39p\cos(\psi_p) \\ 39p\sin(\psi_p) \end{cases} \tag{28}$$

with $|u| \le 1$, $|p| \le 1$. Additionally, $f_{12} = f_2 = f_1$. The model of fuel consumption is given by:

$$\dot{c_2}(t) = \begin{cases} 2, u < 0.25 \\ 128|u|^3, u \ge 0.25 \end{cases} \tag{29}$$

with $c_2(0) = \theta_2 = 12$. The starting and returning point of v_2 is $\beta = (50, 40)$. The circle of radius 30 centered at β encloses the set of points reachable by v_2 within fuel constraints. The target set is the origin of $\mathbb{R}^2$. The value function $\overline{V}(x)$ for the minimum time problem is illustrated on Figure 2, along with the optimal trajectories for randomly chosen initial positions of v_1. The gray area identifies the set of initial positions for which the traveling time of v_1 is reduced if it benefits from cooperation of v_2.

4.3 Example 3

Consider a one-dimensional scenario with $-1 \le f_1(x, u) + g_1(x, p) \le 1$, $-10 \le f_{12}(x, u) \le 10$, $c_2(0) = 0$, $w_2(x, u) = -1$ and $x_{20} = 5$. The u-stable bridge for $S = \{0\}$ is illustrated on Figure 3. The right diagram of Figure 3 corresponds to mode b but it is only a projection of the bridge corresponding to that mode (the fuel variable is not represented). The global stable bridge is the reunion of I and II. Region II is also the stable bridge corresponding to independent operation of v_1.

In order to illustrate the answer to the questions posed in the introduction, consider $t = -10.5$. Without collaboration from *jammer* vehicles, v_1 could only reach the origin in less than 10.5 units of time (u.t.) when starting from $x \in [0, 10.5[$. With collaboration, the range is enlarged to $x \in [0, 15[$. On the other hand, when starting from $x = 15$, v_1 could only reach the origin in 15 (u.t.) without collaboration; with collaboration, that time can be shortened to 10.5 (u.t.). It can be seen that for $x \ge 10$ the gain of collaboration is 4.5 (u.t.). We remark that when v_1 is on the boundary of

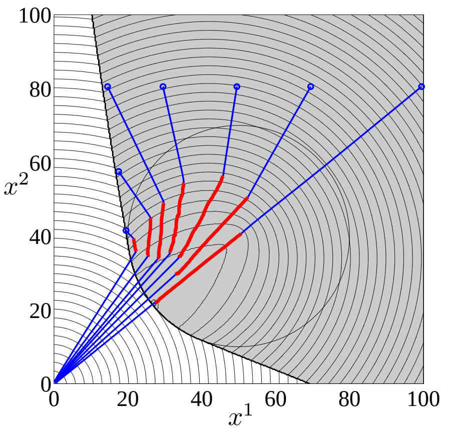

Figure 2. Level sets of $\overline{V}(x)$ for example 3, along with the optimal trajectories for arbitrarily selected starting points. The coordinated flight phase is plotted on red (thick). The circle delimits R_2, the set of points that v_2 can reach and still return to its initial position. The gray area marks the destinations for which coordinated operation is the optimal choice.

the stable bridge, it should use the feedback strategy derived for Problem 1. In the remaining cases, it may use any admissible control.

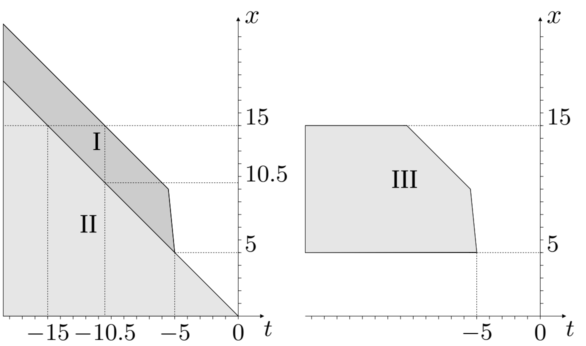

Figure 3. U-stable bridge for Example 4. The global stable bridge is the reunion of I and II. Region II is the stable bridge corresponding to independent operation of v_1. Region III corresponds to mode b (notice that it is limited by $x = 5$ and $x = 15$)

5 Conclusion

The global approach allows a systematic qualitative and quantitative determination of whether cooperation is advantageous or not, along with the determination of the respective optimal trajectory. The value function for the hybrid optimal control problem can be computed as the solution of a HJI PDE involving discrete state variables. As in the conventional approach, the optimal trajectory is easily computed from the value function.

The considered model, assuming additive adversarial behaviour, is suitable to model several physical scenarios. Additionally, it assures the saddle point condition in a small game (Isaacs condition). The fact that the effects of the control and adversarial inputs are separable greatly simplifies the problem formulation. However, there are cases where a multiplicative adversarial action would be more suitable (e.g., to model the reduction of drag when a vehicle is running on the tail of another). We consider that under mild assumptions such formulations could still fulfil the Isaacs condition.

The formulation can be improved in order to deal with more complex dynamic models. However, the concept of joint operation defined here (see (4) and (5)) may prove non-trivial or too limiting on those cases, especially on heterogeneous sets of vehicles.

Due to computational requirements, the numerical computation of the value function is still limited to system of low dimension. Synthesis of optimal feedback controllers without resorting to storing the whole regular grid is also a topic of future work.

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